Thomas, E.
Osaka J. Math.
13 (1976), 163–186

EMBEDDING MANIFOLDS IN EUCLIDEAN SPACE

EMERY THOMAS*

(Received February 17, 1975)

1. Introduction. We consider here the problem of whether a smooth manifold M (compact, without boundary) embeds in Euclidean space of a given dimension. Our results are of two kinds: first we give sufficient conditions for an orientable n-manifold to embed in R^{2n-2}, and we then give necessary and sufficient conditions for RP^n ($=n-$dimensional real projective space) to embed in R^{2n-6}. We obtain these results using the embedding theory of A. Haefliger [6].

Recall that by Whitney [37], every n-manifold embeds in R^{2n}. Combining results of Haefliger [6], Haefliger-Hirsch [9] and Massey-Peterson [16] one knows that every *orientable* n-manifold embeds in R^{2n-1} ($n>4$), and if n is not a power of two, *every* n-manifold embeds in R^{2n-1}. Finally, if n is a power of two ($n>4$), by [9] and [26] one has: a non-orientable n-manifold embeds in R^{2n-1} if and only if $\bar{w}_{n-1}=0$. Here $\bar{w}_i, i\geqslant 0$, denotes the (mod 2) normal Stiefel-Whitney class of a manifold M.

We give two sets of sufficient conditions for embedding an n-manifold in R^{2n-2}; in order to use the theory of Haefliger, we assume $n\geqslant 7$.

Theorem 1.1. *Let M be an orientable n-manifold, with $\bar{w}_{n-3+i}=0$, for $i\geqslant 0$. If either $w_3\neq 0$, or $w_2\neq 0$ and $H_1(M;Z)$ has no 2-torsion, then M embeds in R^{2n-2}.*

Here w_i denotes the i^{th} mod 2 (tangent) Stiefel-Whitney class of M. A necessary condition for M^n to embed in R^{2n-2} is that $\bar{w}_{n-2}=0$. Note, however, that if $n-1$ is a power of two, then RP^n does not embed in R^{2n-2}, even though $\bar{w}_{n-2}=0$. (In this case $\bar{w}_{n-3}\neq 0$ and $H_1(RP^n; Z)=Z_2$).

By Massey-Peterson [16] one has that $\bar{w}_{n-3+i}=0$, $i\geqslant 0$, for M^n, provided one of the following conditions is satisfied: $n\equiv 3 \bmod 4$; $n\equiv 0, 2 \bmod 4$ and $\alpha(n)\geqslant 3$; $n\equiv 1 \bmod 4$ and $\alpha(n)\geqslant 4$. Here $\alpha(n)$ denotes the number of one's in the dyadic expansion of the integer n.

Recall that an orientable manifold is called a spin manifold if $w_2=0$. As a complement to Theorem (1.1) we have:

Theorem 1.2. *Let M be an n-dimensional spin manifold with $\bar{w}_{n-5+i}=0$,*

*) Research supported by the National Science Foundation

$i \geqslant 0$. *Then M^n embeds in R^{2n-2}, provided that $H_1(M; Z)$ has no 2-torsion when $n \equiv 0$ mod 4.*

Again by [16] and [26] we have: let M^n be a spin manifold with $n \equiv i$ mod 8, $4 \leqslant j \leqslant 7$; then, $\bar{w}_{n-j} = 0$. Thus by (1.2) we obtain: *if M^n is a spin manifold with $n \equiv 5, 6, 7$ mod 8, M embeds in R^{2n-2}.*

We consider now the problem of embedding RP^n. Quite good results have been obtained by geometric methods. In particular, the work of Mahowald-Milgram [15], Steer [31], and Rees [25] gives a good picture for large values of $\alpha(n)$. However, we now show that for small values of $\alpha(n)$ the known results are not best possible.

Theorem 1.3. *Let s be a positive integer, not a power of two. Set $n = 8s+t$, $0 \leqslant t \leqslant 7$. Then RP^n embeds in R^{2n-6}, provided $\alpha(n) \geqslant 4$ when $t=1$ or 2.*

To my knowledge this is a new result in the following cases (all congruences are mod 8).

$n \equiv 0$,	$2 \leqslant \alpha(n) \leqslant 8$,
$n \equiv 1$,	$4 \leqslant \alpha(n) \leqslant 6$,
$n \equiv 2$,	$4 \leqslant \alpha(n) \leqslant 7$,
$n \equiv 3, 5$,	$4 \leqslant \alpha(n) \leqslant 5$,
$n \equiv 4$,	$3 \leqslant \alpha(n) \leqslant 7$.

Combining results of [31] and [25] one has (cf., [12, 5.3]): If $n \equiv 7$ mod 8, RP^n embeds in $R^{2n-\alpha(n)-3}$; thus, if $n \equiv 6$ mod 8, RP^n embeds in $R^{2n-\alpha(n)-2}$. Consequently by (1.3) we have:

Corollary 1.4. *Let n be an integer such that $n \geqslant 15$ and $n \not\equiv 2, 4$ mod 8. Then RP^n embeds in R^{2n-6} if, and only if,*

$$\alpha(n) \geqslant 2, \quad \text{and} \quad n \equiv 0$$
$$\alpha(n) \geqslant 4, \quad \text{and} \quad n \equiv 1, 3, 5, 6, 7.$$

Of course, by (1.3), if $n \equiv 2$ and $\alpha(n) \geqslant 4$ or $n \equiv 4$ and $\alpha(n) \geqslant 3$, then RP^n does embed in R^{2n-6}. Note [29], [1], [21], [4], that when $n \equiv 2$ and $\alpha(n) = 3$ or $n \equiv 4$ and $\alpha(n) = 2$, RP^n immerses in R^{2n-6} but not in R^{2n-7}. Thus the following conjecture seems reasonable.

Conjecture 1.5. *If $n \equiv 2$ mod 8 and $\alpha(n) = 3$, or if $n \equiv 4$ mod 8 and $\alpha(n) = 2$, then RP^n does not embed in R^{2n-6}.*

The method of proof developed in this paper also gives one new result for complex projective n-space, CP^n.

Theorem 1.6. *Let n be a positive integer with $n \equiv 3 \mod 4$ and $\alpha(n) \geqslant 4$. Then CP^n embeds in R^{4n-6}.*

For $\alpha(n) \geqslant 5$ this follows by work of Steer [31].

The specific result of Haefliger that we use is the following. For a topological space X let X^2 denote the product $X \times X$ and let Δ denote the diagonal in X^2. The group of order 2, Z_2, acts freely on $X^2 - \Delta$ by interchanging factors; we set $X^* = (X^2 - \Delta)/Z_2$. The projection p: $X^2 - \Delta \rightarrow X^*$ is a 2-fold covering map; denote by ξ the associated line bundle and by $S_q(\xi)$ the $(q-1)$ sphere bundle associated to the q-plane bundle $q\xi$. Haefliger proves (see [5] and [6, §1.7]):

Theorem 1.7 (Haefliger). *Let M be a smooth n-manifold and let q be a positive integer such that $2q \geqslant 3(n+1)$. Then M embeds in R^q if, and only if, the bundle $S_q(\xi)$ has a section.*

REMARK. A similar theorem has been proved by Weber [36] for *PL*-manifolds (and semi-linear embeddings) and by J.A. Lees [41] for topological manifolds with locally flat embeddings (assuming $2q > 3(n+1)$). Thus Theorems (1.1) and (1.2) can be stated for these categories of manifolds. In connection with Theorems (1.3) and (1.6), note the work of Rigdon [27].

Our method of proof is to use various techniques of obstruction theory to show that the bundle $S_q(\xi)$ has a section. Briefly, the following techniques will occur: (i) indeterminancy, (ii) relations, (iii) naturality, (iv) generating class, (v) Whitney product formulae.

The remainder of the paper is organized as follows: in section 2 we develop some facts about the space M^*. Section 3 is a brief survey of obstruction theory, while in section 4 we give the proofs of Theorems (1.1) and (1.2). In section 5 we prove Theorem (1.3) and in section 6, Theorem (1.6). Finally, sections 7 and 8 contain proofs omitted in previous sections.

2. Properties of M^*

In order to use Theorem 1.7, we need to know the cohomology of M^*, especially mod 2. For the rest of the paper all cohomology will be with mod 2 coefficients unless otherwise indicated.

To compute $H^*(M^*)$ (mod 2 coefficients!) we use another result of Haefliger [7], as reworded by Rigdon [26]. We set (cf., [19]).

$$\Gamma M = S^\infty \times_{Z_2} (M^2),$$

where S^∞ is the unit sphere in R^∞, and where Z_2 acts by the diagonal action. Also, let P^∞ denote the infinite dimensional real projective space, and for a manifold M let $P(M)$ denote the projective line bunlde associated to the tangent bundle.

Theorem 2.1 (Haefliger). *Given an n-manifold M, there is a commutative diagram of mod 2 cohomology, as shown below, in which each row is an exact sequence* ($i\geqslant 0$):

$$\begin{tikzcd}
0 \arrow[r] & H^{i-n}(M) \arrow[r, "\varphi_1"] & H^i(M\times M) \arrow[r, "\rho_1"] & H^i(M\times M-\Delta) \arrow[r] & 0 \\
0 \arrow[r] & H^{i-n}(P^\infty\times M) \arrow[u, "r^*"] \arrow[d, equal] \arrow[r, "\varphi"] & H^i(\Gamma M) \arrow[u, "q^*"] \arrow[d, "k^*"] \arrow[r, "\rho"] & H^i(M^*) \arrow[u, "p^*"] \arrow[d, "j^*"] \arrow[r] & 0 \\
0 \arrow[r] & H^{i-n}(P^\infty\times M) \arrow[r, "\varphi_2"] & H^i(P^\infty\times M) \arrow[r, "\rho_2"] & H^i(P(M)) \arrow[r] & 0
\end{tikzcd}$$

All the morphisms in the diagram, except the φ's, are induced by mappings between spaces. φ_1 and φ_2 can be thought of as Gysin maps. Specifically, given $x\in H^{i-n}(M)$, then

(2.2) $\qquad \varphi_1(x)= U\cdot(1\otimes x),\quad$ where $\quad U\in H^n(M^2)$

is the mod 2 "Thom class" of M, as given, e.g., by Milnor [20]. φ_2 is computed as follows. Let $u\in H^1(P^\infty)$ denote the generater. Then, for $x\in H^*(M)$, and $j\geqslant 0$,

(2.3) $\qquad \varphi_2(u^j\otimes x)=\sum_0^n u^{i+j}\otimes w_{n-i}(M)\cdot x\,.$

The key space in 2.1 is ΓM; Steenrod [30] has computed the cohomology of this as follows.

Let t be the involution of $M\times M$ which transposes the factors, and set $\sigma=1+t^*: H^*(M^2)\to H^*(M^2)$. Let K^* and I^* denote, respectively, the kernel and image of σ. Thus K^* and I^* are graded groups with $I^*\subset K^*$; set $\bar{K}^*=K/I$.

Using the obvious projection $\Gamma M\to P^\infty$, we regard $H^*(\Gamma M)$ as an $H(P^*)$-module.

Theorem 2.4 (Steenrod). *There is an isomorphism of $H^\infty(P^*)$-modules,*

$$H^*(\Gamma M)\approx(H^*(P^\infty)\otimes\bar{K}^*)\oplus I^*\,,$$

where $H^*(P^\infty)$ acts trivially on I^*.

Note that $\bar{K}^*$ is zero in odd dimensions. For each $n\geqslant 0$ we have an isomorphism

$$H^n(M)\approx\bar{K}^{2n},\qquad x\to(x)^2$$

where $(x)^2$ denotes the coset of I containing $x\otimes x$.

We now describe the morphisms q^* and k^* in (2.1).

Proposition 2.5.

(i) $q^*|(\bar{K}^*\oplus I^*) = $ *identity.*

(ii) $q^*(u^m\otimes(x)^2) = 0,$ *if* $m>0$

(iii) $k^*(I^*) = 0,$

(iv) $k^*(u^m\otimes(x)^2) = \sum_0^q u^{m+q-i}\otimes \mathrm{Sq}^i(x),$ *if* $\deg x = q.$

For the proof, see Haefliger [7] and Steenrod [30].

Note that by 2.5 (iv), $k^*|(H^*(P^\infty)\otimes\bar{K}^*)$ is injective. Thus, for $y\in H^*(\Gamma M)$,

(2.5) (v) $y = 0$ *if and only if* $q^*(y) = 0$ *and* $k^*(y) = 0.$

Returning to diagram (2.1), the map $r: M\to P^\infty\times M$ is simply the inclusion; the morphism ρ_2 is computed as follows. As before, let ξ denote the canonical line bundle over M^*; set $\eta=j^*\xi$, $v=w_1\eta\in H^1(P(M))$. Recall (e.g. [11]) that $H^*(P(M))$ is a free $H^*(M)$-module on 1, $v\cdots v^{n-1}$, with the relation

$$v^n = \sum_{i=1}^n v^{n-i}\cdot w_i(M). \tag{2.6}$$

Given $x\in H^*(M)$, we have

$$\rho_2(u^m\otimes x) = v^m\cdot x, \quad m\geqslant 0. \tag{2.7}$$

Our goal is to find ways of showing that the obstructions vanish for a section of the bundle $S_q(\xi)$ over M^*. For this we need ways of showing that a class in $H^*(M^*)$ is zero. The following result is useful for this.

Define B^* to be the subspace of $H^*(\Gamma M)$ generated by all classes of the form

$$u^j\otimes(x)^2, \quad \text{with} \quad j+\deg x<\dim M. \tag{2.8}$$

We set $\lambda=\rho_2 k^*=j^*\rho$, in (2.1), and write $\Lambda^*=\lambda(B^*)\subset H^*(P(M))$. Note that $B^*\cap I^*=0$.

Proposition 2.9.

(a) *Kernel* $j^*=\rho(I^*)$,

(b) $\rho\,|\,I^*$ *is injective,*

(c) *Image* $j^*=\lambda(B^*)$ $(=\Lambda^*)$,

(d) ρ *maps* $B^*\oplus I^*$ *isomorphically onto* $H^*(M^*)$.

The proof is given in §7.

Set $w=w_1\xi\in H^1(M^*)$. Since $\rho(u)=w$ and $j^*w=v$, we have (by 2.4),

Corollary 2.10. $w\cdot(\mathit{kernel}\, j^*)=0$.

3. Obstruction theory for sphere bundles

We discuss here the general problem of finding a section to a sphere bundle. At the end of the section we consider the special case posed by Theorem 1.7—the sphere bundle is one associated to a multiple of a line bundle.

Let X be a complex and ω an oriented q-plane bundle over X, $q\geqslant 8$. We assume that $\dim X\leqslant q+5$. Then the (mod 2) obstructions to a section in the associated sphere bundle are the following (see [14], [34]), using the fact that the 4 and 5-stems are zero [35].

$$\begin{aligned}
&\chi(\omega)\in H^q(X;Z),\\
&(\alpha_1,\alpha_3)(\omega)\in H^{q+1}(X)\oplus H^{q+3}(X),\\
&(\beta_2,\beta_3)(\omega)\in H^{q+2}(X)\oplus H^{q+3}(X),\\
&\gamma_3(\omega)\in H^{q+3}(X).
\end{aligned}$$

In our applications, ω comes from a double cover and so the mod 3 obstruction in dim $q+3$ is zero ([3], [28]). Also, for such a bundle, $w_{2i+1}(\omega)=0$, $i\geqslant 0$.

These obstructions have the following indeterminacies: for $j\geqslant 1$, define $\theta_j\colon H^*(X)\to H^*(X)$ by

$$x\to \mathrm{Sq}^j(x)+w_j(\omega)\cdot x.$$

Then, assuming that $w_1(\omega)=w_3(\omega)=0$,

$$\begin{aligned}
(3.1)\qquad &\mathrm{Indet}\,(\alpha_1,\alpha_3)=(\theta_2,\theta_4)H^{q-1}(X;Z).\\
&\mathrm{Indet}\,(\beta_2,\beta_3)=(\theta_2,\mathrm{Sq}^2\mathrm{Sq}^1)H^q(X)+\mathrm{Sq}^1H^{q+2}(X),\\
&\mathrm{Indet}\,(\gamma_3)=\theta_2H^{q+1}(X)+\mathrm{Sq}^1H^{q+2}(X).
\end{aligned}$$

In the case of the β's and γ, this is just the indeterminancy obtained by passing from one stage of the Postnikov resolution to the next—not the "full" indeterminancy in the sense of [18]. At one point we will need the full indeterminancy for (β_2,β_3). Specifically, one can show (see [17], [18]):

$$\mathrm{Indet}\,(\beta_2,\beta_3)(\omega)=\Psi_\omega H^{q-1}(X;Z),$$

where Ψ_ω is a "twisted" secondary cohomology operation [17], [32] defined on Kernel $\theta_2\cap$ Kernel $\theta_4\cap H^{q-1}(X;Z)$, taking values in $H^{q+2}(X)\oplus H^{q+3}(X)$, and with Indet $\Psi_\omega=(\theta_2,\mathrm{Sq}^2\mathrm{Sq}^1)H^q(X)+\mathrm{Sq}^1H^{q+2}(X)$. Note the simple, but important, fact:

$$\text{if Kernel }\theta_2\cap\text{Kernel }\theta_4\cap H^{q-1}(X;Z)=0,$$

then

$$\mathrm{Indet}\,(\beta_2,\beta_3)=\mathrm{Indet}\,\Psi_\omega=(\theta_2,\mathrm{Sq}^2\mathrm{Sq}^1)H^q(X)+\mathrm{Sq}^1H^{q+2}(X).$$

A second useful fact about these obstructions is that they satisfy certain universal relations, see [14], [33], [34, 4.2]. Namely

$$\begin{aligned}&\theta_2\alpha_1(\omega) = 0\\ (3.2)\qquad &\mathrm{Sq}^2\mathrm{Sq}^1\alpha_1(\omega)+\mathrm{Sq}^1\alpha_3(\omega) = 0\\ &\theta_2\beta_2(\omega)+\mathrm{Sq}^1\beta_3(\omega) = 0,\end{aligned}$$

assuming, as above, that $w_1(\omega)=w_3(\omega)=0$. Moreover, if $w_i(\omega)=0$ for $1\leqslant i\leqslant 7$, we then have

$$(3.3)\qquad \mathrm{Sq}^6\alpha_1(\omega)+\mathrm{Sq}^4\alpha_3(\omega) = 0.$$

Suppose now that Y is a second complex and $f\colon Y\to X$ a map. One then has naturality relations for the obstructions: e.g.,

(3.4) (i) *If* $\chi(\omega)=0$, *then* $(\alpha_1, \alpha_3)f^*\omega$ *is defined and* $f^*(\alpha_1, \alpha_3)(\omega)\subset(\alpha_1, \alpha_3)f^*\omega$.

(ii) *If* $\chi(\omega) = 0$ *and* $(\alpha_1, \alpha_3)(\omega)\equiv 0$, *then* $(\beta_2, \beta_3)(f^*\omega)$ *is defined and* $f^*(\beta_2, \beta_3)\omega\subset(\beta_2, \beta_3)f^*\omega$.

We consider now the special case $\omega=q\xi$, ξ a line bundle over X. We take q even, say $q=2s$, so that ω is orientable. Let $v=w_1\xi\in H^1(X)$. Also, denote by δ_2 the Bockstein coboundary associated with the exact sequence $Z\xrightarrow{\times 2}Z\to Z_2$. Since $\chi(2\xi)=\delta_2 v$, one has

$$(3.5)\qquad \chi(q\xi) = \delta_2(v^{q-1}).$$

To compute $\alpha_1(q\xi)$ (assuming $\chi(q\xi)=0$), we use the theory of "twisted" cohomology operations, as developed in [17] and [32]. Write θ_2 for $\theta_2(q\xi)$. One then has a secondary operation Φ_3, of degree 3, asociated with the following relation (see p. 206 in [32]):

$$(3.6)\qquad \Phi_3\colon\ \theta_2\circ\theta_2=0,\quad \text{on integral classes.}$$

Our result is:

Proposition 3.7. *Let* ξ *be a line bundle over* X, *with* $v=w_1\xi$. *Suppose that* $\chi(q\xi)=0$, *for some* $q=2s$, $s\geqslant 2$. *If* $\theta_2H^{q-1}(X; Z)=\theta_2H^{q-1}(X)$, *then* $\alpha_1(q\xi)\equiv \Phi_3(\delta_2(v^{q-3}).\ sv^2)$.

This is proved at the end of the section, using the "generating class" theorem of [32].

One final technique we will need is the Whitney product formula for higher order obstructions: see [22] and [34, 4.3]. We keep the notation of 3.7.

Proposition 3.8.

(i) *Suppose that* $\chi(q\xi)=0$. *Then*
$\alpha_1(q+2)\xi\equiv\alpha_1(q\xi)\cdot v^2$,
$\alpha_3(q+2)\xi\equiv\alpha_3(q\xi)\cdot v^2+\alpha_1(q\xi)\cdot v^4$.

(ii) *Suppose that* $\chi(q\xi)=0$ *and that* $(\alpha_1,\alpha_3)(q\xi)\equiv 0$.
Then,
$\beta_j(q+2)\xi\equiv\beta_j(q\xi)\cdot v^2$, $j=2, 3$.

Proof of 3.7. Let η denote the canonical oriented 2-plane bundle over CP^∞. The sphere bundle associated to $s\eta$ is

$$S^{2s-1}\xrightarrow{i}CP^{s-1}\xrightarrow{\pi}CP^\infty,$$

where π is homotopic to the inclusion. Let $x=\chi(\eta)\in H^2(CP^\infty; Z)$ denote the Euler class of η. Thus, $\chi(s\eta)=x^s\in H^{2s}(CP^\infty; Z)$. Consider now the first stage in a Postnikov resolution of π.

$$\begin{tikzcd} & E \arrow[d, "p"] & \\ CP^{s-1} \arrow[ur, "q"] \arrow[r, "\pi"'] & CP^\infty \arrow[r, "x^s"'] & K(Z, 2s) \end{tikzcd}$$

Let $\alpha\in H^{2s+1}(E)$ denote the second obstruction. Then α arises because of the relation

$$\theta_2(x^s)=0.$$

(Note §§3–5 of [32]). But

$$x^s \bmod 2=\theta_2(x^{s-1}).$$

Thus, in the language of §5 of [32], x^{s-1} is a "generating class" for α; and hence, by Theorem 5.9 of [32].,

$$(*)\qquad \alpha\in\Phi_3(p^*x^{s-1}, sx \bmod 2).$$

To prove 4.4, let $f\colon X\to CP^\infty$ be a map such that $f^*(x)=\delta_2 v$. Then, $q\xi=f^*(s\eta)$ —since $q=2s$. By hypothesis, $\chi(q\xi)=0$ and so f lifts to a map $g\colon X\to E$. Moreover, $g^*\alpha\in\alpha_1(q\xi)$. But by (*), $g^*\alpha\in\Phi_3((\delta_2 v)^{s-1}, sv^2)$. By the hypotheses of 4.4, α_1 and Φ_3 have the same indeterminacy, and so the theorem is proved.

REMARK. A similar result has been obtained independently by Rigdon [26].

4. Embedding n-manifolds in R^{2n-2}

If M is an n-manifold, then M^* has the homotopy type of a $(2n-1)$-complex, and so to see whether $(2n-2)\xi$ has a section (i.e., by 1.7, whether M embeds in R^{2n-2}) we need only consider $\chi(2n-2)\xi$ and $\alpha_1(2n-2)\xi$. To compute χ we use the following important result of Haefliger [7].

Theorem 4.1 (Haefliger). *If M is an n-manifold, then $v^{n+k}=0$ if, and only if, $\bar{w}_{k+i}=0$, $i\geqslant 0$.*

The following result implies Theorem (1.1).

Proposition 4.2. *Let M be an orientable n-manifold. If $Sq^2H^{n-2}(M; Z)=H^n(M)$, then $\theta_2H^{2n-3}(M^*; Z)=H^{2n-1}(M^*)$.*

We give the proof at the end of the section.

Proof of Theorem 1.1. Since $\bar{w}_{n-3+i}=0$, for $i\geqslant 0$, it follows from (4.1) and (3.5) that $\chi(q\xi)=0$, where $q=2n-2$. We will show that $\alpha_1(q\xi)\equiv 0$ by showing that $\mathrm{Sq}^2H^{n-2}(M; Z)=H^n(M)$. For then by (4.2), $H^{2n-1}(M^*)=\mathrm{Indet}\ \alpha_1(q\xi)$ and hence $\alpha_1(q\xi)\equiv 0$.

Let $\mu\in H^n(M)$ denote the generator. Suppose first that $w_3\neq 0$. Then there is a class $y\in H^{n-3}(M)$ such that $y\cdot w_3=\mu$. But by *Wu* [38], $\mu=y\cdot w_3=\mathrm{Sq}^2\mathrm{Sq}^1 y$, and so $\mu\in \mathrm{Sq}^2H^{n-2}(M; Z)$. On the other hand, suppose that $H_1(M; Z)$ has no 2-torsion. Then by Poincare duality, $H^{n-1}(M; Z)$ has no 2-torsion and so $H^{n-2}(M)=H^{n-2}(M; Z)$ mod 2. Assume that $w_2\neq 0$, and let $z\in H^{n-2}(M)$ be a class such that $\mu=z\cdot w_2=\mathrm{Sq}^2 z$. But $z=\hat{z}$ mod 2. for some $\hat{z}\in H^{n-2}(M; Z)$, and so again $\mu\in \mathrm{Sq}^2H^{n-2}(M; Z)$, which completes the proof of the Theorem.

We turn now to the proof of Theorem (1.2). Since M is a spin manifold, $\mathrm{Sq}^2H^{n-2}(M)=0$, and so we cannot use Proposition (4.2); instead we have the following:

Proposition 4.3. *Let M be an n-dimensional spin manifold. If $n\not\equiv 0$ mod 4 or if $H_1(M; Z)$ has no 2-torsion, then $\theta_2H^{2n-3}(M^*; Z)=\theta_2H^{2n-3}(M^*)$.*

Here $\theta_2=\theta_2(2n-2)\xi$. We give the proof at the end of the section.

Proof of Theorem 1.2. Since $\bar{w}_{n-5+i}=0$ $(i\geqslant 0)$, $\chi(2n-2)\xi=0$, using (3.5) and (4.1), and so $\alpha_1(2n-2)\xi$ is defined. By (4.3) and (3.7), $\alpha_1(q\xi)\equiv \Phi_3(\delta_?(v^{q-3}),sv^2)$, where $q=2n-2$, $s=n-1$. Since $\bar{w}_{n-5+i}=0$ $(i\geqslant 0)$, then by 4.1 $v^{2n-5}=v^{q-3}=0$, and so $\alpha_1(q\xi)\equiv 0$, which gives an embedding of M^n in R^{2n-2}, by Theorem (1.7).

Proof of Proposition 4.2. Note that by (2.8), $B^{2n-1}=0$, and hence by (2.9), $H^{2n-1}(M^*)=\rho I^{2n-1}$.

Let $\hat{\xi}$ denote the line bundle over ΓM with $w_1\hat{\xi}=u$, and let $\hat{\theta}_2=\theta_2(q\hat{\xi})$. Then,

$$\rho\hat{\theta}_2 = \theta_2\rho,\ \hat{\theta}_2(I^*) = \mathrm{Sq}^2(I^*).$$

To prove (4.2), let $y\in H^{2n-1}(M^*)$ and let $z\in I^{2n-1}$ with $\rho(z)=y$. Then, $q^*z=\sigma(\mu\otimes b)$, for some $b\in H^{n-1}(M)$. By hypothesis, there is a class $\hat{a}\in H^{n-2}(M;\,Z)$ with $\mathrm{Sq}^2\hat{a}=\mu$. Also, since M is orientable, $b=\hat{b}$ mod 2 for some $\hat{b}\in H^{n-1}(M;\,Z)$. By analyzing $H^*(\Gamma M;\,Z)$ (e.g. [2]), one sees that there is a class $\hat{x}\in H^{2n-3}(\Gamma M;\,Z)$ such that $q^*(\hat{x})=\hat{a}\otimes\hat{b}+\hat{b}\otimes\hat{a}\in H^{2n-3}(M^2;\,Z)$, and $\hat{x}$ mod $2\in I^{2n-3}$. Thus,

$$q^*(\mathrm{Sq}^2\hat{x}) = \mathrm{Sq}^2(\hat{a}\otimes\hat{b}+\hat{b}\otimes\hat{a}) = \sigma(\mu\otimes b) = q^*(z).$$

Since z, $\mathrm{Sq}^2\hat{x}\in I^*$ this means that $z=\mathrm{Sq}^2\hat{x}$ and so

$$y = \rho(z) = \rho\mathrm{Sq}^2\hat{x} = \rho\hat{\theta}_2\hat{x} = \theta_2\rho(\hat{x}),$$

as desired.

Proof of Proposition 4.3. Let $\theta_2=\theta_2(2n-2)\xi$, $\hat{\theta}_2 = \theta_2(2n-2)\hat{\xi}$, as above. Let $x\in H^{2n-3}(M^*)$. Then (see (2.9)), one may choose $b\in H^{2n-3}(\Gamma M)$ so that $\rho(b)=x$ and

$$k^*(b) = u^{n-1}\otimes h+u^{n-2}\otimes \mathrm{Sq}^1h,$$

for some $h\in H^{n-2}(M)$. Since M is spin, $\mathrm{Sq}^2h=0$ and $q^*\hat{\theta}_2(b)=0$. Moreover,

$$k^*\hat{\theta}_2(b) = \hat{\theta}_2k^*(b) = \binom{n}{2}u^{n+1}\otimes h+\binom{n-2}{2}u^n\otimes\mathrm{Sq}^1h.$$

Note that

$$u^{n+1}\otimes h = \varphi_2(u\otimes h) = k^*\varphi(u\otimes h),$$

and $q^*(u\otimes h)=0$. Set

$$\beta = \hat{\theta}_2(b)-\varphi\left(\binom{n}{2}(u\otimes h)\right).$$

Then,

$$(*)\qquad \rho(\beta) = \theta_2(x),\ q^*(\beta) = 0,\ k^*(\beta) = \binom{n-2}{2}u^n\otimes\mathrm{Sq}^1h.$$

Case I, $n\not\equiv 0$ mod 4. If $n\equiv 2$, 3 mod 4, then $\binom{n-2}{2}=0$ mod 2 and so $\beta=0$, which means that $\theta_2(x)=0$. If $n\equiv 1$ mod 4, then $u^n\otimes\mathrm{Sq}^1h=\hat{\theta}_2\mathrm{Sq}^1(u^{n-2}\otimes h)$, and $k^*\mathrm{Sq}^1(1\otimes(h)^2)=\mathrm{Sq}^1(u^{n-2}\otimes h)$. Set

$$\beta' = \beta-\hat{\theta}_2\delta_2(1\otimes(h)^2).$$

Then, $q^*\beta'=q^*\beta=0$, $k^*\beta'=0$, so $\beta'=0$. Thus,

$$\theta_2(x) = \rho(\beta) = \theta\delta_2\rho(1\otimes(h)^2)\in\theta_2H^{2n-3}(M^*; Z);$$

this completes the proof in this case.

Case II, $H_1(M; Z)$ has no 2-torsion. By Poincare duality, $H^{n-1}(M; Z)$ has no 2-torsion and so $\mathrm{Sq}^1H^{n-2}(M)=0$. Thus, in equation (*), $\mathrm{Sq}^1h=0$ and so $\beta=0$. This means that $\theta_2(x)=0$, which completes the proof.

One can deduce other embedding results from (4.2), such as:

Theorem 4.4. *Let M be an n-dimensional, non-orientable manifold, such that $\bar{w}_{n-3+i}=0$, $i\geqslant 0$. If $Sq^2H^{n-3}(M)=H^{n-1}(M)$ and if $w_3=w_1^3$, then M embeds in R^{2n-2}.*

Note that this gives as a special case the result of Handel [10]: if $n=4k+2$, $k\geqslant 2$ then RP^n embeds in R^{2n-2}.*)

5. Embedding real projective space

Before proving Theorem (1.3) we develop some preliminary material. For convenience we write P^n for RP^n, $n\geqslant 1$. In order to use Theorem (1.7), we need some rather detailed information about $H^*(P^{n*})$. We obtain this mainly by studying Λ^* and I^*—see (2.4) and (2.8).

We begin with some notation. In $H^*(P^\infty\times P^n)$, we set

(5.1) $\qquad [d, e] = \sum_{i=0}^{e} u^{d-i}\otimes \mathrm{Sq}^i x^e,$

where d, e are positive integers and x generates $H^1(P^n)$. By an abuse of notation, we use the same symbol to denote the image of $[d, e]$ by ρ_2: thus, in $H^*(P(P^n))$,

$$[d, e] = \sum_{i=0}^{e} v^{d-i}\cdot \mathrm{Sq}^i x^e.$$

Note that by (2.5) (iv),

(5.2) $\qquad k^*(u^d\otimes(x^e)^2) = [d+e, e].$

Also, from (2.8), (2.9) we have

(5.3) $\qquad$ *In $H^*(P(P^n))$, Λ^* is spanned by the classes $[d, e]$, where $e\leqslant d<n$.*

In §8 we prove:

Proposition 5.4. *In $H^*(P^\infty\times P^n)$ and $H^*(P(P^n))$,*

$$\mathrm{Sq}^1[d, e] = d[d+1, e],$$

$$\mathrm{Sq}^2[d, e] = \binom{d}{2}[d+2, e]+e[d+1, e+1].$$

---

*) Remark (added in proof). These results overlap some with recent work of D. Bausam (Trans. A.M.S. 213 (1975), 263–303).

Similarly, in $I^*\subset H^*(\Gamma M)$, we write $\sigma(d, e)$ for $\sigma(x^d\otimes x^e)$. By the Cartan formula we have:

Proposition 5.5. $Sq^1\sigma(d, e)=d\sigma(d+1, e)+e\sigma(d, e+1)$,

$$Sq^2\sigma(d, e)=\binom{d}{2}\sigma(d+2, e)+de\sigma(d+1, e+1)+\binom{e}{2}\sigma(d, e+2).$$

Combining (5.4) and (5.5) we prove in §8:

Proposition 5.6.

$$Sq^2H^{4k-1}(P^{n*})=Sq^2Sq^1H^{4k-2}(P^{n*}).$$

At one point we will need to know something about the integral cohomology of $P(P^n)$. The following result (proved in §8) suffices.

Proposition 5.7. *If n is even and $k\equiv 1$ mod 4, then*

$$H^k(P^{n*}; Z)=\delta_2H^{k-1}(P^{n*}).$$

We now can give the proof of Theorem 1.3. We do this by a series of lemmas that fit together to prove all parts of the Theorem.

Lemma 5.8. *Let q be a power of two, $q\geqslant 8$. Then,*

$$\alpha_1(q+4)\xi\equiv 0 \quad in \quad H^{q+5}(M^*),\ M=P^{q-1}.$$

Proof. Since q is a power of two, $\overline{w}_i(P^{q-1})=0$, $i>0$, and so by 4.1, $v^q=0$ in $H^q(M^*)$, $M=P^{q-1}$. Thus by (3.5) $\chi(q+4)\xi=0$, and so $\alpha_1(q+4)\xi$ is defined. But by (3.7) and (5.6),

$$\alpha_1(q+4)\xi\equiv\Phi_3(\delta_2v^{q+1}, 0)\equiv 0,$$

since $v^{q+1}=0$. This completes the proof.

Now let s be an integer that is not a power of two, as in (1.3), and set $k=s-1$, so that

$$8s+t=8k+8+t, \quad 0\leqslant t\leqslant 7.$$

Let q be the largest power of two such that $q/2<8s$. Then, $q+4\leqslant 16k+4$, and so using the embedding $P^{8k+15}\subset P^{q-1}$, together with (3.8), we have:

Corollary 5.9.

$$\alpha_2(16k+4)\xi\equiv 0 \quad in \quad H^{16k+5}(M^*),\ M=P^{8k+15}.$$

Recall the map $j\colon P(M)\to M^*$, given in diagram (2.1).

Lemma 5.10. *Taking $M=P^{8k+15}$, we have: There is a class a_3 such that*

$(0, a_3)\in(\alpha_1, \alpha_3)(16k+8)\xi$ *and* $j^*a_3=r[8k+10, 8k+1]$, *in* $H^{16k+11}(P(M))$, $r\in Z_2$.

Proof. By (5.9), $\alpha_1(16k+8)\xi\equiv 0$; let $a_3\in H^{16k+11}(M^*)$ be such that $(0, a_3)\in(\alpha_1, \alpha_3)(16k+8)\xi$. To prove (5.10) we show that a_3 can be chosen so that $j^*a_3=r[8k+10, 8k+1]$, $r\in Z_2$.

Note that $w_i(16k+8)\xi=0$, $1\leqslant i\leqslant 7$, and so by (3.2), and (3.3),

$$\mathrm{Sq}^1 a_3 = 0, \quad \mathrm{Sq}^4 a_3 = 0.$$

Using (5.3) and (5.4), we have (since $\mathrm{Sq}^1 j^*a_3=0$), $j^*a_3=\sum_{i=0}^{4} c_i[8k+6+2i, 8k+5-2i]$, where $c_i\in Z_2$. Now $\mathrm{Sq}^4(j^*a_3)=0$, and so the proof of (5.10) is complete when we show:

A) $\mathrm{Sq}^4[8k+10, 8k+1]=0$

B) Sq^4 is injective on the subspace spanned by $[8k+14, 8k-3]$, $[8k+12, 8k-1]$, $[8k+8, 8k+3]$, $[8k+6, 8k+5]$.

We will use one more piece of notation: we set $(i, j)=v^i\cdot x^j$ in $H^{i+j}(P(P^n))$. Thus,

$$\begin{aligned}[8k+10, 8k+1] = {} & (8k+10, 8k+1)+(8k+9, 8k+2)+ \\ & k(8k+2, 8k+9)+k(8k+1, 8k+10),\end{aligned}$$

and so $\mathrm{Sq}^4[8k+10, 8k+1]=0$, as claimed.

To prove (B), note that

$$\begin{aligned}
\mathrm{Sq}^4[8k+14, 8k-3] &= [8k+14, 8k+1]+\cdots \\
\mathrm{Sq}^4[8k+12, 8k-1] &= [8k+12, 8k+3]+\cdots \\
\mathrm{Sq}^4[8k+8, 8k+3] &= [8k+11, 8k+4]+\cdots \\
\mathrm{Sq}^4[8k+6, 8k+5] &= [8k+10, 8k+5]+\cdots,
\end{aligned}$$

where in each case the terms omitted have a left-hand coordinate smaller than that of the term shown. Thus Sq^4 is injective as claimed, which completes the proof of (5.10).

REMARK. In doing calculations such as above, we continually use the fact that, by (2.6),

$$(5.11) \qquad (n, 0) = \sum_{i=1}^{n}\binom{n+1}{i}(n-i, i), \quad \text{in} \quad P(P^n).$$

Also, note that $(i,0)\cdot[d, e]=[d+i, e]$.

Lemma (5.10) will suffice to calculate the obstructions (α_1, α_3) in all the cases of Theorem (1.3).

We now jump ahead to compute the obstruction γ_3.

Lemma 5.12. *For $n\geqslant 15$, if $\gamma_3(2n-6)\xi$ is defined on P^{n*}, then $\gamma_3(2n-6)\xi\equiv 0$.*

This follows at once from (3.1), using the following fact, which we prove in §8.

(5.13) $\quad H^{2n-3}(P^{n*}) = \theta_2 H^{2n-5}(P^{n*})+\mathrm{Sq}^1 H^{2n-4}(P^{n*}),\quad$ where $\quad \theta_2=\theta_2(2n-6)\xi$.

We now come to the proof of Theorem (1.3): we divide the proof into three cases. As before, set $n=8s+t=8k+8+t$, $0\leqslant t\leqslant 7$, s not a power of two.

Case I. $n\equiv 3, 4, 5 \bmod 8$.

Let $q=8k+15$, we do all our calculation on P^{q*}.

(5.14) $\quad$ On P^{q*}, $(\alpha_1, \alpha_3)(16k+16)\xi\equiv 0$.

By (5.10) and the Whitney formula, (3.8), there is a class $a_3\in H^{16k+17}(P^{q*})$ such that $(0, a_3)\in(\alpha_1, \alpha_3)(16k+14)\xi$ and $j^*(0, a_3)=(0, r[8k+16,8k+1])$. But by (5.11),

$$[8k+16, 8k+1] = (8k+16, 8k+1)+(8k+15, 8k+2)+$$
$$k(8k+8, 8k+9)+k(8k+7, 8k+10)=0,$$

and so $j^*a_3=0$. Thus, by (3.8) and (2.10), $(\alpha_1, \alpha_3)(16k+16)\xi\equiv 0$, as desired.

We now show

(5.15) $\quad (\beta_2, \beta_3)(16k+16)\xi\equiv 0,\quad$ on $\quad P^{q*}$, $q=8k+15$.

Note first that

(C) On $P(P^q)$, $q=8k+15$,

$$\Lambda^{16k+19} = \mathrm{Sq}^1\Lambda^{16k+18}+\mathrm{Sq}^2\mathrm{Sq}^1\Lambda^{16k+16}.$$

This is a simple calculation using (5.4) and (5.3)-e.g., $[8k+14, 8k+5]=\mathrm{Sq}^1[8k+13, 8k+5]$, $[8k+13, 8k+6]=\mathrm{Sq}^2\mathrm{Sq}^1[8k+11, 8k+5]$. Thus, by (3.1), $j^*\beta_3(16k+16)\xi\equiv 0$, on $P(P^q)$. Choose classes $(b_2, b_3)\in(\beta_2, \beta_3)(16k+16)\xi$ such that $j^*b_3=0$. Since $\theta_2(16k+16)\xi=\mathrm{Sq}^2$, we have by (3.2), $j^*\mathrm{Sq}^2 b_2=0$. By (5.3), $j^*b_2=\sum_{i=0}^5 c_i[8k+9+i, 8k+9-i]$, $c_i\in Z_2$. Using (5.4), one finds that Kernel Sq^2 on Λ^{16k+18} is generated by $[8k+12, 8k+6]$. Since,

$$\mathrm{Sq}^2[8k+10, 8k+6] = [8k+12, 8k+6],$$
$$\mathrm{Sq}^2\mathrm{Sq}^1[8k+10, 8k+6]=0,$$

this means that one can alter b_2 to a class b_2' (without changing b_3), so that $(b_2', b_3)\in(\beta_2, \beta_3)(16k+16)\xi$ and $j^*b_2'=j^*b_3=0$. Hence, by (2.9) there are classes $(c_2, c_3)\in I^*$ with $\rho(c_2)=b_2'$, $\rho(c_3)=b_3$. Using (5.5) one easily shows:

(5.16) On ΓP^q, $I^{16k+19} = \mathrm{Sq}^1 I^{16k+18} + \mathrm{Sq}^2\mathrm{Sq}^1 I^{16k+16}$.

This shows that $\rho(c_3) \in \mathrm{Indet}\ \beta_3$, and so we may choose c_3 to be zero-i.e., $\rho(c_2, 0) \in (\beta_2, \beta_3)(16k+16)\xi$. By (3.2), $\rho(\mathrm{Sq}^2 c_2)=0$, and so by (2.9), $\mathrm{Sq}^2 c_2=0$. Using (5.5) one finds that on I^{16k+18}, Kernel Sq^2 is generated by $\sigma(8k+14, 8k+4)$ and $\sigma(8k+12, 8k+6)+\sigma(8k+10, 8k+8)$. Since $\mathrm{Sq}^2\sigma(8k+14, 8k+2)= \sigma(8k+14, 8k+4)$ and $\mathrm{Sq}^2\sigma(8k+10, 8k+6)=\sigma(8k+12, 8k+6)+\sigma(8k+10, 8k+8)$, we see that $(b_2, b_3) \in \mathrm{Indet}\ (\beta_2, \beta_3)$, and so $(\beta_2, \beta_3)(16k+16)\xi \equiv 0$, as claimed.

Combining (5.14), (5.15) and (5.12), we see (cf. §2) that on $(P^{8k+11})^*$ the sphere bundle associated to $(16k+16)\xi$ has a section and hence, by (1.7), P^{8k+11} embeds in R^{16k+16}. Similarly, using (3.8), we see that P^{8k+12} embeds in R^{6k+18} and P^{8k+13} in R^{16k+20}, thus proving Theorem (1.3) in Case I.

Case II. $n \equiv 0 \bmod 8$.

We first prove:

(5.17) $(\alpha_1, \alpha_3)(16k+8)\xi \equiv 0$, on P^{8k+8*}.

By (5.10) there is a class a_3 such that $(0, a_3) \in (\alpha_1, \alpha_3)(16k+8)\xi$ (on P^{8k+8*}) and $j^*a_3 = r[8k+10, 8k+1]$, $r \in Z_2$. But by (5.11),

$$[8k+10, 8k+1] = (8k+10, 8k+1)+(8k+9, 8k+2) = 0,$$

which shows that $j^*a_3=0$.

We now use the fact [15], [25] that P^{8k+7} embeds in R^{16k+8}; thus, if $i\colon P^{8k+7*} \to P^{8k+8*}$ is induced by the inclusion $P^{8k+7} \subset P^{8k+8}$, we have (by (1.7)), $i^*(\alpha_1, \alpha_3)(16k+8)\xi \equiv 0$ and hence, by (3.1), there is a class $y \in H^{16k+7}(P^{8k+7*}; Z)$ such that

$$(0, i^*a_3) \in (\mathrm{Sq}^2, \mathrm{Sq}^4)(y).$$

Let z be a (mod 2) class in $B^* \oplus I^*$ such that $\rho(z)=y$ mod 2, and let

$$z = b+e,\ b \in B^*,\ e \in I^*.$$

We have

$$b = s(u^5 \otimes (x^{8k+1})^2)+t(u^3 \otimes (x^{8k+2})^2)+q(u \otimes (x^{8k+3})^2),$$

and so $k^*(b)=\sum_{i=1}^{6} r_i(8k+7-i, 8k+i)$, $r_i \in Z_2$. Consequently,

$$\begin{aligned} k^*(\mathrm{Sq}^4 b) &= s_1(8k+10, 8k+1)+s_2(8k+9, 8k+2)+ \\ &\qquad s_3(8k+8, 8k+3),\ s_i \in Z_2 \\ &= k^*\varphi(s_1(u^3 \otimes x^{8k+1})+s_2(u^2 \otimes x^{8k+2})+s_3(u \otimes x^{8k+3})). \end{aligned}$$

Thus, by (2.5) (ii) and (iv),

$$\mathrm{Sq}^4 b = 0, \text{ mod Image } \varphi.$$

Since I^*=kernel k^*, we have $\mathrm{Sq}^4 I^* \subset I^*$. A simple calculation shows that $\mathrm{Sq}^4 I^{16k+7}=0$, in $H^*(\Gamma P^{8k+7})$. Consequently, $\mathrm{Sq}^4 z=0$, mod image, φ and so

$$i^* a_3 = \mathrm{Sq}^4 y = \rho(\mathrm{Sq}^4 z) = 0.$$

Recall that back on P^{8k+8*}, $j^*a_3=0$. Hence there is a class $d \in I^{16k+7}$ (on ΓP^{8k+8}), with $\rho(d)=a_3$; since i^*d also is in I^*, and since $\rho(i^*d)=i^*a_3=0$, it follows that $i^*d=0$. Thus, $d=r\sigma(8k+8, 8k+3)$, $r \in Z_2$. But by (3.2), since $\alpha_1 \equiv 0$,

$$\rho(\mathrm{Sq}^1 d) = \mathrm{Sq}^1 a_3 = 0,$$

and hence, $\mathrm{Sq}^1 d=0$. Since $\mathrm{Sq}^1\sigma(8k+8, 8k+3)=\sigma(8k+8, 8k+4)\neq 0$, this shows that $r=0$, and so $a_3=0$, completing the proof of (5.17).

By (5.17), $(\beta_2, \beta_3)(16k+8)\xi$ is defined, on P^{8k+8*}. We now show:

(5.18) $\quad j^*(\beta_2, \beta_3)(16k+8)\xi \equiv 0.$

This follows easily from (3.1), (3.2) and (5.3). We leave the details to the reader.

Finally, by (2.10) and (3.8), (5.18) implies that $(\beta_2, \beta_3)(16k+10)\xi \equiv 0$, and so, by (5.12) and (1.7), P^{8k+8} embeds in R^{16k+10}, as desired. This completes the proof for Case II.

Case III. $n \equiv 1, 2 \bmod 8$, $\alpha(n) \geqslant 4$.

We do all the argument on P^{8k+10}; set $m=8k+10$. The firstr result is:

(5.19) $\quad (\alpha_1, \alpha_3)(16k+10)\xi \equiv 0$, on P^{m*}.

As before, by (5.10), there is a class a_3 such that $(0, a_3) \in (\alpha_1, \alpha_3)(16k+8)\xi$ (on P^{m*}) and $j^*a_3=r[8k+10, 8k+1]$ in $H^{16k+11}(P(P^m))$, $r \in Z_2$. Thus by (3.8), $j^*(\alpha_1, \alpha_3)(16k+10)\xi \equiv (0, r[8k+12, 8k+1])$. Let $l\colon P^{m-1*} \to P^{m*}$, $\hat{l}\colon P(P^{m-1}) \to P(P^m)$ denote the maps induced by the inclusion $P^{m-1} \subset P^m$. We now use the fact that P^{8k+9} immerses in R^{16k+10}, see [23]. (It is at this point that we require $\alpha(n) \geqslant 4$.) Thus the bundle $\hat{l}^* i^*(16k+10)\xi$ has a (nowhere zero) section on $P(P^{m-1})$, by Haefliger-Hirsch [8]. Consequently, by (3.1),

(*) $\quad (0, r[8k+12, 8k+1]) \in (\theta_2, \mathrm{Sq}^4)H^{16k+9}(P(P^{8k+9}); Z).$

Using (5.11) one sees that $[8k+12, 8k+1]=[8k+8, 8k+5]$ on $P(P^{8k+9})$. Moreover, by (5.4), θ_2 is an injection on Kernel $\mathrm{Sq}^1 \cap H^{16k+9}(P(P^{8k+9}))$, $\theta_2=\theta_2(16k+10)\xi$. Since $[8k+12, 8k+1]\neq 0$, it follows from (*) that $r=0$. Thus, $j^*a_3=0$ and so $j^*(\alpha_1, \alpha_3)(16k+8)\xi \equiv 0$; consequently, by (3.8) and (2.10), $(\alpha_1, \alpha_3)(16k+10)\xi \equiv 0$, on P^{m*}, which proves (5.19).

The obstruction $(\beta_2, \beta_3)(16k+10)\xi$ is consequently defined, and we now show:

(5.20) $j^*(\beta_2, \beta_3)(16k+10)\xi \equiv 0$, on P^{m*}, $m=8k+10$.

The first step is to show:

A) We may choose classes $(b_2, b_3) \in (\beta_2, \beta_3)(16k+10)\xi$ so that

$$j^*(b_2, b_3) = (r[8k+6, 8k+6], 0), \ r \in Z_2.$$

Note that

$$\begin{aligned} j^*b_2 &= r[8k+6, 8k+6]+s[8k+7, 8k+5]+ \\ &\qquad t[8k+8, 8k+4]+q[8k+9, 8k+3], \\ j^*b_3 &= c[8k+7, 8k+6]+d[8k+8, 8k+5]+e[8k+9, 8k+4], \end{aligned}$$

where the coefficients all lie in Z_2. Since

$$\begin{aligned} &\mathrm{Sq}^1[8k+7, 8k+5] = [8k+8, 8k+5], \\ &\mathrm{Sq}^2\mathrm{Sq}^1[8k+7, 8k+3] = [8k+9, 8k+4], \quad \text{and} \\ &\mathrm{Sq}^2\mathrm{Sq}^1[8k+5, 8k+5] = [8k+8, 8k+5]+[8k+7, 8k+6], \end{aligned}$$

we see that b_3 can be chosen so that $j^*b_3=0$. Thus, by (3.2), $j^*(\theta_2 b_2)=0$, where $\theta_2=\theta_2(16k+10)\xi$. Using (5.4) and (5.11) one finds that this implies: $s=0$, $t=q$. But $\theta_2[8k+8, 8k+2]=[8k+9, 8k+3]+[8k+8, 8k+4]+[8k+6, 8k+6]$. Hence, b_2 can be altered (without changing b_3) so that $j^*b_2=r[8k+6, 8k+6]$, as claimed.

To complete the proof of (5.20), we use the map i: $P^{m-2*} \to P^{m*}$. In Case II we proved that $i^*(\beta_2, \beta_3)(16k+10)\xi \equiv 0$ on P^{m-2*}, and so $i^*(b_2, b_3) \in \Psi_\omega H^{16k+9}(P^{m-2*}; Z)$, (see discussion following (3.1)), where $\omega=(16k+10)\xi$. Now by (5.7), a class in $H^{16k+9}(P^{m-2*}; Z)$ is determined by its mod 2 reduction. Suppose then that y is in domain Ψ_ω, and let $\bar{y}=y$ mod 2. Then (see §3), $\mathrm{Sq}^1\bar{y}=0$, $\theta_2\bar{y}=0$, $\mathrm{Sq}^4\bar{y}=0$. But a calculation shows that

$$H^{16k+9}(P^{m-2*}) \cap \text{Kernel } \mathrm{Sq}^1 \cap \text{Ker } \theta_2 \cap \text{Ker } \mathrm{Sq}^4 = 0,$$

and so $\bar{y}$ mod $2=0$. Thus, $y=0$, and so by §3,

$$i^*(b_2, b_3) \in \text{indet } \Psi_\omega = (\theta_2, \mathrm{Sq}^1\mathrm{Sq}^1)H^{16k+10}(P^{m-2*}) + \mathrm{Sq}^1 H^{6k+12}(P^{m-2*}).$$

Also, by what we have already proved, $j^*i^*(b_2, b_3)=(r[8k+6, 8k+6], 0)$, in $P(P^{m-2})$. Thus, there is a class $y \in H^{16k+10}(P^{m-2*})$ with $\theta_2(j^*y)=r[8k+6, 8k+6]$. A simple calculation using (5.4) shows that this is possible only if $j^*y=0$ and $r=0$. Hence, back on P^{m*}, $j^*(\beta_2, \beta_3)(16k+10)\xi \equiv 0$, as claimed.

Therefore, by (3.8) and (5.11), $(\beta_2, \beta_3)(16k+12)\xi \equiv 0$ on P^{m*}, and hence on P^{m-1*}. Thus by (5.12) and (1.7), P^n embeds in R^{2n-6}, for $n=8k+9$ and $8k+10$, completing the proof of Theorem (1.3).

6. Embedding complex projective space

Our goal is to show that if $n=4s+3$, s not a power of two, then CP^n embeds in R^{4n-6}. We do this by showing that the sphere bundle over CP^{n*}, associated to $(4n-6)\xi$, has a section. Since the methods here are very similar to those used in §5, we only sketch the proof.

We use the following notation: $y \in H^2(CP^n)$ denotes the generator, and in $H^*(P(CP^n))$ we set

$$[d, 2j] = \sum_{i=0}^{j} v^{d-2i} \cdot \mathrm{Sq}^{2i}(y^j).$$

As before, $\Lambda^* \subset H^*(P(CP^n))$ denotes $j^*H^*(CP^{n*})$, and as in (5.3) we have:

(6.1) The classes $[d,2i]$ generate Λ^*, where $2i \leqslant d \leqslant 2n-1$.

Finally, we set $s=k+1$, so that $n=4k+7$. The first step in the proof of (1.5) is to show:

(6.2) the obstruction $(\alpha_1, \alpha_3)(16k+12)\xi$ is defined and there are classes $(a_1, a_3) \in (\alpha_1, \alpha_3)(16k+12)\xi$ such that

$$j^*a_1 = r[8k+13, 8k]+s[8k+9, 8k+4]$$
$$j^*a_3 = s[8k+11, 8k+4],\ r, s \in Z.$$

This is proved using (3.1) and (3.2). Now $w_4(16k+12)\xi \neq 0$, while $w_i(16k+12)\xi=0$, for $1 \leqslant i \leqslant 7$, $i \neq 4$. Thus, one has a relation analogous to (3.3):

$$\mathrm{Sq}^6 a_1 + \theta_4 a_3 = 0.$$

Using this on (6.2) one finds that $s=0$ (in (6.2)). But by a formula analogus to (5.11), $[8k+15, 8k]=0$, and hence $j^*(\alpha_1, \alpha_3)(16k+14)\xi \equiv 0$, using (3.8). Therefore, by (2.10) and (3.8), $(\alpha_1, \alpha_3)(16k+16)\xi \equiv 0$.

The next step is to show:

(6.3) $(\beta_2, \beta_3)(16k+22)\xi \equiv 0$.

Starting with classes $(b_2, b_3) \in (\beta_2, \beta_3)(16k+18)\xi$, one finds that by using the indeterminancy of β_2(i.e., θ_2), b_2 can be chosen so that $j^*b_2=r[8k+12, 8k+8]$. And by (3.2), one has $j^*b_3=s[8k+13, 8k+8]$. But

$$[8k+14, 8k+8] = [8k+15, 8k+8] = 0,$$

and so $j^*(\beta_2, \beta_3)(16k+20)\xi \equiv 0$. Consequently, by (2.10) and (3.8), (β_2, β_3)

$(16k+22)\xi\equiv 0$, as desired.

By an indeterminancy argument (use θ_2) one shows that $j^*\gamma_3(16k+22)\xi\equiv 0$. But $I^{16k+25}=0$, and so by 2.9, $\gamma_3(16k+22)\xi\equiv 0$, which means by (1.7) that CP^n embeds in R^{2n-6}.

7. The cohomology of M^*

This section contains the proofs that were omitted in sections 2 and 4. We begin with the proof of Proposition (2.9.)

(a) Kernel $j^* = \rho(I^*)$.

This follows at once from the exactness of (2.1), given that kernel $k^*=I^*$ (see (2.5)).

(b) $\rho|I^*$ is injective.

Set $D^*=H^*(P^\infty)\otimes\bar{K}^*$, see (2.4). Note that $D^*\cap I^*=0$ and that $\varphi(u^i\otimes x)\in D^*$, if $i>0$ and $x\in H^*(M)$. Suppose that $e\in I^*$ with $\rho(e)=0$. Then, by (2.1) and the above remarks, $e=\varphi(1\otimes y)$, for some $y\in H^*(M)$. By (2.5) and (2.1), since $e\in I^*$,

$$0 = k^*(e) = k^*\varphi(1\otimes y) = \varphi_2(1\otimes y).$$

But φ_2 is injective, and so $y=0$, which proves $e=0$, as claimed.

(c) Image $j^* = \lambda(B^*) = \Lambda^*$.

Note that by (2.3), Image $\varphi_2=u^n\otimes H^*(M)\oplus u^{n+1}\otimes H^*(M)\oplus\cdots$, where $n=\dim M$. Thus, if we set $C=\sum_{i=0}^{n-1} u^i\otimes H^*(M)$, we have that ρ_2 maps C isomorphically onto $H^*(P(M))$. Set $\tilde{C}=C\cap\rho_2{}^{-1}(\text{Image } j^*)$. Note that $k^*(B^*)\subset C$ and hence $k^*(B^*)\subset\tilde{C}$, we show:

$$(*) \qquad k*(B^*) = \tilde{C},$$

which proves (c). Moreover, by (*), λ maps B^* isomorphically onto Image j^* and hence $\rho|B^*$ is an inverse to j^*, which proves (d), in (2.9).

To prove (*) all we need show is that k^* maps B^* onto $\tilde{C}$. This is a consequence of the following:

Proposition 7.1. *Given $y\in H^*(\Gamma M)$, there is a class $b\in B^*$ such that $\lambda(b)=\lambda(y)$.*

Before proving this we develop some preliminary material. Given a class y in $H^*(\Gamma M)$ we associate with it a unique class in $H^*(P^\infty)\otimes\bar{K}$, called the leading term of y. Suppose that degree $y=d$, and set $s=[d/2]$. Then we can write $y=\sum_0^s u^{d-2i}\otimes(x_i)^2+l$, where $l\in I$ and where degree $x_i=i$, $0\leqslant i\leqslant s$. Let j be the integer such that $x_j \neq 0$ and $x_i = 0$ for $i < j$. We define *leading term* $y = u^{d-2j} \otimes (x_j)^2$. If $y = l$, we set *leading term* $y = 0$.

We will need the following key fact.

(7.2) *Let $x \in \bar{H}^q(M)$, $x \neq 0$, and let j be a non-negative integer. Then, leading term* $\phi(u^j \otimes x) = u^{n-q+j} \otimes (x)^2$.

Proof. Write $d = n+q+j$, and set $s = [d/2]$. There are classes $l \in I$ and $y_i \in \bar{K}^{2i}$ such that

$$\phi(u^j \otimes x) = \sum_0^s u^{d-2i} \otimes y_i + l.$$

Also, by 2.3,

$$\phi_2(u^j \otimes x) = \sum_0^n u^{n+j-i} \otimes w_i M \cdot x = \sum_0^n u^{d-q-i} \otimes w_i M \cdot x.$$

Thus the term in $\phi_2(u^j \otimes x)$ with highest power of u is $u^{d-q} \otimes x$. But $\phi_2 = k^* \phi$, and so $y_i = 0$ for $i < q$ and

$$k^*(u^{d-2q} \otimes y_q) = u^{d-q} \otimes x + \text{terms with lower degree in } u.$$

Using 2.5 (iv), and recalling that $\mathrm{Sq}^0(x) = x$, we have $y_q = (x)^2$ (mod I), which implies that

$$\begin{aligned} \text{leading term } \phi(u^j \otimes x) &= u^{d-2q} \otimes (x)^2 \\ &= u^{n-q+j} \otimes (x)^2, \end{aligned}$$

as claimed. This completes the proof of (7.2).

Proof of 7.1. Let $y \in H^*(\Gamma M)$. Since $\lambda(I) = 0$, we may suppose that $y \in H^*(P^\infty) \otimes \bar{K}$. Let leading term $y = u^k \otimes (x)^2$, where $k \geqslant 0$ and degree $x = q$, say. If $k+q < n$, then $y \in B^*$ and there is nothing to prove, so suppose that $k+q \geqslant n$. Let $y_1 = \phi(u^{k+q-n} \otimes x)$. Then, by (2.1)

$$\lambda(y) = \lambda(y - y_1).$$

But by (7.2), y and y_1 have the same leading term, and so

$$\text{leading term } (y - y_1) = u^{k_1} \otimes (x^{(1)})^2,$$

where $k_1 \leqslant k-2$ and $k_1 + 2 \deg x^{(1)} = \deg y$. Thus,

$$k_1 + \deg x^{(1)} < k + \deg x.$$

Continuing in this way we obtain classes $y_1, y_2, \cdots, y_r$, say, such that

$$\lambda(y) = \lambda(y - (y_1 + \cdots + y_r)), \quad \text{and} \quad y - (y_1 + \cdots + y_r) \in B^*,$$

thus completing the proof of (7.1) and hence of (2.9).

8. The cohomology of P^{n*}

This section contains the proofs that were omitted in §5. We begin with the following useful fact.

Lemma 8.1. *Let $d\in H^q(\Gamma P^n)$, $q>0$, and let $k^*(d)=\sum_{i=0}^q a_i(u^{q-i}\otimes x^i)$, where $a_i\in Z_2$. If $a_i=0$ for $2i\leqslant q$, then $k^*(d)=0$.*

This is an immediate consequence of (5.2). Using this we have:

Proof of 5.4. We do the proof in $H^*(P^\infty\times P^n)$, giving the details only for Sq^2. Suppose then that d and e are positive integers with $d\geqslant e$. Note that

$$\mathrm{Sq}^2[d+4,\ e] = \mathrm{Sq}^2(u^4\cdot[d,\ e]) = u^4\cdot\mathrm{Sq}^2[d,\ e],$$

and so to prove (5.4) we may assume $e\leqslant d\leqslant e+3$, since $\binom{d}{2}\equiv\binom{d+4}{2}$ mod 2. Now for $j\geqslant 0$, $[e+j,\ e]\in$ Image k^*, by (5.2). Also, if $d\leqslant e+3$, we find that

$$\mathrm{Sq}^2[d,\ e]+\binom{d}{2}[d+2,\ e]+e[d+1,\ e+1] = \sum a_i(u^i\otimes x^j),$$

where the sum is over all, $i+j=d+e+2$ and where $a_i=0$ for $i\geqslant i$. Thus by (8.1), $\mathrm{Sq}^2[d,\ e]+\binom{d}{2}[d+2,\ e]+e[d+1,\ e+1]=0$ as claimed.

The proof for Sq^1 is similar, using the fact that

$$\mathrm{Sq}^1[d+2,\ e] = \mathrm{Sq}^1(u^2\cdot[d,\ e]) = u^2\cdot\mathrm{Sq}^1[d,\ e].$$

Hence, we need only take $d=e$, $e+1$.

REMARK. A proof for Sq^1 is given in [40], and [2, section 7]; note also [13].

Proof of 5.6. Since $H^*(P^{n*})=\rho H^*(\Gamma P^n)$, and since $H^*(\Gamma P^n)$ is determined by k^* and q^*, (5.6) will follow when we show:

(8.2) (i) $\mathrm{Sq}^2k^*H^{4k-1}(\Gamma P^n) = \mathrm{Sq}^2\mathrm{Sq}^1k^*H^{4k-2}(\Gamma P^n)$

(ii) $\mathrm{Sq}^2q^*I^{4k-1} = \mathrm{Sq}^2\mathrm{Sq}^1q^*I^{4k-2}$.

Now (i) follows at once from (5.4), while (ii) may be proved by an inductive argument using (5.5). We omit the details.

To prove (5.6), let $y\in H^{4k-1}(\Gamma P^n)$. By (8.2) (i), we may choose $d\in H^{4k-2}(\Gamma P^n)$ so that $k^*(\mathrm{Sq}^2\mathrm{Sq}^1d)=k^*(\mathrm{Sq}^2y)$. Set $\bar{y}=y-\mathrm{Sq}^1d$. By (8.2)(ii), there is a class $e\in I^*$ such that $q^*(\mathrm{Sq}^2\mathrm{Sq}^1e)=q^*\mathrm{Sq}^2\bar{y}$. Since $\mathrm{Sq}^2\mathrm{Sq}^1I^*\subset I^*$ and $k^*I^*=0$, we see that $k^*\mathrm{Sq}^2\mathrm{Sq}^1(d+e)=k^*\mathrm{Sq}^2y$, $q^*\mathrm{Sq}^2\mathrm{Sq}^1(d+e)=q^*\mathrm{Sq}^2y$, and hence $\mathrm{Sq}^2y=\mathrm{Sq}^2\mathrm{Sq}^1(d+e)$, completing the proof of (5.6).

We will need the following well-known fact in the proof of (5.7).

Lemma 8.3. *Let X be a space and k a positive integer such that $H^k(X; Z)$ is finitely generated and has no odd torsion. Then, $H^k(X; Z)=\delta_2 H^{k-1}(X; Z_2)$ if, and only if,*

$$\textit{Kernel}\ \mathrm{Sq}^1 = \textit{Image}\ \mathrm{Sq}^1 \quad \textit{on} \quad H^k(X; Z_2).$$

Proof of 5.7. Note that $\mathrm{Sq}^1 I^* \subset I^*$, and since k is odd, $\mathrm{Sq}^1 B^k \subset B^{k+1}$. Thus by (8.3), (5.7) is proved when we show:

$$\mathrm{Sq}^1 B^{k-1} = \ker \mathrm{Sq}^1 \cap B^k, \quad \mathrm{Sq}^1 I^{k-1} = \mathrm{Ker}\ \mathrm{Sq}^1 \cap I^k.$$

Since $\lambda: B^* \approx \Lambda^*$ (see 2.9), we do the argument for B^* in Λ^*. Define $V \subset \Lambda^*$ to be the subspace spanned by generators $[d, e]$, with $e \leqslant d \leqslant n-2$. Since n is even (in 5.7), $\mathrm{Sq}^1 V \subset V$, by (5.4). Let k (in 5.7) be written, $k=4s+1$. We assume $k>n$, since this is the only case of interest to us. Then,

$$\begin{aligned}
&\Lambda^k = \{[n-1, k-n+1]\} \oplus V. \quad \text{But} \\
&\mathrm{Sq}^1[n-1, k-n+1] = [n, k-n+1] = \\
&[n-1, k-n+2]+v, \quad \text{where} \quad v \in V.
\end{aligned}$$

(We use here 5.11 and the fact that $k-n+1$ is even.) Thus $\mathrm{Sq}^1[n-1,k-n+1] \notin V$ and so $\mathrm{Ker}\ \mathrm{Sq}^1 \cap \Lambda^k = \mathrm{Ker}\mathrm{Sq}^1 \cap V$. An easy calculation shows that $\mathrm{Ker}\ \mathrm{Sq}^1 \cap V \subset \mathrm{Sq}^1 \Lambda^{k-1}$. Finally, since

$$k^* \mathrm{Sq}^1(1 \otimes (x^r)^2) = q^* \mathrm{Sq}^1(1 \otimes (x^r)^2) = 0,$$

where $r=(k-1)/2$, we see that $\mathrm{Sq}^1 B^{k-1} \subset B^k$ and hence $\mathrm{Ker}\ \mathrm{Sq}^1 \cap B^k = \mathrm{Sq}^1 B^{k-1}$, as claimed. Similarly, one shows that $\mathrm{Sq}^1 I^{k-1} = \mathrm{Ker}\ \mathrm{Sq}^1 \cap I^k$, thus proving (5.7).

Proof of (5.13). For this it suffices to show:

$$\begin{aligned}
&\mathrm{Sq}^2 I^{2n-5} + \mathrm{Sq}^1 I^{2n-4} = I^{2n-3}, \\
&\theta_2 \Lambda^{2n-5} + \mathrm{Sq}^1 \Lambda^{2n-4} = \Lambda^{2n-3},
\end{aligned}$$

recalling that $\theta_2 = \mathrm{Sq}^2$ on I^*. Now the first equation follows by a straightforward calculation (consider the cases, n odd and n even); for the second equation, note that Λ^{2n-3} is generated by $[n-1, n-2]$. But if n is odd, then $\mathrm{Sq}^1[n-2, n-2]=[n-1, n-2]$, while if n is even, one shows that $\theta_2[n-2, n-3] = [n-1, n-2]$. This completes the proof of (5.13).

UNIVERSITY OF CALIFORNIA, BERKELEY

References

[1] J. Adem and S. Gitler: *Non-immersion theorems for real projective spaces*, Bol. Soc. Mat. Mexicana **9** (1964), 37–50.

[2] D. Bausum: *Embeddings and immersions of manifolds in Euclidean space*, thesis, Yale Univ., 1974.

[3] A. Copeland and M. Mahowald: *The odd primary obstructions...*, Proc. Amer. Math. Soc. **19** (1968), 1270–1272.

[4] S. Gitler: *The projective Stiefel manifolds*, II, Topology **7** (1968), 47–53.

[5] A. Haefliger: *Differentiable embeddings*, Bull. Amer. Math. Soc. **67** (1961), 109–112.

[6] A. Haefliger: *Plongements differentiable dans le domaine stable*, Comment. Math. Helv. **37** (1962), 155–176.

[7] A. Haefliger: *Points multiples d'une application et produit cyclique reduit*, Amer. J. Math. **83** (1961), 57–70.

[8] A. Haefliger and M. Hirsch: *Immersions in the stable range*, Ann. of Math. **74** (1962), 231–241.

[9] A. Haefliger and M. Hirsch: *On the existence and classification on differentiable embeddings*, Topology **2** (1963), 125–130.

[10] D. Handel: *An embedding theorem for real projective spaces*, ibid. **7** (1968), 125–130.

[11] D. Husemoller: Fiber Bundles, McGraw-Hill, New York, 1966.

[12] I. James: *Two problems studied by H. Hopf*, Lecture Notes in Mathematics, No. 279, Springer-Verlag, Berlin, 1972, 134–174.

[13] L.L. Larmore: *The cohomology of* $(\Lambda^2X, \Delta X)$, Canad. J. Math. **5** (1973), 908–921.

[14] M. Mahowald: *On obstruction theory in orientable fiber bundles*, Trans. Amer. Math. Soc. **110** (1964), 315–349.

[15] M. Mahowald and J. Milgram: *Embedding real projective spaces*, Ann. of Math. **87** (1968), 411–422.

[16] W. Massey and F. Peterson: *On the dual Stiefel-Whitney classes of a manifold*, Bol. Soc. Mat. Mexicana **8** (1963), 1–13.

[17] J.F. McClendon: *Higher order twisted cohomology operations*, Invent. Math. **7** (1969), 183–214.

[18] J.F. McClendon: *Obstruction theory in fiber spaces*, Math. Z. **120** (1971), 1–17.

[19] J. Milgram: Unstable Homotopy from the Stable Point of View, Lecture Notes in Math., No. 368, Springer-Verlag, Berlin, 1974.

[20] J. Milnor and J. Stasheff: Characteristic Classes, Annals of Math. Studies, No. 76, Princeton Univ., Press, Princeton, 1974.

[21] F. Nussbaum: *Non-orientable obstruction theory*, thesis, Northwestern Univ., 1970.

[22] F. Peterson and N. Stein: *Secondary characteristic classes*, Ann. of Math. **76** (1962), 510–523.

[23] D. Randall: *Some immersion theorems for projective spaces*, Trans. Amer. Math. Soc. **147** (1970), 135–151.

[24] D. Randall: *Some immersion theorems for manifolds*, ibid. **156** (1971), 45–58.

[25] E. Rees: *Embeddings of real projective spaces*, Topology **10** (1971), 309–312.

[26] R. Rigdon: *Immersions and embeddings of manifolds in Euclidean space*, thesis, Univ. of Calif., Berkeley, 1970.

[27] R. Rigdon: *p-equivalences and embeddings of manifolds*, Proc. London Math. Soc. to appear.

[28] R. Rigdon: to appear.

[29] B. Sanderson: *Immersions and embeddings of projective spaces*, Proc. London Math. Soc. **14** (1964), 137–153.

[30] N. Steenrod and D. Epstein: Cohomology Operations, Annals of Math. Studies, No. 50, Princeton Univ, Press, Princeton, 1962.

[31] B. Steer: *On the embeddings of projective spaces in euclidean space*, Proc. London Math. Soc. **21** (1970), 489–501.

[32] E. Thomas: *Postnikov invariants and higher order cohomology operations*, Ann. of Math. **85** (1967), 184–217.

[33] E. Thomas: Seminar on Fiber Spaces, Lecture Notes in Math., No. 13, Springer-Verlag, Berlin, 1966.

[34] E. Thomas: *Whitney-Cartan product formulae*, Math. Z. **118** (1970), 115–138.

[35] H. Toda: Composition Methods in Homotopy Groups of Spheres, Ann. of Math. Studies, No. 49, Princeton Univ. Press, Princeton, 1962.

[36] C. Weber: *Plongements de polyèdres dans le domaine metastable*, Comment. Math. Helv. **42** (1967), 1–27.

[37] H. Whitney: *The self-intersections of a smooth n-manifold in 2n-space*, Ann. of Math. **45** (1944), 220–246.

[38] W. Wu: *Classes caracteristique et i-carres d'une variete*, C.R. Acad. Sci. Paris **230** (1950), 508–511.

[39] G. Yo: *Cohomology operations and duality in a manifold*, Sci. Sinica **12** (1963), 1469–1487.

[40] G. Yo: *Cohomology mod p of deleted cyclic product of a manifold*, ibid. **12** (1963), 1779–1794.

[41] J.A. Lees: *A classification of locally flat imbeddings of topological manifolds* (preprint).