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ON THE DEFINING PROPERTIES OF TEICHMÜLLER MAP

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0. Introduction

The concept of Teichmüller mapping seems to be first explicitly introduced by Bers [4] in 1960, significance of which lies, of course, in the fact that it describes a necessary and sufficient condition for a homeomorphism of a closed Riemann surface to be extremal quasiconformal within individual homotopy classes. A few years later the remarkable counter-example was presented by Strebel [14] which showed that the Teichmüller character is no necessary condition for the extremal quasiconformality of mappings between disks with prescribed boundary correspondence: his extremal quasiconformal mapping also plays a part as an example illustrating the non-uniqueness of extremal quasiconformality for the non-compact problem.

Let R, S be a pair of topologically equivalent Riemann surfaces. It does not matter whether they are compact or not. A quasiconformal homeomorphism f of R onto S is customarily called Teichmüller mapping if the Beltrami coefficient μ_f of f satisfies an equation $\mu_f=\kappa\Phi/|\Phi|$ with a positive constant $\kappa(<1)$ and with an analytic differential Φ of type (2, 0) on R at every point of R where $\Phi\neq 0$ (cf. Bers [4], Strebel [14]). Or equivalently, the Teichmüller mapping f is defined as a diffeomorphic solution to the Beltrami differential equation with the coefficient μ_f which equals a constant κ $(0<\kappa<1)$ in modulus and whose argument agrees with the trajectories $\Phi>0$ for some analytic quadratic differential Φ except possibly at its zeros on R.

Here our special attention will be focussed upon the analyticity associated indirectly with Teichmüller mappings. According to Ahlfors [1] it appears to derive from the vanishing of the first variation of maximal dilatation as a functional, so far as the algebraic Riemann surfaces are concerned. On the other hand we know a very simple transcendental example of Teichmüller mapping which is not extremal quasiconformal. What does then characterize the Teichmüller maps at all? The present study arose from an attempt to answer this question, which is also written as a continuation of my previous work [13] in a certain sense. Major part of this paper is devoted to the study on those defining conditions of Teichmüller maps. The main result reads in a broad way that a constancy of dilatation in a modified sense implies the analyticity.

Our reasoning rests on the two fundamental facts as a whole. The one is the classical existence and uniqueness theorem on ordinary differential equations of the first order together with the dependence of the solutions on their initial data and the other the regularity that the solutions of Beltrami equations with adequately smooth coefficients enjoy.

We recall in §1 some notions as well as conventions employed in [13]. This section also prepares the concept such as characteristic directions at non-singular points of a quasiconformal mappings which was motivated by the eigen vectors of linear transformations. Aiming at prolongation of the characteristic direction at every point of the domain considered, we arrive at an idea of characteristic arcs and characteristic quadrilaterals in §2. A noteworthy relationship between characteristic arcs and a kind of closed differentials is obtained there as a by-product. It is well known that the local K-quasiconformality gives rise to the global one to the effect that an upper bound of the dilatations in a neighbourhood of each point becomes their upper bound in the large, too. What can we say if the upper bound K is replaced by a lower bound of dilatation-quotients for a smooth quasiconformal mapping? §3 deals with this problem, where the characteristic quadrilateral plays an essential rôle. This enables us to do with our principal theorem under the superfluous assumption of smoothness. The ultimate goal will be achieved in §4 in terms of maximal and minimal dilatations at points.

1. Preliminaries

Let $w=T_1(z)$ be a quasiconformal (but not conformal) mapping defined in a domain G of $\boldsymbol{C}$ and let $\mathfrak{T}[G]$ the space of all the quasiconformal homeomorphisms of G onto $G'=T_1(G)$ endowed with the topology of normal convergence in G. The class of j-times continuously differentiable topological mappings in G shall be denoted by $C^j[G]$. In the subsequent argument I shall refer to the notations and terminologies below, which were employed in my former paper [13]:

$N^\alpha(z)$: α-neighbourhood of the point z

$\dot{N}^\alpha(z)$: the deleted α-neighbourhood of the point z

mod Ω: modulus of the quadrilateral Ω

$$\left.\begin{array}{l} S=T^{-1} \text{ (the inverse mapping of } T) \\ p_T(z)=\partial T/\partial z, \quad q_T(z)=\partial T/\partial \bar{z} \\ Q_T(z)=\dfrac{|p_T(z)|+|q_T(z)|}{|p_T(z)|-|q_T(z)|} \\ \bar{D}_T(z)\text{: the maximal dilatation of } T \text{ at } z \\ \underline{D}_T(z)\text{: the minimal dilatation of } T \text{ at } z \end{array}\right\} \quad T\in\mathfrak{T}[G]$$

$T\in\mathfrak{I}[G]$ is said to be non-singular at a point $z_0\in G$, if $T(z)$ is totally differentiable at z_0 and further the Jacobian $|p_T(z_0)|^2-|q_T(z_0)|^2$ is positive. The Beltrami coefficient $\mu_T(z)=q_T(z)/p_T(z)$ for $T(z)$ can be determined at non-singular points. At such a point z_0 we have the differential inequalities

$$(|p_T(z_0)|-|q_T(z_0)|)|dz|\leq|dw|\leq(|p_T(z_0)|+|q_T(z_0)|)|dz|.$$

The extrema of $[|dw|/|dz|]_{z=z_0}$ are attained by the directions $\arg dz=(\arg\mu_T(z_0))/2$ and $\arg dz=[(\arg\mu_T(z_0))/2]+(\pi/2)$, which one calls characteristic directions of $T(z)$ at z_0: For geometric interpretation of this fact we observe the infinitesimal ellipse $E(z_0)$ centred at z_0 with the properties: (i) its minor axis points to the direction $\arg dz=(\arg\mu_T(z_0))/2$: (ii) magnitude of the major and minor axes is in the proportion $(|p_T(z_0)|+|q_T(z_0)|)/(|p_T(z_0)|-|q_T(z_0)|)$. Then $T(z)$ takes $E(z_0)$ to the infinitesimal circle centred at $w_0=T(z_0)$ to within infinitesimals of higher order. Let us agree to name the direction $\arg dz=(\arg\mu_T(z_0))/2$ to be *minor-axial*. The *major-axial* direction is orthogonal to the minor-axial direction. By means of $T(z)$ the characteristic directions of $T(z)$ at z_0 correspond to those of the inverse mapping $T^{-1}(w)$ at w_0, but the major-axial and minor-axial directions interchange with one another. $Q_T(z_0)=(|p_T(z_0)|+|q_T(z_0))|/(|p_T(z)_0|-|q_T(z_0)|)$ is the so-called dilatation-quotient of T at z_0.

The following familiar proposition will serve our infinitesimal consideration afterwards:

Lemma 1 (cf. Hedrick-Ingold-Westfall [10]). *Let $z_0\in G$ be a non-singular point for a $T\in\mathfrak{I}[G]\cap C^1[G]$. If $T(z)$ is not conformal at $z=z_0$, $T(z)$ preserves, among all angles with the vertex at z_0, only the four right angles formed by a pair of characteristic directions through z_0.*

We include also the proposition for later use which rephrases the second part of Theorem 6 in [13]:

Lemma 2. *Let $T\in\mathfrak{I}[G]\cap C^1[G]$ be non-singular everywhere in G and let $\bar{G}_0$ an arbitrary closed subregion of the domain G. Let θ be the characteristic direction of T at a point $z\in\bar{G}_0$ and let R denote the square with vertices z, $z+re^{i\theta}$, $z+(1+i)re^{i\theta}$, $z+ire^{i\theta}$ with a real positive r. Given any $\varepsilon>0$, there exists some $\delta=\delta(\varepsilon)>0$, such that $r<\delta$ implies the inequality*

$$|\max\{\text{mod } T(R),\ 1/\text{mod } T(R)\}-Q_T(z)|<\varepsilon$$

uniformly with respect to $z\in\bar{G}_0$.

In contrast with the significant counter-example due to Strebel there is an

EXAMPLE (Sasaki [12]). $w=T(z)=|z|^{\alpha}\exp\{i(\arg z-\sqrt{\alpha^2-1}\log|z|)\}$ ($\alpha>1$ being *const.*) is a Teichmüller mapping of $0<|z|<1$ onto $0<|w|<1$ but not extremal quasiconformal.

2. Characteristic quadrilaterals

Roughly speaking, the dilatations of a quasiconformal mapping are the quantities which measure the proportion between the moduli of quadrilaterals and those of their images under the mapping, regardless of whether in the infinitesimal, in the small or in the large. One has been with reason interested mainly in the estimate on such quantities from above, since the quasiconformality itself was originally defined by means of the inequalities of those types. If one wishes, however, to look for meaningful estimates of moduli from below, one will necessarily be obliged to distinguish the two cases according as the choice of permutations in which their vertices are ordered. In case of a regular smooth mapping it amounts to take care merely of a lower bound for maximal ratios of stretching to shrinking at all points in question. To this end we shall have to trace the loci of characteristic directions throughout the domain.

DEFINITION 1. Suppose that a domain G consists only of non-singular points for a $T(z)$ belonging to $\mathcal{Q}[G]\cap C^1[G]$ and that $T(z)$ is nowhere conformal in G. Then a smooth open arc C comprised in G is called to be *minor-axially* (resp. *major-axially*) *characteristic* for T, if the tangent vector dz to C satisfies $\arg dz=(\arg \mu_T(z))/2$ (resp. $\arg dz=[(\arg \mu_T(z))/2]+(\pi/2))$ (mod 2π) at every point of C. The arc subject to either of the above two requirements shall be generically referred to as *characteristic arc* (or briefly *characteristics*).

Let T be of $\mathcal{Q}[G]\cap C^2[G]$. The subset G_0 of G where $T(z)$ is non-singular and non-conformal must be relatively open. We suppose, for a moment, that G_0 contains a point $z_1=x_1+iy_1$ satisfying $|\arg \mu_T(z_1)|<\pi$ (mod 2π). Then there exists an open interval $I=\{(x, y): |x-x_1|<\alpha, |y-y_1|<\beta\}\subset G_0$ such that every point $z=x+iy\in I$ satisfies $|\arg \mu_T(z)| \leq \theta\pi$ $(0<\theta<1)$. Under these circumstances the differential equation

$$\frac{dy}{dx}=\tan\frac{\arg \mu_T(z)}{2} \tag{1}$$

is well defined in I. One and only one solution curve for (1) passes through any point of the closed interval $I_1=\{(x, y): |x-x_1|\leq\min\{\alpha/2, \beta/2\tan(\theta\pi/2)\}, |y-y_1|<\beta/2\}$. In other words a suitable neighbourhood of z_1 is filled by the family of characteristic arcs $p_T(z)\overline{q_T(z)}dz^2>0$ exactly once. The situation is all the same for another characteristic arcs $p_T(z)\overline{q_T(z)}dz^2<0$. Even if G_0 may contain a point z_1' where $|\arg\mu_T(z_1')|=\pi$ (mod 2π), we see the classical existence and uniqueness theorem for the ordinary differential equations utilized above still apply after an appropriate rotation of the coordinate axes to obtain the similar conclusion in some neighbourhood of z_1'. We summarize the results in

Theorem 1. *Let $T(z)$ be of $\mathcal{Q}[G]\cap C^2[G]$. Then the open set $G_0\subset G$ where*

T is non-singular and non-conformal is filled up by two families of orthogonally interesecting arcs, i.e., the minor-axial and major-axial characteristic arcs. They coincide respectively with the trajectories and the orthogonal trajectories of the quadratic differential $p_T(z)\overline{q_T(z)}dz^2$.

DEFINITION 2. A quadrilateral $\Omega \subset G$ is termed to be *characteristic* for a non-conformal quasiconformal mapping $T(z)$ belonging to $\mathfrak{T}[G] \cap C^2[G]$, if Ω consists exclusively of non-singular points of $T(z)$ and if the families of major-axial and minor-axial characteristic arcs cover the interior as well as the boundary of Ω exactly once in such a fashion that each member of them is a simple arc which connects the opposite sides of Ω on itself.

REMARK 1. The condition of twice continuous differentiability is not so indifferent as it looks like in defining the characteristic arcs and charcateristic quadrilaterals.

Lemma 3. *Let a $T \in \mathfrak{T}[G] \cap C^2[G]$ be non-singular and non-conformal in G. Then there exists a conformal metric $ds^2 = \rho(z)|dz|^2$ on G, such that any pair of minor-axial characteristic arcs cuts off sub-arcs of the equal length measured with this metric from all the major-axial characteristic arcs.*

Proof. We fix a characteristic quadrilateral $\Omega_0 = \Omega_0(0, a, z_0, b)$ of T with the minor-axial characteristic side $\widehat{0, a}$ once and for all, so that every horizontal line in the z-plane may intersect its major-axial characteristic cross-cuts at most once; this situation can always be realized at least locally. Let $z_1 \in \operatorname{int} \Omega_0$ be arbitrary. Then the point z_1 determines the major-axial (resp. minor-axial) cross-cut γ' (resp. γ'') of Ω_0 through z_1. Put $z_1'' = \gamma'' \cap \widehat{0, b}$ (resp. $z_1' = \gamma' \cap \widehat{0, a}$). Take an arbitrary point $z_1 + \Delta z_1$ on γ'. On denoting the minor-axial characteristic arc through $z_1 + \Delta z_1$ by $\tilde{\gamma}''$, we set $z_1'' + \Delta z_1'' = \tilde{\gamma}'' \cap \widehat{0, b}$. We want to show that $\lim_{\Delta z_1 \to 0} |\Delta z_1''|/|\Delta z_1|$ exists and is continuous in z_1.

To this end we set $z_2 = \{z\colon \operatorname{Re} z = \operatorname{Re} z_1\} \cap \tilde{\gamma}''$, $z_2'' = \{z\colon \operatorname{Re} z = \operatorname{Re} z_1''\} \cap \tilde{\gamma}''$. Then $\lim_{z_2 \to z_1} |z_2'' - z_1''|/|z_2 - z_1|$ exists and is continuous in z_1, because the solutions of the differential equation (1) depend smoothly on the initial data, while $\lim_{z_2'' \to z_1''} |\Delta z_1''|/|z_2'' - z_1''| = \cos[(\arg \mu_T(z_1''))/2]$ is continuous in z_1 owing to their continuous dependence on these initial data (cf. e.g., Petrovski [11], pp. 55–64). Therefore the point-function $g(z_1) = \lim_{\Delta z_1 \to 0} |\Delta z_1''|/|\Delta z_1| = \lim_{\Delta z_1 \to 0} (|z_2'' - z_1''|/|z_2 - z_1|)$ $\cos[(\arg \mu_T(z_1''))/2]/\cos[(\arg \mu_T(z_1))/2]$ must be continuous in z_1. Thus we obtain the desired metric $ds^2 = \rho(z)|dz|^2$ with $\rho(z) = [g(z)]^2$.

The point $z_1 \in \Omega_0$ determines uniquely the oriented path $C(z_1) = \overrightarrow{\widehat{0, z_1''}} \circ \overrightarrow{\widehat{z_1'', z_1}}$, where the symbol $\circ$ denotes the addition of the two singular 1-simplexes

$\overrightarrow{0, z_1''}$, $\overrightarrow{z_1'', z_1}$ taken along $\partial\Omega_0$, γ'' respectively. With the above notations in mind we next define the mapping

$$u+iv=\phi(z)=\int_{C(z)}\sqrt{\chi(\zeta)p_T(\zeta)\overline{q_T(\zeta)}}\,d\zeta$$

with $\chi(z)=\rho(z)/|p_T(z)\overline{q_T(z)}|$. The ϕ is one-valued, injective and of class $C^1[\Omega_0]$, so it induces a Riemannian metric Λ in regard to which the minor-axial characteristics are geodesic. The classical transversality theorem (cf. e.g., Eisenhart [6], p. 174) assures that γ' cuts off the equal lengths from γ'' and $\tilde{\gamma}''$ measured with Λ, accordingly with $\rho(z)|dz|^2$. Therefore every minor-axial (resp. major-axial) characteristic cross-cut of Ω_0 goes into the horizontal (resp. vertical) line under the mapping ϕ. The dilatation-quotient of this C^1-diffeomorphism equals 1, hence ϕ is biholomorphic.

DEFINITION 3. The new coordinate $\phi(z)=u+iv$ of the point $z\in\Omega_0$ shall be referred to as *natural coordinate* of z with respect to the characteristic quadrilateal Ω_0 of $T\in\mathfrak{Q}[G]\cap C^2[G]$.

We have proved

Lemma 4. *Let Ω be a characteristic quadrilateral of a non-singular and non-conformal mapping T of $\mathfrak{Q}[G]\cap C^2[G]$ and let $\phi(z)=u+iv$ the natural coordinate of $z\in\Omega$ with respect to this quadrilateral. Then $\phi(z)$ preserves the modulus of every characteristic sub-quadrilateral of Ω.*

Theorem 2. *For any T of $\mathfrak{Q}[G]\cap C^2[G]$ there exists a positive continuous function $\chi(z)$ which makes the quadratic differential $\chi(z)(\partial T/\partial z)(\overline{\partial T/\partial \bar{z}})dz^2$ analytic in the whole open sub-set of G where T is non-singular.*

This theorem can be paraphrased into

Corollary 1. *A sufficiently smooth quasiconformal mapping with non-vanishing Jacobian is harmonic with respect to some conformal metric*[1].

As another deduction of Theorem 2 we mention among other things

Corollary 2. *Let C: $z=z(t)=x(t)+iy(t)$ $(0<t<1;\ z_0=z(0)\neq z(1)=z_1)$ be a simple arc lying in $\mathbf{C}$ such that $z(t)$ is continuously differentiable and $dz/dt\neq 0$. Then the point-set C can be made into an analytic arc through a suitable change of parametrization.*

Proof. For a point z of C we set $\theta(z)=\mathrm{Arctan}(\dot{y}(t)/\dot{x}(t))$. Then this

1) As to the definition of harmonic mappings into Riemannian manifolds, see Eells-Sampson [5] and Gerstenhaber-Rauch [7].

function is prolongable continuously up to the whole Gaussian plane $\boldsymbol{C}$. Let $\kappa(z)$ (<1) be a positive continuous function supported by a compact region whose interior comprises C. On setting $\mu(z)=\kappa(z)e^{2i\theta(z)}$, we can have a quasi-conformal homeomorphism $w=T(z)$ of $\boldsymbol{C}$ which satisfies the Beltrami equation $\partial w/\partial\bar{z}=\mu(z)\partial w/\partial z$. It is possible to find a positive continuous function $\chi(z)$ such that the function

$$t'=\phi_0(z)=\int_{z_0}^{z}\sqrt{\chi(\zeta)p_T(\zeta)\overline{q_T(\zeta)}}\,d\zeta\,\Big/\int_{z_0}^{z_1}\sqrt{\chi(\zeta)\,|\,p_T(\zeta)\overline{q_T(\zeta)}\,|}\,|d\zeta|$$

is biholomorphic in a simply connected domain comprising C (the integral in the denominator being taken along C) (Theorem 2). The inverse map ϕ_0^{-1} sends the interval $\{t'\colon 0<t'<1\}$ holomorphically onto C, since C constitutes a part of the minor-axial characteristic arcs of $T(z)$. q.e.d.

Lemma 5. *Let $T\in\mathfrak{T}[G]\cap C^2[G]$ be non-singular and non-conformal in G and let $\bar{G}_0$ an arbitrary closed subregion of G. Then, given any $\varepsilon>0$, there exists a $\delta>0$ depending only on ε and $\bar{G}_0$ such that for any characteristic quadrilateral $\Omega=\Omega(z_1, z_2, z_3, z_4)$ of T which is not disjoint with $\bar{G}_0$,* diam $\Omega<\delta$ *implies*

$$|\max\left\{\frac{\operatorname{mod} T(\Omega)}{\operatorname{mod}\Omega},\ \frac{\operatorname{mod}\Omega}{\operatorname{mod} T(\Omega)}\right\}-Q_T(z_j)|<\varepsilon \tag{2}$$

at some vertex z_j ($j=1, 2, 3, 4$) of Ω.

Proof. Let $\delta_0=\operatorname{dist}(\bar{G}_0, \partial G)$ and let $\bar{G}_1$ denote an arbitrary closed region such that $\bar{G}_0\subset G_1\subset\bar{G}_1\subset G$ and $\operatorname{dist}(\bar{G}_1, \partial G)<\delta_0/2$. We take a characteristic quadrilateral $\Omega_0=\Omega_0(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\subset G_1$ for T and introduce the natural coordinate $\phi(z)=u+iv$ with the origin at ζ_1 and with the u-axis (resp. v-axis) $\widehat{\zeta_1, \zeta_2}$ (resp. $\widehat{\zeta_1, \zeta_4}$). The composite mapping $T^*=T\circ\phi^{-1}$ is C^1-diffeomorphic and non-singular in the rectangular domain $R_0=\phi(\Omega_0)$: its characteristics are the horizontal and vertical lines (Lemma 1). Let $R=R(w_1, w_2, w_3, w_4)\subset R_0$ denote a characteristic sub-quadrilateral for T^*: if diam R is small, say $<\delta_1$, we have

$$|\max\left\{\frac{\operatorname{mod} T^*(R)}{\operatorname{mod} R},\ \frac{\operatorname{mod} R}{\operatorname{mod} T^*(R)}\right\}-Q_{T^*}(w_j)|<\varepsilon \tag{3}$$

for some index j, where δ_1 depends only on ε (Lemma 2). We can find a $\delta_2>0$ by definition of the metric such that diam $\Omega<\delta_2$ implies diam $\phi(\Omega)<\delta_1$ for any characteristic sub-quadrilateral $\Omega\subset\Omega_0$ of T. It follows from (3) that

$$|\max\left\{\frac{\operatorname{mod} T(\Omega)}{\operatorname{mod}\Omega},\ \frac{\operatorname{mod}\Omega}{\operatorname{mod} T(\Omega)}\right\}-Q_T(z_j)|<\varepsilon$$

at $z_j=\phi^{-1}(w_j)$ (Lemma 4). Repetition of the same argument yields the existence of some positive $\delta<\delta_0/2$ such that any characteristic quadrilateral Ω of T with diameter smaller than δ satisfies (2). q.e.d.

3. Estimates of dilatations from below

DEFINITION 4. Let $\{\gamma_j\}_{j=1,2\cdots,m-1}$ (resp. $\{\gamma_k'\}_{k=1,2,\cdots,n-1}$) denote any major-axial (resp. minor-axial) characteristic arcs connecting the minor-axial (resp. major-axial) sides of a characteristic quadrilateral Ω_0. The totality of them divides Ω_0 into the union of mn characteristic sub-quadrilaterals Ω_{jk} $(j=1, 2,\ldots, m;\ k=1, 2,\ldots, n)$. Then we say that we have a *characteristic subdivision* Δ of the characteristic quadrilateral Ω_0 and that each Ω_{jk} belongs to Δ.

Theorem 3. *Let $T(z)$ be non-singular and non-conformal twice continuously differentiable quasiconformal mapping of a domain G and let $\Omega=\Omega(z_1, z_2, z_3, z_4)\subset G$ a characteristic quadrilateral of T. If there exists some characteristic subdivision Δ of Ω such that each characteristic sub-quadrilateral Ω_{jk} belonging to Δ satisfies*

$$\max\left\{\frac{\mathrm{mod}\ T(\Omega_{jk})}{\mathrm{mod}\ \Omega_{jk}},\ \frac{\mathrm{mod}\ \Omega_{jk}}{\mathrm{mod}\ T(\Omega_{jk})}\right\} \geq K_0,$$
$$(j = 1, 2,\ldots,\ m;\ k = 1, 2,\ldots, n)$$

with some constant K_0 (>1), then we have

$$\max\left\{\frac{\mathrm{mod}\ T(\Omega)}{\mathrm{mod}\ \Omega},\ \frac{\mathrm{mod}\ \Omega}{\mathrm{mod}\ T(\Omega)}\right\} \geq K_0.$$

Proof. We may assume without losing the generality that the major-axial characteristic cross-cuts $\{\gamma_j\}_{j=1,2,\cdots,m-1}$ connecting the sides $\widehat{z_1, z_2}$ and $\widehat{z_3, z_4}$ divide Ω into the union of characteristic sub-quadrilaterals Ω_j $(j=1, 2,\ldots, m)$ and that each Ω_j is divided by the minor-axial characteristics $\{\gamma_k'\}_{k=1,2,\cdots,n-1}$ into the union of n characteristic sub-quadrilaterals $\Omega_{jk}(k=1, 2,\ldots, n)$. For any point z of Ω we introduce the natural coordinate $\phi(z)=u+iv$ with respect to Ω. It follows from Lemma 4 that we have through a suitable choice of the moduli of those quadrilaterals considered

$$\mathrm{mod}\ \Omega(z_1, z_2, z_3, z_4) = \sum_{j=1}^{m} \mathrm{mod}\ \Omega_j = \sum_{j=1}^{m} \frac{1}{\sum_{k=1}^{n} \frac{1}{\mathrm{mod}\ \Omega_{jk}}}$$

$$\leq \sum_{j=1}^{m} \frac{1}{K_0 \sum_{k=1}^{n} \frac{1}{\mathrm{mod}\ T(\Omega)_{jk}}} = \frac{1}{K_0} \sum_{k=1}^{n} \mathrm{mod}\ T(\Omega_{jk})$$

$$= \frac{1}{K_0}\ \mathrm{mod}\ T(\Omega(z_1, z_2, z_3, z_4)).$$

q.e.d.

Theorem 4. *Let $T(z)$ be a twice continuously differentiable quasiconformal mapping in a domain G. If $T(z)$ has the dilatation-quotient $Q_T(z)$ bounded below by a constant K_0 (>1) everywhere in G, we have*

$$\max\left\{\frac{\operatorname{mod} T(\Omega)}{\operatorname{mod}\Omega}, \frac{\operatorname{mod}\Omega}{\operatorname{mod} T(\Omega)}\right\} \geq K_0$$

for any characteristic quadrilateral $\Omega\subset G$ of T.

Proof. Given any $\varepsilon>0$ there exists some characteristic subdivision Δ of the Ω such that every characteristic sub-quadrilateral Ω' belonging to Δ satisfies

$$\max\left\{\frac{\operatorname{mod} T(\Omega')}{\operatorname{mod}\Omega'}, \frac{\operatorname{mod}\Omega'}{\operatorname{mod} T(\Omega')}\right\} > K_0-\varepsilon$$

(Lemma 5). Hence we get by Theorem 3 that

$$\max\left\{\frac{\operatorname{mod} T(\Omega)}{\operatorname{mod}\Omega}, \frac{\operatorname{mod}\Omega}{\operatorname{mod} T(\Omega)}\right\} > K_0-\varepsilon\,.$$

Letting $\varepsilon\to 0$, we complete the proof.

In the subsequent consideration we shall confine ourselves only to a bounded domain G, but this will be readily seen to imply no essential restriction of generality. Now given any $T\in\mathfrak{T}[G]$, it is possible to find a complex-valued function $\tilde{\mu}(z)$ subject to the requirements: (i) $\tilde{\mu}(z)$ is of class $C^2[\boldsymbol{C}]$; (ii) supp $\tilde{\mu}(z)$ is a uniformly bounded closed region whose interior comprises $\bar{G}$; (iii) $|\tilde{\mu}(z)|\leq|\mu_T(z)|$ $(z\in G)$; (iv) $|\tilde{\mu}(z)|\leq$ ess $\sup|\mu_T(z)|$ $(z\in\boldsymbol{C})$ (cf. [13]). The Beltrami equation $\partial w/\partial\bar{z}=\tilde{\mu}(z)\partial w/\partial z$ has a solution $w=\tilde{T}(z)$ which provides a homeomorphism of $\boldsymbol{C}$ onto itself with the property $p_T(z)\neq 0$ (Ahlfors [2]). $\tilde{T}$ belongs to $C^2[G]$ (Ahlfors [3], pp. 85–88), hence it has well-determined characteristic arcs everywhere in G (Theorem 1). The totality of such $\tilde{T}$ shall be denoted by $\tilde{\mathfrak{T}}=\tilde{\mathfrak{T}}[G]$. For a $\delta>0$ we set $\mathcal{A}_\delta[T;\tilde{\mathfrak{T}}]=\{\tilde{T}\in\tilde{\mathfrak{T}}: \|\mu_T-\mu_{\tilde{T}}\|_2<\delta\}$, where $\|\cdot\|_2$ stands for the L^2-norm over $\boldsymbol{C}$; it is known that $\mathcal{A}_\delta[T;\tilde{\mathfrak{T}}]\neq\phi$ for any $\delta>0$.

Next fix a point $z_0\in G$ and a number $\alpha>0$ arbitrarily besides the above $T\in\mathfrak{T}[G]$. We take a point ζ in $N^\alpha(z_0)\cap G$ and a positive real $r<$dist $\{\zeta,\partial(N^\alpha(z_0)\cap G)\}$. Whatever $\tilde{T}$ we may choose out of $\tilde{\mathfrak{T}}$, a pair of major-axial (resp. minor-axial) characteristic arcs γ_j $(j=1,2)$ (resp. γ_j' $(j=1,2)$) is completely determined by the requirement that the four arcs should be tangent to the circle $|z-\zeta|=r$. Let C, C' denote a couple of major-axial and minor-axial characteristic arcs through ζ. Two cases may occur: (i) C, C' forms, together with one of the $\{\gamma_j\}$ and one of the $\{\gamma_j'\}$ $(j=1, 2)$, at least one characteristic quadrilateral Ω_k $(1\leq k\leq l$: $l=1, 2, 3$ or $4)$ comprised in G; (ii) otherwise, the symbol mod Ω_j, mod $T(\Omega_j)$ etc. do not make sense, but we set formally mod $T(\Omega_j)/$mod $\Omega_j=+\infty$ $(j=1, 2, 3, 4)$.

DEFINITION 5. $\underset{\sim}{D}^{\alpha}_{T}(z_0)=\underset{\sim}{D}^{\alpha}_{T}(z_0;\tilde{\mathfrak{I}})=$
$\inf_{\zeta}\inf_{r}\lim_{\delta\to 0}\sup\max_{1\le j\le 4}\max\left\{\frac{\operatorname{mod}\tilde{T}(\Omega_j)}{\operatorname{mod}\Omega_j},\ \frac{\operatorname{mod}\Omega_j}{\operatorname{mod}\tilde{T}(\Omega_j)}\right\}$, $(z_0\in G)$, where the supremum shall be taken in reference to all $\tilde{T}$ belonging to $\mathcal{A}_{\delta}[T;\tilde{\mathfrak{I}}]$.

Lemma 6. $\lim_{\alpha\to 0}\underset{\sim}{D}^{\alpha}_{T}(z;\tilde{\mathfrak{I}})=\underline{D}_T(z)\quad(z\in G)$.

Proof. We shall first show that $\lim_{\alpha\to 0}\underset{\sim}{D}^{\alpha}_{T}(z;\tilde{\mathfrak{I}})\ge\underline{D}_T(z)$ everywhere in G. Suppose, contrary to the assertion, G contain a point z_0 such that $\lim_{\alpha\to 0}\underset{\sim}{D}^{\alpha}_{T}(z_0;\tilde{\mathfrak{I}})<\underline{D}_T(z_0)$. Then there is a constant c satisfying

$$\lim_{\alpha\to 0}\underset{\sim}{D}^{\alpha}_{T}(z_0;\tilde{\mathfrak{I}})<c<\underline{D}_T(z_0)\,. \tag{4}$$

The first inequality requires the existence of some $\zeta\in N^{\alpha}(z_0)\cap G$ as well as some positive $r<\operatorname{dist}\{\zeta,\partial(N^{\alpha}(z_0)\cap G)\}$ for every $\alpha>0$, such that the characteristic quadrilateral $\Omega(\zeta,\zeta_1,\zeta_2,\zeta_3)$ for all $\tilde{T}\in\mathcal{A}_{\delta}[T;\tilde{\mathfrak{I}}]$ satisfies the conditions: (i) the sides $\widehat{\zeta_1,\zeta_2}$, $\widehat{\zeta_2,\zeta_3}$ of Ω lie on some characteristic arcs of $\tilde{T}$ touching the circle $|z-\zeta|=r$, (ii) $\max\{\operatorname{mod}\tilde{T}(\Omega)/\operatorname{mod}\Omega,\ \operatorname{mod}\Omega/\operatorname{mod}\tilde{T}(\Omega)\}<c$ if δ is small. On the other hand we see from the second inequality of (4) that if $\alpha>0$ is small, the square $R=R(\zeta,\zeta+re^{i\theta},\zeta+(1+i)re^{i\theta},\zeta+ire^{i\theta})$ satisfies $\operatorname{mod}\tilde{T}(R)>c+\eta$ $(\eta>0)$ with the above ζ, r and with some θ $(0\le\theta\le 2\pi)$; in fact, this assertion is trivial for $\zeta\in\dot{N}^{\alpha}(z_0)$ and we can do with the other case by continuity of modulus. If $\delta>0$ is sufficiently small, we must have $\operatorname{mod}\tilde{T}(R)>c+(\eta/2)$ with any $\tilde{T}\in\mathcal{A}_{\delta}[T;\tilde{\mathfrak{I}}]$. But this contradicts the condition (ii) as $\alpha\to 0$ on account of Lemmas 2 and 5.

Similar argument yields the opposite inequality $\lim_{\alpha\to 0}\underset{\sim}{D}^{\alpha}_{T}(z_0)\le\underline{D}_T(z_0)$ and the proof is completed.

Lemma 7. *Let $T\in\mathfrak{I}[G]$ be arbitrary and let G_0 a subdomain of G. If a finite number of neighbourhoods $N_j=N^{\alpha_j}(z_j)$ $(z_j\in G_0;\ \alpha_j>0;\ j=1,2,\dots,m)$ covers G_0 satisfying $\underset{\sim}{D}^{\alpha_j}_{T}(z_j;\tilde{\mathfrak{I}})\ge K$ with some constant K $(j=1,2,\dots,m)$, we have $\underset{\sim}{D}^{\alpha}_{T}(z;\tilde{\mathfrak{I}})\ge K$ for any $z\in G_0$ and any $\alpha>0$.*

Proof. Let $\varepsilon>0$ be given. Then there is a $\delta_j=\delta_j(\varepsilon)>0$ such that $\mathcal{A}_{\delta_j}[T;\tilde{\mathfrak{I}}]$ contains at least one $\tilde{T}$, every characteristic quadrilateral $\Omega_{j\beta}\subset N_j$ of which satisfies

$$\max\left\{\frac{\operatorname{mod}\tilde{T}(\Omega_{j\beta})}{\operatorname{mod}\Omega_{j\beta}},\ \frac{\operatorname{mod}\Omega_{j\beta}}{\operatorname{mod}\tilde{T}(\Omega_{j\beta})}\right\}>K-\varepsilon\,,$$
$$j=1,2,\dots,m,$$
β belonging to some set of indices.

Take a $\delta>0$ smaller than $\min_{1\le j\le m}\delta_j$. $\mathcal{A}_{\delta}[T;\tilde{\mathfrak{I}}]$ still contains a $\tilde{T}'$ which satisfies the following condition: we can spread such a fine net $\mathfrak{N}$ over G_0 that each mesh of $\mathfrak{N}$ is some characteristic quadrilateral Ω_β' of the $\tilde{T}'$ with diameter smaller than $\min_{1\leq j\leq m} \alpha_j$ and that

$$(5) \qquad \max\left\{\frac{\operatorname{mod} \tilde{T}'(\Omega_\beta')}{\operatorname{mod} \Omega_\beta'}, \frac{\operatorname{mod} \Omega_\beta'}{\operatorname{mod} \tilde{T}'(\Omega_\beta')}\right\} > K-\varepsilon .$$

Let $\alpha>0$ be arbitrary. Taken any $z_0 \in G_0$, $\zeta \in N^\alpha(z_0) \cap G_0$ and a positive $r<\operatorname{dist}\{\zeta, \partial(N^\alpha(z_0)\cap G_0)\}$, there can be determined four characteristic arcs of $\tilde{T}'$ tangent to the circle $|z-\zeta|=r$. We lose no generality in assuming that these four characteristics constitute a characteristic quadrilateral Ω' for the $\tilde{T}'$ comprised completely in G_0. If we denote by Ω_l' $(l=1, 2, 3, 4)$ the characteristic sub-quadrilaterals of Ω' into which a couple of major-axial and minor-axial characteristics through ζ divides, we obtain immediately

$$\max\left\{\frac{\operatorname{mod} \tilde{T}'(\Omega_l')}{\operatorname{mod} \Omega_l'}, \frac{\operatorname{mod} \Omega_l'}{\operatorname{mod} \tilde{T}'(\Omega_l')}\right\} > K-\varepsilon, \quad (l=1, 2, 3, 4)$$

from (5) with the aid of Theorem 3. Letting $\delta\to 0$, we have $\underset{\sim}{D}^\alpha_T(z_0; \tilde{\mathfrak{I}})>K-\varepsilon$, which was to be proved, since $\varepsilon>0$ is arbitrary.

The following statement finds itself in a situation to supplement Theorem 3 in [13] from an opposite direction:

Theorem 5. $\inf_{z\in G} \underset{\sim}{D}_T(z)$ *is an upper semi-continuous functional in* T *of* $\mathfrak{I}[G]$.

Proof. Let α, α' be any positive reals such that $\alpha<\alpha'$. Taking an arbitrary sequence $\{T_n\}_{n=1,2,\dots}$ of $\mathfrak{I}[G]$, we first show that $\limsup_{n\to\infty} \underset{\sim}{D}^\alpha_{T_n}(z) \leq \underset{\sim}{D}^\alpha_T(z)$. Suppose, contrary to the assertion, that $\{T_n\}_{n=1,2,\dots}$ contain some subsequence, denoted by the same symbol for convenience' sake, satisfying $\lim_{n\to\infty} \underset{\sim}{D}^\alpha_{T_n}(z_0) > \underset{\sim}{D}^\alpha_T(z_0)$ at some $z_0\in G$. Then some constants c, $\varepsilon_0>0$ exist such that

$$(6) \qquad \underset{\sim}{D}^\alpha_T(z_0) < c-\varepsilon_0 < c < \lim_{n\to\infty} \underset{\sim}{D}^\alpha_{T_n}(z_0) .$$

Let $\bar{G}_0$ denote a closed subregion of G which contains z_0. The first part of (6) persists in the existence of some $\zeta\in N^\alpha(z_0)\cap\bar{G}_0$ and some positive $r<\operatorname{dist}\{\zeta, \partial(N^\alpha(z_0)\cap\bar{G}_0)\}$ such that the characteristic quadrilateral of Ω all $\tilde{T}\in\mathcal{A}_\delta[T; \tilde{\mathfrak{I}}]$ touching the circle $|z-\zeta|=r$ gives

$$(7) \qquad \max_{1\leq j\leq 4} \max\left\{\frac{\operatorname{mod} \tilde{T}(\Omega_j)}{\operatorname{mod} \Omega_j}, \frac{\operatorname{mod} \Omega_j}{\operatorname{mod} \tilde{T}(\Omega_j)}\right\} < c-\varepsilon, \quad (0<\varepsilon<\varepsilon_0)$$

for sufficiently small $\delta(\varepsilon)>0$, where Ω_j $(j=1, 2, 3, 4)$ are the characteristic sub-quadrilaterals belonging to the characteristic subdivision of Ω with one vertex at ζ. According to the third part of (6), however, there is an index ν, such that

$\underset{\sim}{D}^{\alpha}_{T_n}(z_0)>c$ for all $n>\nu$. Hence a sufficiently small $\delta_n>0$ assures the presence of some $\tilde{T}_n\in\mathcal{A}_{\delta_n}[T_n;\tilde{\mathfrak{T}}]$ whose characteristic quadrilateral Ω_n tangent to the above circle $|z-\zeta|=r$ fulfills

$$\max_{1\leq j\leq 4}\max\left\{\frac{\operatorname{mod}\tilde{T}_n(\Omega_{nj})}{\operatorname{mod}\Omega_{nj}},\ \frac{\operatorname{mod}\Omega_{nj}}{\operatorname{mod}\tilde{T}_n(\Omega_{nj})}\right\}>c$$

(the relationship between Ω_n and Ω_{nj} being the same as the one between Ω and Ω_j). Fixing δ and letting $n\to\infty$, $\delta_n\to 0$, we arrive at the inclusion $\mathcal{A}_{\delta_n}[T_n;\tilde{\mathfrak{T}}]\subset\mathcal{A}_{\delta}[T;\tilde{\mathfrak{T}}]$, which is a contradiction on account of (7).

Let $z\in\bar{G}_0$ be arbitrary. Putting the neighbourhood $N^{\alpha}(z)$ into correspondence with the point z, we can cover $\bar{G}_0$ with a finite sub-collection $\bigcup_j N^{\alpha}(z_j)$ out of $\bigcup_{z\in\bar{G}_0}N^{\alpha}(z)$ $(z_j\in\bar{G}_0)$. Let c be any constant smaller than $\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha}_T(z)$. It follows from Lemma 7 that $\underset{\sim}{D}^{\alpha'}_T(z)>c$ for all z on $\bar{G}_0$. Hence $\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha'}_T(z)\geq c$. Thus we have $\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha}_T(z)\leq\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha'}_T(z)$, which implies $\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha}_T(z)=\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha'}_T(z)$, since the opposite inequality is trivial.

Next we shall see $\inf_{z\in\bar{G}_0}\underline{D}_T(z)=\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha}_T(z)$. Suppose that $\inf_{z\in\bar{G}_0}\underset{\sim}{D}_T(z)>\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha}_T(z)$ for some α. Then there would be a constant c satisfying $\inf_{z\in\bar{G}_0}\underline{D}_T(z)>c>\inf_{z\in\bar{G}_0}\underset{\sim}{D}^{\alpha}_T(z)$ for all α. The first inequality implies that for any $z\in\bar{G}_0$ there is an α satisfying $\underset{\sim}{D}^{\alpha}_T(z)>c$ (Lemma 6), while the second implies the presence of some $z'\in\bar{G}_0$ satisfying $\underset{\sim}{D}^{\alpha}_T(z')<c$ for any α. This is a cintradiction. Hence $\inf_{z\in\bar{G}_0}\underline{D}_T(z)$ is an upper semi-continuous functional in $T\in\mathfrak{T}[G]$. So is $\inf_{z\in G}\underline{D}_T(z)$. q.e.d.

4. Necessary and sufficient condition for a quasiconformal mapping to be Teichmüller

As an immediate consequence of Theorem 2 we mention

Theorem 6. *Let $T(z)$ be a twice continuously differentiable quasiconformal mapping of a domain G. If $T(z)$ has a constant dilatation-quotient everywhere in G, $T(z)$ is a Teichmüller mapping, unless conformal.*

Proof. If $Q_T(z)\equiv K\equiv 1$, $T(z)$ is clearly conformal in G. So we assume that $K>1$. Then $T(z)$ is non-singular in G, since it has the dilatation-quotient at every point. Theorem 2 assures the existence of some continuous function $\chi(z)>0$ in G which makes $T(z)$ into the Teichmüller map with $\phi_T=\chi(z)p_T(z)\overline{q_T(z)}dz^2$ and the given constant K.

The remainder part of this section is devoted to get rid of the smoothness assumption in Theorem 6.

Theorem 7. *A quasiconformal homeomorphism $T_0(z)$ of a domain G is a*

Teichmüller mapping, if and only if $\bar{D}_{T_0}(z)\equiv \underline{D}_{T_0}(z)\equiv const.$ *everywhere in* G.

Proof. Let $T_0(z)$ be a Teichmüller mapping of the domain G defined with a constant K (>1) and an analytic quadratic differential ϕ_{T_0}. At the point where $\phi_{T_0}\neq 0$ it is known to hold $\bar{D}_{T_0}(z)\equiv Q_{T_0}(z)\equiv K$ (cf. [13]). But we prefer to do as follows, regardless of whether ϕ_{T_0} vanishes or not. Given any $z\in G$ and $\alpha>0$, Grötzsch-Teichmüller quasi-invariance theorem on moduli (Grötzsch [8], Teichmüller [15]) provides $\operatorname{mod} T(\Omega)/\operatorname{mod}\Omega\leq K$ for any quadrilateral $\Omega\subset N^{\alpha}(z)\cap G$, hence $\underline{D}_{T_0}(z)\leq \bar{D}_{T_0}(z)\leq K$. On the other hand there is an $\alpha=\alpha(z)>0$ such that $Q_{T_0}(\zeta)=K$ at every $\zeta\in \dot{N}^{\alpha}(z)\cap G$: hence for an arbitrary $\varepsilon>0$ there is a positive $\alpha'(z;\, \varepsilon)\leq\alpha$ such that the square $R=R(\zeta+re^{i\theta},\, \zeta+ire^{i\theta},\, \zeta-re^{i\theta},\, \zeta-ire^{i\theta})$ satisfies $\max\{\operatorname{mod} T(R),\, 1/\operatorname{mod} T(R)\}>K-\varepsilon$ for any positive $r<\operatorname{dist}\{\zeta,\, \partial(N^{\alpha'}(z)\cap G)\}$ and some real θ (Lemma 2). It follows that $\underline{D}_{T_0}(z)\geq \underline{D}^{\alpha'}_{T_0}(z)>K-\varepsilon$. Letting $\varepsilon\to 0$, we conclude $\underline{D}_{T_0}(z)=\bar{D}_{T_0}(z)=K$.

Conversely, suppose that a quasiconformal mapping $w=T_0(z)$ of $\mathcal{Q}[G]$ satisfies $\underline{D}_{T_0}(z)=\bar{D}_{T_0}(z)=K$. Let G_0 be an arbitrary subdomain which is compact relative to G. Given any $\varepsilon>0$, we can assign to each point $z\in\bar{G}_0$ its neighbourhood $N^{\alpha(z)}(z)$ such that $\underset{\sim}{D}^{\alpha(z)}_{T_0}(z;\, \tilde{\mathcal{Q}})>K-(\varepsilon/2)$ (Lemma 6). A finite sub-collection $\bigcup_j N^{\alpha_j}(z_j)$ of $\bigcup_{z\in\bar{G}_0} N^{\alpha(z)}(z)$ covers $\bar{G}_0$. Hence $\underset{\sim}{D}^{\alpha}_{T_0}(z;\, \tilde{\mathcal{Q}})>K-(\varepsilon/2)$ for any $z\in\bar{G}_0$ and any $\alpha>0$ (Lemma 7). Let $\{\delta_n\}_{n=1,2,\cdots}$ be a sequence of positive reals such that $\lim_{n\to\infty}\delta_n=0$ and let G_1 and simply connected subdomain of G_0. Then there is an index m depending on ε and G_1 such that some $\tilde{T}_n\in\mathcal{A}_{\delta_n}[T_0;\, \tilde{\mathcal{Q}}]$ ($n=m,\, m+1,\, \cdots$) possesses the following properties (i) $\tilde{T}_n(G_0)\supset T_0(G_1)$; (ii) any characteristic quadrilateral $\Omega_n\subset G_1$ of $\tilde{T}_n$ satisfies the inequality $\max\{\operatorname{mod}\tilde{T}_n(\Omega_n)/\operatorname{mod}\Omega_n,\, \operatorname{mod}\Omega_n/\operatorname{mod}\tilde{T}_n(\Omega_n)\}>K-\varepsilon$. We specify the quadrilateral $\Omega_n=\Omega_n(z_1^{(n)},\, z_2^{(n)},\, z_3^{(n)},\, z_4^{(n)})$ so that its four sides may touch a fixed circle in G_1. The path-independent integral $\int_{z_1^{(n)}}^{z}\sqrt{\sigma_n(\zeta)p_{T_n}(\zeta)\overline{q_{T_n}(\zeta)}}\,d\zeta$ defines a holomorphic function $\phi_n(z)$ in G_1 with an appropriate continuous function $\sigma_n(\zeta)>0$, where the radical sign indicates either of the one-valued branches in G_1 (Theorem 2). Multiplying $\sigma_n(\zeta)$ by a suitable constant, we can normalize $\phi_n(z)$ ($n=m,\, m+1,\ldots$) so that they may be uniformly bounded and that $\phi_n(\Omega_n)$ may comprise the unit square $\Sigma=\{Z\colon 0<\operatorname{Re} Z<1,\, 0<\operatorname{Im} Z<1\}$. If $\{\phi_n(z)\}_{n=m,\, m+1,\cdots}$ contains a subsequence tending uniformly to a constant, T_0 itself must reduce to a constant map, which is absurd. Therefore $\{\phi_n'(z)^2\}_{n=m,\, m+1,\cdots}$ contains a subsequence, say again $\{\phi_n'(z)^2\}_{n=m,\, m+1,\cdots}$ for shortness' sake, which converges to a holomorphic function $a(z)\not\equiv 0$ normally in G_1. We may assume without losing the generality that $a(z)\neq 0$ everywhere in G_1, since we have only to restrict the domain considered if necessary. The sequence

$\{\phi_n(z)\}_{n=m, m+1, \ldots}$ converges to a non-constant univalent holomorphic function $Z=\phi(z)$ normally in G_1 such that $\phi'(z)^2=a(z)$: $\{\phi_n^{-1}\}_{n=m, m+1, \ldots}$ converges uniformly to ϕ^{-1} on some compact region whose interior comprises the unit square Σ. Passing to the limit we have verified the existence of the local holomorphic injection ϕ^{-1} defined on Σ such that the quasiconformal mapping $F(\Sigma)=T_0\circ\phi^{-1}(Z)$ satisfies the inequality $\max\{\mathrm{mod}\, F(\Sigma), 1/\mathrm{mod}\, F(\Sigma)\}\geq K-\varepsilon$. Let $W=\psi(w)$ map the interior of the quadrilateral $F(\Sigma)$ conformally onto a rectangular domain R so that the vertices of $F(\Sigma)$ may correspond to those of R. The last estimate together with another condition $[1+|\mu_{\psi\circ F}(Z)|]/[1-|\mu_{\psi\circ F}(Z)|]\leq \bar{D}_{T_0}(\phi^{-1}(Z))=K$ implies

$$\psi\circ T_0\circ\phi^{-1}(Z)=a[(K+1)Z+(K-1)\bar{Z}]+b$$

with complex constants a, b (Grötzsch [9], Ahlfors [1]). Although $\phi(z)$ has been defined only in the small, the differential $\phi_{T_0}=a(z)dz^2$ is prolongable analytically throughout the whole domain G. Restricted to the subdomain where $\phi_{T_0}\neq 0$, $T_0(z)$ turns out to be smooth, hence $Q_{T_0}(z)=K$ (Theorems 2, 6 in [13]). Therefore $T_0(z)$ must be a Teichmüller mapping of G (Theorem 6). q.e.d.

REMARK 2. It is seen without any difficulty that Theorem 7 still holds for quasiconformal homeomorphisms of Riemann surfaces.

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