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ON THE EXISTENCE OF A REPRODUCING KERNEL ON HARMONIC SPACES AND ITS PROPERTIES

SHIRO OGAWA AND TADASHI MURAZAWA

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Introduction. Let B be a finite plane domain with the smooth boundary and $\Lambda^2(B)$ the class of all solutions φ of the differential equation $\Delta\varphi - p\varphi = 0$ such that

$$D[\varphi] = \iint_B \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + p\varphi^2\right] dxdy < +\infty ,$$

where $p=p(x, y)$ is a positive analytic function of real variables x and y in B. S. Bergman [6] proved the existence of a function K which has the characteristic reproducing property of a kernel function, with respect to the Dirichlet integral

$$D[\varphi, \psi] = \iint_B \left[\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + p\varphi\psi\right] dxdy .$$

From the point of view of the axiomatic harmonic function theory, B is a space with the pre-sheaf: $U \rightarrow \Lambda^2(U)$, where U is any open subset of B.

The aim of this paper is to show that there exists a reproducing kernel of a space formed by harmonic functions on harmonic spaces in the sense of H. Bauer, to study some properties of the kernel function and to obtain the Cauchy-type representation of harmonic functions by an integral kernel obtained from the reproducing kernel. The results are immediately applicable to the classical harmonic functions on R^n and the family of all solutions of the heat equation on R^{n+1}, and moreover to that of all solutions of more general differential equations on Riemannian manifolds which satisfies Bauer's axioms.

In the paragraph 1, we construct a Hilbert space $R^2(U)$, formed by harmonic functions, with a certain scalar product, and in the paragraph 2, by applying the existence theorem of a kernel function, we discuss that there exists a reproducing kernel of $R^2(U)$. In the paragraph 3, we show the monotonicity of the kernel function with respect to the domain of its definition on harmonic spaces, which is an imprtant property of a class of kernel functions. In the last paragraph, using an integral kernel obtained by the reproducing kernel we study an integral representation of harmonic functions in Cauchy-type.

1. The spaces $L^2(\sigma)$ and $R^2(U)$

Lex X be a locally compact Hausdorff space with a countable base and suppose that X is a harmonic space relative to a sheaf $\mathcal{H}$ of real valued continuous functions which satisfies the Bauer's four axioms and the following one more axiom: *The constant* 1 *is superharmonic.* μ_x^U is the harmonic measure with respect to a relatively compact open subset U in X and a point x of U, that is, the balayaged measure of Dirac mass at x to the complementary set of U. Let ν be a positive measure, defined on a dense subset U' in U, whose support $S\nu$ is the closure of U. In fact, as X is a locally compact space with a countable base, surely there exists such a measure ν. Then by the superharmonicity of the constant 1 we can define a positive measure σ on ∂U, the boundary of U, by $\sigma(e)=\int_U \mu_x^U(e)d\nu(x)$, where e is any Borel set on ∂U. Denote by $L^2(\sigma)$ the family of all real valued σ-measurable functions f on ∂U such that $\int_{\partial U} f^2 d\sigma$ is finite. We define the bilinear functional $(f, g)_\sigma$ and the non-negative functional $||f||_\sigma$ on $L^2(\sigma)$ as follows:

$$(f, g)_\sigma = \int_{\partial U} fg d\sigma \qquad \text{for any } f, g \in L^2(\sigma)\,,$$
$$||f||_\sigma = \left(\int_{\partial U} f^2 d\sigma\right)^{1/2} \qquad \text{for any } f \in L^2(\sigma)\,.$$

Then $(f, g)_\sigma$ satisfies the condition of scalar product and, under the condition that f is equal to g (denoted by $f=g$) if and only if $||f-g||_\sigma=0$, $||f||_\sigma$ satisfies the condition of a norm. It is well known that $L^2(\sigma)$ has the structure of a Hilbert space relative to the scalar product $(f, g)_\sigma$ and the norm $||f||_\sigma$.

The following lemmas are very useful for coming arguments.

Lemma 1.1 (H. Bauer [4]). *Suppose that f is a real valued function, defined on ∂U, which is μ_x^U-integrable for any point x in a dense subset of U. Then f is μ_x^U-integrable for all points x of U and the function*

$$x \rightarrow \int_{\partial U} f d\mu_x^U$$

is harmonic on U.

Lemma 1.2. *For $f, g \in L^2(\sigma)$, f is equal to g if and only if $f(\theta)=g(\theta)$ μ_x^U-a.e. for all points x of U.*

Proof. By the definition, $f=g$ signifies $||f-g||_\sigma=0$. On the other hand, we obtain following equalities:

$$||f-g||_\sigma^2 = \int_{\partial U} (f-g)^2 d\sigma$$

$$= \int_U \int_{\partial U} (f(\theta)-g(\theta))^2 d\mu_x^U(\theta) d\nu(x)$$
$$= 0\,,$$

which implies that, for every point x of a dense subset U'' in U,

$$\int_{\partial U} (f(\theta)-g(\theta))^2 d\mu_x^U(\theta) = 0\,.$$

By Lemma 1.1, it follows that

$$\int_{\partial U} (f(\theta)-g(\theta))^2 d\mu_x^U(\theta) = 0 \qquad \text{for all } x \in U\,,$$

which implies

$$f(\theta) = g(\theta) \qquad \mu_x^U\text{-a.e.} \qquad \text{for all } x \in U\,.$$

The inverse is evident. This completes the proof.

Here consider the following spaces of real valued functions for a natural number p:

$$L^p(\sigma) = \left\{ f : \int_{\partial U} |f|^p d\sigma < +\infty \right\},$$
$$L^p(\mu_x^U) = \left\{ g : \int_{\partial U} |g|^p d\mu_x^U < +\infty \right\}.$$

Then we have

Lemma 1.3. *For any natural number p, there is the following relation between $L^p(\sigma)$ and $L^p(\mu_x^U)$,*

$$L^p(\sigma) \subset \bigcap_{x \in U} L^p(\mu_x^U)\,.$$

Proof. For any function $f \in L^p(\sigma)$, we have

$$\int_{\partial U} |f(\theta)|^p d\sigma(\theta) = \int_U \int_{\partial U} |f(\theta)|^p d\mu_x^U(\theta) d\nu(x) < +\infty\,,$$

which implies that, in a dense subset U''' of U,

$$\int_{\partial U} |f(\theta)|^p d\mu_x^U(\theta) < +\infty\,.$$

By Lemma 1.1, we obtain, for any point x of U,

$$\int_{\partial U} |f(\theta)|^p d\mu_x^U(\theta) < +\infty\,.$$

Therefore we have that $L^p(\sigma) \subset \bigcap_{x \in U} L^p(\mu_x^U)$.

Let us denote by $R^2(U)$ the family

$$\left\{H_f(x)\colon H_f(x)=\int_{\partial U} f d\mu_x^U \text{ for all } f\in L^p(\sigma)\right\}.$$

Then there exists the following relation between $L^2(\sigma)$ and $R^2(U)$.

Lemma 1.4. *$R^2(U)$ is a subspace of the space $\mathcal{H}_U$ of all harmonic functions defined on U, and the correspondence*

$$f\in L^2(\sigma)\to H_f\in R^2(U)$$

is isomorphic.

Proof. Since $L^2(\sigma)\subset L^1(\sigma)$, any function f of $L^2(\sigma)$ is σ-integrable, which implies, by virtue of Lemma 1.3, that f is μ_x^U-integrable for all x of U. By the resolutivity theorem [4], $H_f(x)=\int_{\partial U} f d\mu_x^U$ is harmonic on U for all f of $L^2(\sigma)$. It is evident that $R^2(U)$ is a vector space and it holds that, for any pair $f, g\in L^2(\sigma)$ and real numbers a and b,

$$af+bg\to H_{af+bg}=aH_f+bH_g\ .$$

Moreover Lemma 1.2 follows that, for $f, g\in L^2(\sigma)$, f is equal to g if and only if $H_f(x)=\int_{\partial U} f d\mu_x^U$ is equal to $H_g(x)=\int_{\partial U} g d\mu_x^U$ for all x of U. This fact implies that the correspondence between $f\in L^2(\sigma)$ and $H_f\in R^2(U)$ is one-to-one and it is evident that this mapping is onto. This completes the proof.

On $R^2(U)$ we define the scalar product (H_f, H_g) and the norm $\|H_f\|$ as follows;

$$(H_f, H_g)=(f, g)_\sigma \qquad \text{for } H_f, H_g\in R^2(U)\,,$$
$$\|H_f\|=\|f\|_\sigma \qquad \text{for } H_f\in R^2(U)\,.$$

Then by Lemma 1.4 and the fact that $L^2(\sigma)$ is a Hibert space with respect to the scalar product $(f, g)_\sigma$ and the norm $\|f\|_\sigma$, we have immediately the following theorem.

Theorem 1.5. *$R^2(U)$ is a Hilbert space with respect to the scalar product (H_f, H_g) and the norm $\|H_f\|$.*

2. Representation of a function of $R^2(U)$ by a reproducing kernel of $R^2(U)$

In this paragraph showing that there exists a non-negative reproducing kernel of $R^2(U)$, we are going to consider the representation of every function of $R^2(U)$ by the reproducing kernel. In order to prove our theorem, the following theorem proved by H. Bauer [4] is very useful.

Theorem 2.1 (H. Bauer). *Suppose that U is an open subset in X, μ a positive meausre in U and F any compact subset in $\mathring{A}_{S\mu} \cap U$, where $\mathring{A}_{S\mu}$ is the interior of the smallest absorption set containing $S\mu$, the support of μ. Then there exists a non-negative constant α depending upon F and μ such that, for all non-negative harmonic function u defined on U,*

$$\sup u(F) \leqq \alpha \int u d\mu .$$

We can obtain the following analoguous theorem concerning $R^2(U)$ to Theorem 2.1.

Theorem 2.2. *Let U be a relatively compact open subset in X, ν and σ the positive measures mentioned in the paragraph 1 and F any compact subset in U. Then there exists a non-negative constant γ depending on F and σ such that*

$$\sup |u(F)| \leqq \gamma \|u\| \qquad \textit{for all } u \in R^2(U) .$$

Proof. By the hypothesis of ν, $\mathring{A}_{S\nu}$ is equal to U and thus $\mathring{A}_{S\nu} \cap U = U$. By Theorem 2.1 it holds that, for any compact subset F in U, there exists a non-negative constant α depending on F and ν such that, for all non-negative harmonic function h in $R^2(U)$,

$$\sup h(F) \leqq \alpha \int h d\nu . \tag{2.1}$$

On the other hand, by virtue of Lemma 1.4, there exists for each function u of $R^2(U)$ a unique function f in $L^2(\sigma)$ such that $u = H_f$. Thus we have, for any point x of F,

$$|u(x)| = |H_f(x)| = \left| \int f d\mu_x^U \right| \leqq \int |f| d\mu_x^U = H_{|f|}(x) . \tag{2.2}$$

Noting that $f \in L^2(\sigma)$ implies $|f| \in L^2(\sigma)$ and applying (2.1) to $h = H_{|f|}$, it holds that

$$H_{|f|}(x) \leqq \sup H_{|f|}(F) \leqq \alpha \int H_{|f|} d\nu . \tag{2.3}$$

Taking account of the fact that

$$\int H_{|f|} d\nu = \int |f| d\sigma \leqq \left(\int d\sigma \right)^{1/2} \|f\|_\sigma$$

and

$$\|f\|_\sigma = \|H_f\| = \|u\| ,$$

we have, by (2.2) and (2.3), the following results,

$$|u(x)| \leqq \gamma \|u\| , \tag{2.4}$$

$$\sup|u(F)| \leqq \gamma ||u|| , \tag{2.5}$$

where we denote by γ the constant $\alpha\left(\int d\sigma\right)^{1/2}$. We complete the proof.

Here let us recall into our mind something about the reproducing kernel of a Hilbert space.

Let M be an abstract set and let a system $\mathfrak{F}$ of complex valued functions defined on M constitute a Hilbert space by the scalar product

$$(f, g) = (f(x), g(x))_x ,$$

and the norm

$$||f|| = ((f, f))^{1/2} .$$

A complex valued function $K_0(x, y)$ defined on $M \times M$ is called a reproducing kernel of $\mathfrak{F}$ if it satisfies the condition: for any fixed point y of M, $K_0(x, y) \in \mathfrak{F}$ as a function of x,

$$f(y) = (f(x), K_0(x, y))_x$$

and

$$\overline{f(y)} = (K_0(x, y), f(x))_x .$$

As for the existence of reproducing kernels, we have

Theorem 2.3 (N. Aronszajn [1], S. Bergman [6]). *$\mathfrak{F}$ has a reproducing kernel if and only if there exists, for any x of M, a non-negative constant C_x, deending on x, such that*

$$|f(x)| \leqq C_x ||f|| \qquad \textit{for all } f \in \mathfrak{F} .$$

Let us go back to our argument and show that there exists a reproducing kernel of $R^2(U)$. Then we have the following theorems.

Theorem 2.4. *There exist a reproducing kernel $K(x, y)$ of $R^2(U)$ with the relation*

$$u(y) = (u(x), K(x, y)) \qquad \textit{for all } u \in R^2(U) , \tag{a}$$

and a complete orthonormal countable base $\{u_n\}$ of $R^2(U)$ such that

$$K(x, y) = \sum u_n(x) u_n(y) , \tag{b}$$

which implies the symmetricity of $K(x, y)$, $K(x, y) = K(y, x)$.

Proof. From Theorem 1.5, 2.2 and 2.3 immediately follow the existence of a reproducing kernel $K(x, y)$ of $R^2(U)$ with the relation (a). Since the basic space X is separable, there exists, in $R^2(U)$, a complete orthonormal countable base $\{u_n\}$ with the property (b), applying theorem 1 in O. Lehto [10] or Satz, III_7 in H. Meschkowski [11].

Theorem 2.5. *The reproducing kernel $K(x, y)$ of $R^2(U)$ is non-negative.*

Proof. For any function u of $R^2(U)$, there exists a unique function f of $L^2(\sigma)$ such that

$$u(x) = \int_{\partial U} f(\theta) d\mu_x^U(\theta)$$

and we define $\tilde{u}$ by

$$\tilde{u}(x) = \int_{\partial U} |f(\theta)| d\mu_x^U(\theta) .$$

Then the function $\tilde{u}$ belongs to $R^2(U)$ and we have the relations

$$u(x) \leqq \tilde{u}(x) \qquad \text{for all } x \in U$$

and

$$||u|| = ||\tilde{u}|| = ||f||_\sigma .$$

Let us put

$$u^+(x) = \frac{1}{2}(\tilde{u}+u)$$

and

$$u^-(x) = \frac{1}{2}(\tilde{u}-u) .$$

Then $u^+(x)$ and $u^-(x)$ are obviously non-negative functions of $R^2(U)$ with the properties

$$u = u^+ - u^-$$
$$(u^+, u^-) = 0 ,$$

and therefore it holds that, for any u of $R^2(U)$,

$$(2.6) \qquad \begin{aligned}(u^-, u) &= (u^-, u^+) - (u^-, u^-) \\ &= -(u^-, u^-) \\ &= -||u^-||^2 \leqq 0 .\end{aligned}$$

As, for every $y \in U$, $K_y(x) = K(x, y)$ is a function of $R^2(U)$, $K_y(x)$ satisfies the above relation (2.6), that is,

$$0 \leqq K_y^-(y) = (K_y^-, K_y) = -||K_y^-||^2 ,$$

which implies that $K_y^- = 0$ and so $K_y = K_y^+ \geqq 0$. This completes the proof.

3. Monotonicity of the reproducing kernel with respect to the domain of its definition on harmonic spaces

S. Bergman [6] proved the following relation related to the monotonicity of a kernel function with respect to the domain of its definition on the complex plane: *Let B_1 and B_2 be respectively finite domains with the smooth boundaries in the complex plane. If the domain B_2 is included in B_1, then*

$$K_{B_1}(z, z) \leqq K_{B_2}(z, z)$$

at any point (z, z) in $B_2 \times B_2$, where $K_{B_1}(z, z')$ and $K_{B_2}(z, z')$ denote respectively reproducing kernels of $\mathcal{L}^2(B_1)$ and $\mathcal{L}^2(B_2)$, where $\mathcal{L}^2(B)$ denotes the class of all functions $f(z)$ which are regular and single valued in B and

$$\int_B |f(z)|^2 dxdy < \infty \,.$$

In the previous paragraph, we have proved the existence of a non-negative reproducing kernel $K(x, y)$ of $R^2(U)$ in a harmonic space. The purpose of this paragraph is to prove the above Bergman's Theorem for our reproducing kernel of $R^2(U)$. To do so, it is necessary to prepare some lemmas.

Lemma 3.1. *Let U be a relatively compact open subset of X. Denote by $(u, v)_U$ and $||u||_U = \sqrt{(u, u)_U}$ respectively the inner product and the norm of $R^2(U)$ defined in the paragraph 1. Suppose that x is a point in U. Then there exists a function u_0 of $R^2(U)$ such that*

$$\begin{aligned} ||u_0||_U &= \min\{||u||_U : u \in R^2(U),\, u(x) = 1\} \\ &= 1/\sqrt{K_U(x, x)}\,, \end{aligned}$$

where $K_U(x, y)$ is the reproducing kernel of $R^2(U)$.

Proof. In order to prove this lemma, it is sufficient to apply to $R^2(U)$ the procedure of the minimizing problem to $\mathcal{L}^2(B)$ which S. Bergman [6] discussed. In fact, by Theorem 2.4, there exists a complete orthonormal countable base $\{u_n\}$ with $\sum_{n=1} |u_n(y)|^2 < \infty$ in U. Hence, for any u of $R^2(U)$, we have the representation

$$u(y) = \sum_{n=1}^{\infty} a_n u_n(y) \qquad \text{in } U\,,$$

where $a_n = (u, u_n)_U$. Then, by following the same method as that of p. 21 in [6], we can prove that there exists the minimum function $u_0(y)$, belonging to $R^2(U)$ with $u_0(x) = 1$, such that the norm $||u||_U$ is minimum, that is,

$$||u_0||_U = \min\{||u||_U : u \in R^2(U),\, u(x) = 1\}\,,$$

and that

$$u_0(y) = \frac{\sum_{n=1}^{\infty} u_n(y)u_n(x)}{\sum_{n=1}^{\infty} |u_n(x)|^2} \qquad \text{in } U\,.$$

On the other hand, as

$$K_U(y, x) = \sum_{n=1}^{\infty} u_n(y)u_n(x)\,,$$

u_0 can be denoted by

$$u_0(y) = \frac{K_U(y, x)}{K_U(x, x)} \qquad \text{in } U$$

and it holds that

$$\begin{aligned} ||u_0||_U^2 &= \left(\frac{K_U(y, x)}{K_U(x, x)},\ \frac{K_U(y, x)}{K_U(x, x)}\right)_U \\ &= \frac{1}{K_U(x, x)}\,. \end{aligned}$$

Therefore we obtain the minimum value $||u_0||_U = 1/\sqrt{K_U(x, x)}$.

From now on in this paragraph we suppose that U_1 and U_2 are relatively compact open subsets in X such that U_1 includes U_2 and σ_1 and σ_2 are the positive maesures defined by

$$\sigma_i(e) = \int_{U_i} \mu_x^{U_i}(e) d\nu_i(x) \qquad (i = 1, 2)$$

where $\nu_i (i=1,2)$ is a positive measure defined on a dense subset U_i' of U_i, whose support is the closure of U_i, and ν_2 is the restriction of ν_1 on U_2.

Let us denote by H_f^U the general solution of the Dirichlet problem with respect to an open subset U of X and a resolutive function f on ∂U. Then we have the following:

Lemma 3.2. *If g and h are the following boundary functions on ∂U_2 concerning every function f of $L^2(\sigma_1)$:*

$$g(\theta) = \begin{cases} (H_f^{U_1}(\theta))^2 & \text{on } \partial U_2 \cap U_1 \\ (f(\theta))^2 & \text{on } \partial U_2 \cap \partial U_1 \end{cases}$$

$$h(\theta) = \begin{cases} H_{f^2}^{U_1}(\theta) & \text{on } \partial U_2 \cap U_1 \\ (f(\theta))^2 & \text{on } \partial U_2 \cap \partial U_1\,, \end{cases}$$

then we obtain that

$$H_g^{U_2}(y) \leqq H_h^{U_2}(y) = H_{f^2}^{U_1}(y) \qquad \text{in } U_2\,.$$

Proof. Since f belongs to $L^2(\sigma_1)$ and necessarily to $L^1(\sigma_1)$, by applying

Lemma 1.3, the following function

$$u(\theta)=\int_{\partial U_1} f(\eta)d\mu_\theta^{U_1}(\eta)$$

is well defined in U_1 and we can write by

$$u(\theta)=H_f^{U_1}(\theta) \qquad \text{in } U_1 .$$

Then we have, by Schwarz's inequality and the superharmonicity of constants, that

$$(3.1) \qquad (H_f^{U_1}(\theta))^2 \leqq \int_{\partial U_1} (f(\eta))^2 d\mu_\theta^{U_1}(\eta) \qquad \text{in } U_1 ,$$

where, by virtue of Lemma 1.3, the function of the right hand is well defined and harmonic in U_1 and the following representation is possible:

$$(3.2) \qquad H_{f^2}^{U_1}(\theta)=\int_{\partial U_1} (f(\eta))^2 d\mu_\theta^{U_1}(\eta) \qquad \text{in } U_1 .$$

It is well known that, using Corollary 4.2.5 of Bauer's book [4],

$$(3.3) \qquad H_{f^2}^{U_1}(y)=\int_{\partial U_2} h(\theta)d\mu_y^{U_2}(\theta)=H_h^{U_2}(y) \qquad \text{in } U_2 ,$$

which implies that h is μ_y^U-integrable for all y of U_2. On the other hand, by (3.1) and (3.2), it holds that, in U_2

$$(3.4) \qquad \int_{\partial U_2} g(\theta)d\mu_y^{U_2}(\theta) \leqq \int_{\partial U_2} h(\theta)d\mu_y^{U_2}(\theta)=H_h^{U_2}(y) .$$

This means the fact that g is also μ_y^U-integrable for all y of U_2 and so we can denote as follows:

$$(3.5) \qquad H_g^{U_2}(y)=\int_{\partial U_2} g(\theta)d\mu_y^{U_2}(\theta) \qquad \text{in } U_2 .$$

It follows from (3.3), (3.4) and (3.5) that

$$H_g^{U_2}(y) \leqq H_h^{U_2}(y)=H_{f^2}^{U_1}(y) \qquad \text{in } U_2 ,$$

which completes the proof.

Lemma 3.3. *For any u of $R^2(U_1)$, the restriction of u on U_2, denoted by $u|U_2$, belongs to $R^2(U_2)$.*

Proof. In the first place, consider the case that $U_1 \supset \bar{U}_2 \supset U_2$. Since the restriction of u on ∂U_2, $u|\partial U_2$, is continuous on ∂U_2 and hence belongs to $L^2(\sigma_2)$, we have the following representation:

$$u(y)=\int_{\partial U_2} u|\partial U_2(\theta)d\mu_y^U(\theta) \qquad \text{in } U_2 .$$

Thus we obtain that the restriction of u on U_2, $u|U_2$, belongs to $R^2(U_2)$. In the case that $U_1 \supset U_2$ and $\partial U_1 \cap \partial U_2$ is not null, we consider the boundary function f on ∂U_2,

$$\tilde{f}(\theta) = \begin{cases} H_f^{U_1}(\theta) & \text{on } \partial U_2 \cap U_1 \\ f(\theta) & \text{on } \partial U_2 \cap \partial U_1 , \end{cases}$$

where f is the function of $L^2(\sigma_1)$ in Lemma 1.4 such that

$$u(y) = \int_{\partial U_1} f(\theta) d\mu_y^{U_1}(\theta) .$$

Then it is well known that $\tilde{f}$ is a resolutive function on ∂U_2 and

$$u(y) = \int_{\partial U_2} \tilde{f}(\theta) d\mu_y^{U_2}(\theta) \qquad \text{in } U_2 .$$

We are going to prove that $\tilde{f}(\theta)$ is a function of $L^2(\sigma_2)$. In fact, it is evident that

$$(\tilde{f}(\theta))^2 = g(\theta) \qquad \text{on } \partial U_2 ,$$

where $g(\theta)$ is the function in Lemma 3.2 and then by Lemma 3.2 it holds that

$$H_{\tilde{f}^2}^{U_2}(y) \leqq H_{f^2}^{U_1}(y) \qquad \text{in } U_2$$

and integrating by the measure ν_2 the above inequality, we have

$$\int_{\partial U_2} (\tilde{f}(\theta))^2 d\sigma_2(\theta) \leqq \int_{U_2} H_{f^2}^{U_1}(y) d\nu_2(y) \leqq \int_{\partial U_1} (f(\theta))^2 d\sigma_1(\theta) ,$$

which implies $\tilde{f}$ is a function of $L^2(\sigma_2)$.

We obtain immediately the following corollary of this lemma.

Corollary. *For a fixed point x in U_2, it holds that*

$$\frac{K_{U_1}(y, x) | U_2}{K_{U_1}(x, x)} \in \{u \in R^2(U_2) : u(x) = 1\}$$

and

$$||u_0||_{U_2}^2 \leqq \left(\frac{K_{U_1}(y, x) | U_2}{K_{U_1}(x, x)} , \frac{K_{U_1}(y, x) | U_2}{K_{U_1}(x, x)} \right)_{U_2} ,$$

where u_0 is the minimum function in Lemma 3.1 *to* $R^2(U_2)$.

We now prove the following lemma which plays the essentially important role in studing our purpose of this paragraph.

Lemma 3.4. *It holds that, for every u of $R^2(U_1)$,*

$$||u|U_2||_{U_2} \leqq ||u||_{U_1} .$$

Proof. For every u of $R^2(U_1)$, there exists a unique function f of $L^2(\sigma_1)$ such that

$$u(x)=\int_{\partial U_1} f(\eta)d\mu_x^{U}(\eta) \qquad \text{in } U_1 .$$

Denoting by $\tilde{f}(\theta)$ the same function used in the proof of Lemma 3.3, it holds that, by Lemma 3.3, $\tilde{f}$ belongs to $L^2(\sigma_2)$ and $u\,|\,U_2$ does to $R^2(U_2)$ and

$$u(y)=\int_{\partial U_2} \tilde{f}(\theta)d\mu_y^{U_2}(\theta) \qquad \text{in } U_2 .$$

Then by applying Lemma 3.2, we have the following:

$$\begin{aligned}
||u\,|\,U_2||_{U_2}^2 &= \int_{\partial U_2}(\tilde{f}(\theta))^2 d\sigma_2(\theta) \\
&= \int_{U_2}\int_{\partial U_2} g(\theta)d\mu_y^{U_2}(\theta)d\nu_2(y) \\
&= \int_{U_2} H_g^{U_2}(y)d\nu_2(y) \\
&\leqq \int_{U_2} H_{f^2}^{U_1}(y)d\nu_2(y) \\
&\leqq \int_{U_1} H_{f^2}^{U_1}(y)d\nu_1(y) \\
&= \int_{\partial U_1}(f(\eta))^2 d\sigma_1(\eta) \\
&= ||u||_{U_1}^2 ,
\end{aligned}$$

where $g(\theta)$ means the same function as that of Lemma 3.2. This completes the proof.

We have immediately the following corollary of this Lemma 3.4.

Corollary. *We obtain that*

$$\frac{||K_{U_1}(y, x)\,|\,U_2||_{U_2}^2}{(K_{U_1}(x, x))^2} \leqq \frac{||K_{U_1}(y, x)||_{U_1}^2}{(K_{U_1}(x, x))^2} ,$$

where x is a fixed point in U_2.

Now we are going to prove our main theorem in this paragraph.

Theorem 3.5. *Let U_1 and U_2 be relatively compact open subsets such that U_1 includes U_2. Then the following relation between the reproducing kernels $K_{U_1}(y, x)$ and $K_{U_2}(y, x)$ is held in $U_2\times U_2$:*

$$K_{U_1}(x, x)\leqq K_{U_2}(x, x) .$$

Proof. By Corollary of Lemma 3.3, for a fixed point x of U_2, we obtain that

$$\frac{K_{U_1}(y, x) \mid U_2}{K_{U_1}(x, x)} \in \{u \in R^2(U_2) \colon u(x) = 1\} .$$

We have, by the minimum property and Corollary of Lemma 3.4, that

$$\begin{aligned} \|u_0\|_{U_2}^2 &\leqq \frac{\|K_{U_1}(y, x) \mid U_2\|_{U_2}^2}{(K_{U_1}(x, x))^2} \\ &\leqq \frac{\|K_{U_1}(y, x)\|_{U_1}^2}{(K_{U_1}(x, x))^2} \\ &= \frac{1}{K_{U_1}(x, x)} . \end{aligned}$$

On the other hand, by Lemma 3.1, we obtain the minimum value

$$\|u_0\|_{U_2}^2 = \frac{1}{K_{U_2}(x, x)} .$$

Hence it holds that, in $U_2 \times U_2$,

$$K_{U_1}(x, x) \leqq K_{U_2}(x, x) .$$

This completes the proof of this theorem.

4. Integral representation of harmonic functions in Cauchy-type

In this paragraph it is very useful to recall into our mind Lemma 1.4: *The correspondence*

$$f \in L^2(\sigma) \to H_f \in R^2(U)$$

is isomorphic. For every y of U, the reproducing kernel $K(x, y)$ of $R^2(U)$ belonging to $R^2(U)$ as a function of x, there exists uniquely the function $k(\theta, y)$ of $L^2(\sigma)$ such that

$$K(x, y) = \int_{\partial U} k(\theta, y) d\mu_x^U(\theta) .$$

Then we have the following Cauchy-type integral representation, for every function u of $R^2(U)$, with respect to the integral kernel $k(\theta, y)$.

Theorem 4.1. *Let U be a relatively compact open subset of X and σ the positive measure mentioned in the paragraph 1. Then for any function u of $R^2(U)$, there exists a unique function f of $L^2(\sigma)$ and u can be represented in the following manner —— so called, in the Cauchy-type integral representation*:

$$u(y) = \int_{\partial U} k(\theta, y) f(\theta) d\sigma(\theta) .$$

Conversely, for any function f of $L^2(\sigma)$, *the function of y*

$$\int_{\partial U} k(\theta, y) f(\theta) d\sigma(\theta)$$

belongs to $R^2(U)$.

Proof. For any function u of $R^2(U)$, by Lemma 1.4, there exists uniquely the function of $L^2(\sigma)$ such that

$$u(x) = \int_{\partial U} f(\theta) d\mu_x^U(\theta) \qquad \text{in } U\,.$$

Taking account of the relation between the inner product of $L^2(\theta)$ and that of $R^2(U)$ and the isomorphism between $L^2(\theta)$ and $R^2(U)$, we have immediately that

$$\begin{aligned} u(y) &= (K(x, y), u(x)) \\ &= (k(\theta, y), f(\theta))_\sigma \\ &= \int_{\partial U} k(\theta, y) f(\theta) d\sigma(\theta)\,. \end{aligned}$$

Conversely, for any function f of $L^2(\sigma)$, consider the function of y

$$\int_{\partial U} k(\theta, y) f(\theta) d\sigma(\theta)$$

and denote this by $u(y)$. It is sure that this function $u(y)$ is well defined, since $k(\theta, y)$ and $f(\theta)$ belong to $L^2(\sigma)$. On the other hand, we consider the following function $u_0(y)$ of $R^2(U)$ associated with this given function f of $L^2(\sigma)$,

$$u_0(y) = \int_{\partial U} f(\theta) d\mu_y^U(\theta)\,.$$

We are going to prove that $u(y)$ is equal to $u_0(y)$. By Lemma 1.4 and the reproducing property of $K(x, y)$ in the space $R^2(U)$, we have the followings:

$$\begin{aligned} u(y) &= \int_{\partial U} k(\theta, y) f(\theta) d\sigma(\theta) \\ &= (k(\theta, y), f(\theta))_\sigma \\ &= (K(x, y), u_0(x)) \\ &= u_0(y)\,. \end{aligned}$$

We now define the spaces

$$L(U) = \bigcap_{x \in U} L^1(\mu_x^U)$$

$$R(U) = \left\{ H_f : H_f(x) = \int_{\partial U} f(\theta) d\mu_x^U(\theta) \textit{ for all } f \in L(U) \textit{ and for all } x \in U \right\}.$$

By the Brelot's resolutivity theorem, $L(U)$ is constructed by all resolutive functions relative to Dirichlet problem for U and $R(U)$ is the set of all general solutions to all resolutive functions. Lemma 1.3 follows that $L(U)\supset L^2(\sigma)$ and so $R(U)\supset R^2(U)$. We are going to discuss the Cauchy-type integral representation of every function u of $R(U)$ concerning a non-negative integral kernel $\tilde{k}(\theta,y)$. To do so we must prepare some lemmas.

Lemma 4.2. *For any Borel subset e of ∂U and any point x of U, we have that*

$$\mu_x^U(e) = \int_e k(\theta, x)d\sigma(\theta)\,,$$

where $k(\theta, x)$ is the same function that appeared in Theorem 4.1.

Proof. In the procedure of the proof of Theorem 4.1, we have that

$$\int_{\partial U} f(\theta)d\mu_x^U(\theta) = \int_{\partial U} f(\theta)k(\theta, x)d\sigma(\theta) \qquad \text{for any } f\in L^2(\sigma)\,.$$

And hence it is evident that the above relation holds for all continuous functions f on ∂U. This fact implies immediately the result of this lemma.

Furthermore we can improve slightly Lemma 4.2 as follows:

Lemma 4.3. *For any Borel subset e of ∂U and any point x of U, there exists a non-negative function $\tilde{k}(\theta, x)$ such that*

$$\mu_x^U(e) = \int_e \tilde{k}(\theta, x)d\sigma(\theta)$$

and

$$k(\theta, x) = \tilde{k}(\theta, x) \qquad \textit{in } L^2(\sigma)\,.$$

Proof. If we note that the measures μ_x^U and σ are positive measures, from Lemma 4.2 immediately it follows that for any x of U

$$k(\theta, x)\geqq 0 \qquad \sigma\text{-a.e. on } \partial U.$$

We define the non-negative function $\tilde{k}(\theta, x)$ by

$$\tilde{k}(\theta, x) = \begin{cases} k(\theta, x) & \text{on } \partial U - E_x \\ 0 & \text{on } E_x\,, \end{cases}$$

where we put for each x of U

$$E_x = \{\theta\in\partial U\colon k(\theta, x)<0\}\,.$$

Then we have immediately

$$\tilde{k}(\theta, x)\in L^2(\sigma) \qquad \text{for all } x\in U\,,$$

$$\mu_x^U(e) = \int_e \tilde{k}(\theta, x) d\sigma(\theta)$$

and

$$\begin{aligned} K(x, y) &= \int_{\partial U} k(\theta, y) d\mu_x^U(\theta) \\ &= \int_{\partial U} k(\theta, y) \tilde{k}(\theta, x) d\sigma(\theta) \\ &= \int_{\partial U} \tilde{k}(\theta, x) k(\theta, y) d\sigma(\theta) \\ &= \int_{\partial U} \tilde{k}(\theta, x) d\mu_y^U \\ &= K(y, x)\,, \end{aligned}$$

which implies that, by Lemma 1.4,

$$k(\theta, x) = \tilde{k}(\theta, x) \qquad \text{in } L^2(\sigma)\,.$$

This lemma means that the measure μ_x^U has the density function $\tilde{k}(\theta, x)$ with respect to the measure σ.

Thus we obtain the following extension of Theorem 4.1.

Theorem 4.4. *Let U be a relatively compact open subset of X and σ the positive meaure mentioned in the paragraph* 1. *Then any function u of $R(U)$ is represented in the Cauchy-type integral representation with respect to the integral kernel $\tilde{k}(\theta, x)$, a function f of $L(U)$ and the measure σ, that is,*

$$u(y) = \int_{\partial U} \tilde{k}(\theta, y) f(\theta) d\sigma(\theta)\,.$$

Conversely, for each function f of $L(U)$, the function of y in U,

$$\int_{\partial U} \tilde{k}(\theta, y) f(\theta) d\sigma(\theta)$$

belongs to $R(U)$.

Proof. For any function u of $R(U)$, there exists, by the definition, a function f of $L(U)$ such that

$$u(y) = \int_{\partial U} f(\theta) d\mu_y^U(\theta)\,.$$

By virtue of Lemma 4.3, we have the following expression,

$$u(y) = \int_{\partial U} \tilde{k}(\theta, y) f(\theta) d\sigma(\theta)\,.$$

Conversely, for any function f of $L(U)$, by the resolutivity theorem, we can define the following function u of $R(U)$

$$u(y) = \int_{\partial U} f(\theta) d\mu_y^U(\theta) .$$

Using again Lemma 4.3, this function $u(y)$ is equal to the function of y in U,

$$\int_{\partial U} \tilde{k}(\theta, y) f(\theta) d\sigma(\theta) .$$

This completes the proof of this theorem.

In the last place, let us note that we obtain as a special case of Theorem 4.4 the following result in the investigation by H.S. Bear and A.M. Gleason [5].

Theorem 4.5. *Let U be a relatively compact open subset in X, Γ the topological boundary of U and $H(U)$ the set of all harmonic functions on U such that there exist their continuous extensions over the closure of U, denoted by $\bar{U}$. Then, for any u of $H(U)$ there exist a Borel probability measure λ on Γ and a non-negative measurable function $q(\theta, y)$ on $\Gamma \times U$ such that in U*

$$u(y) = \int_{\Gamma} q(\theta, y) f(\theta) d\lambda(\theta) ,$$

where f denotes the restriction of the continuous extension of u over $\bar{U}$ on the boundary.

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