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ON $S\underset{R}{\otimes}S$-MODULE STRUCTURE OF S/R-AZUMAYA ALGEBRAS

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Introduction. Let R be a commutative ring and S a commutative R-algebra. An R-Azumaya algebra A is called an *S/R-Azumaya algebra* if A contains S as a maximal commutative subalgebra and is left S-projective. Kanzaki [10] has determined the structure of S/R-Azumaya algebras by using generalized crossed products when S/R is a separable Galois extension. He then has derived directly the so called seven terms exact sequence due to Chase, Harrison and Rosenberg [4], [5]. And recently Hattori [9] has also derived the seven terms exact sequence by another method. In this paper, we shall generalize the notion of cohomology over Hopf algebras introduced by Sweedler [12] and then investigate $S\underset{R}{\otimes}S$-module structure of S/R-Azumaya algebras when S/R is a Hopf Galois extension.

In §1, we shall define the cohomology of algebras over Hopf algebras. Secondly, in §2 we shall give a criterion for S/R-Azumaya algebras to be $S\underset{R}{\otimes}S$-projective. And we shall characterize smash product algebras in §3. Finally we shall give a criterion for $S\underset{R}{\otimes}S$-projective modules to be S/R-Azumaya algebras. In appendix, we shall give a direct proof of the exactness of the following seven terms sequence for an H-Hopf Galois extension S/R;

$$0 \to H^1(H, S/R, U) \to Pic(R) \to H^0(H, S/R, Pic) \to H^2(H, S/R, U) \to$$
$$Br(S/R) \to H^1(H, S/R, Pic) \to H^3(H, S/R, U)$$

where $Br(S/R)$ denotes the Brauer group of R-Azumaya algebras split by S, U denotes the units functor and Pic denotes the Picard group functor.

Throughout, R is a fixed commutative ring with 1. Algebras mean R-algebras, each $\otimes$, Hom, etc. is taken over R unless otherwise stated. Repeated tensor products of an algebra T are denoted by exponents, $T^q = T\otimes\cdots\otimes T$ with q-factors (T^0 means R).

0. Preliminaries

We shall quote for the sake of convenience some definitions, notations and fundamental facts on Hopf algebras and Hopf Galois extensions. For details the reader will be expected to refer Chase-Sweedler [6] and Sweedler [13].

Let H be a Hopf algebra. We denote its diagonalization by Δ_H (or simply by Δ), its augmentation by ε_H (or by ε) and its antipode by λ_H (or by λ) and for $h\in H$ we use the following notations;

$$\Delta(h)=\sum_{(h)} h_{(1)}\otimes h_{(2)},\ (1\otimes\Delta)\Delta(h)=(\Delta\otimes 1)\Delta(h)=\sum_{(h)} h_{(1)}\otimes h_{(2)}\otimes h_{(3)},\ \text{etc.},\ \lambda(h)=h^{-1}.$$

Then $h=\sum_{(h)}\varepsilon(h_{(1)})h_{(2)}=\sum_{(h)}\varepsilon(h_{(2)})h_{(1)},\ \varepsilon(h)=\sum_{(h)}\varepsilon(h_{(1)})\varepsilon(h_{(2)})=\sum_{(h)}h_{(1)}h_{(2)}^{-1}$ $=\sum_{(h)}h_{(1)}^{-1}h_{(2)}$.

A Hopf algebra H is called to be *finite cocommutative* if H is a finitely generated projective R-module and the diagonalization is commutative, i.e., $\sum_{(h)}h_{(1)}\otimes h_{(2)}=\sum_{(h)}h_{(2)}\otimes h_{(1)}$. In this paper, H denotes a finite cocommutative Hopf algebra.

Let A be an algebra, then Hom(H, A) has a natural algebra structure (its multiplication is denoted by $*$) defined by $(f*g)(h)=\sum_{(h)}f(h_{(1)})g(h_{(2)})$, $1_{\mathrm{Hom}(H,A)}(h)=\varepsilon(h)1_A$, where $f, g\in\mathrm{Hom}(H, A)$, $h\in H$. We call this algebra a convolution algebra of H and A.
Furthermore if $A=R$, then Hom$(H, R)=H^*$ has also a Hopf algebra structure defined by $\Delta_{H^*}(f)(g\otimes h)=f(gh)$, $\varepsilon_{H^*}(f)=f(1_H)$, $f\in H^*$, $g, h\in H$.

Let S be an R-algebra with the left H-module structure map $\psi\colon H\otimes S\to S$, then we call S an H-module algebra if ψ satisfies the following conditions;

(i) $\psi(h\otimes xy)=\sum_{(h)}\psi(h_{(1)}\otimes x)\psi(h_{(2)}\otimes y)$

(ii) $\psi(h\otimes 1)=\varepsilon(h)1_S, \quad h\in H,\ x, y\in S$.

We call ψ a measuring and write $h\cdot x$ for $\psi(h\otimes x)$.
Further, we assume S is commutative and define the (trivial) smash product algebra $S\# H$ of S and H as follows; As an R-module, $S\# H=S\otimes H$, except that we write $s\# h$ rather than $s\otimes h$, for $s\in S$, $h\in H$. Multiplication in $S\# H$ is defined by the formula

$$(x\# g)(y\# h)=\sum_{(g)}xg_{(1)}\cdot y\# g_{(2)}h, \qquad x, y\in S,\ g, h\in H.$$

$S\# H$ is an algebra with unit $1\# 1$, S and H become subalgebras of $S\# H$ via the canonical imbeddings.
Now, we regard S as a left $S\# H$-module by setting

$$(s\# h)x=sh\cdot x, \qquad s, x\in S,\ h\in H.$$

So, we have an R-algebra homomorphism $S\# H\to\mathrm{Hom}(S, S)$.

DEFINITION (cf. Chase-Sweedler [6] 9.3). *Let S be a commutative H-module algebra, which is finitely generated faithful projective as an R-module, then we call the extension S/R is an H-Hopf Galois extension if the homomorphism $S\#H \to Hom(S, S)$ is an isomorphism.*

REMARK. If S/R is an H-Hopf Galois extension in our sense, it is an H^*-Hopf Galois extension in Chase-Sweelder's sense, and conversely. So, we have an isomorphism $S\otimes S \cong S\otimes H^*$. We adopt this definition for the sake of cohomological descriptions.

The following lemma will be useful.

Lemma 0.1 (Chase-Sweedler [6] 9.8). *Let S/R be an H-Hopf Galois extension and T be a commutative R-algebra. Then $T\otimes S$ is a $T\otimes H$-Hopf Galois extension of T.*

1. Cohomology and smash product algebras

Let S be a commutative H-module algebra, then we have commutative algebras Hom (H^q, S), $q=0, 1, \cdots$, and homomorphisms d_i: Hom $(H^q, S) \to$ Hom (H^{q+1}, S), $i=0, 1, \cdots, q+1$, given by $d_0(f)(h_1\otimes\cdots\otimes h_{q+1}) = h_1\cdot f(h_2\otimes\cdots\otimes h_{q+1})$, $d_i(f)(h_1\otimes\cdots\otimes h_{q+1}) = f(h_1\otimes\cdots\otimes h_{i-1}\otimes h_i h_{i+1}\otimes h_{i+2}\otimes\cdots\otimes h_{q+1})$ for $i=1, \cdots, q$, $d_{q+1}(f)(h_1\otimes\cdots\otimes h_{q+1}) = f(h_1\otimes\cdots\otimes h_q)\varepsilon(h_{q+1})$ where $f\in$ Hom (H^q, S), $h_1\otimes\cdots\otimes h_{q+1}\in H^{q+1}$.

Let F be a covariant functor from the category of commutative algebras to the category of abelian groups. We form a complex as follows; The object of q-th degree is F(Hom (H^q, S)), the coboundary operator $D^q = D^q(H, S/R, F) = F(d_0)F(d_1)^{-1}\cdots F(d_{q+1})^{(-1)^{q+1}}$.

The cohomology of H in S with respect to F is defined to be the homology of the above complex and the q-th group ($Ker\ D^q/Im\ D^{q-1}$ for $q>0$ and Ker D^0 for $q=0$) is denoted by $H^q(H, S/R, F)$.

Next let 1_i: Hom $(H^{q+1}, S) \to$ Hom (H^q, S), $i=1, 2, \cdots, q+1$, be the homomorphisms given by $1_i(f)(h_1\otimes\cdots\otimes h_q) = f(h_1\otimes\cdots\otimes h_{i-1}\otimes 1\otimes h_i\otimes\cdots\otimes h_q)$, $f\in$ Hom (H^{q+1}, S). We define a subcomplex as follows; The object of q-th degree is the intersection of kernel $F(1_i)$'s if $q>0$, and F(Hom (R, S)) if $q=0$. This complex is a normal subcomplex and the inclusion map induces an isomorphism between two chomologies.

Theorem 1.1 (cf. Sweedler [12]). *If S/R is an H-Hopf Galois extension, then the above cohomology coincides with the Amistur cohomology.*

Proof. Consider the maps α_q: $S^{q+1} \to$ Hom (H^q, S) defined by $\alpha_q(x_1\otimes\cdots\otimes x_{q+1})(h_1\otimes\cdots\otimes h_q) = x_1 h_1\cdot(x_2 h_2\cdot(\cdots(x_q h_q\cdot x_{q+1})\cdots)$. α_q is an algebra homomorphism as is easily verified. To see α_q is an isomorphism, we use an induction on q. For $q=0$, α_0: $S\to\mathrm{Hom}(R, S)$ is an isomorphism. For $q=1$, the composition of the isomorphism $S^2\cong S\otimes H^*$ and the canonical isomorphism $S\otimes H^*\cong \mathrm{Hom}(H, S)$ is α_1, so α_1 is an isomorphism. Now let α_{q-1} be an isomorphism and $\mathfrak{m}$ be an arbitrary maximal ideal of R. We shall show that the induced $R/\mathfrak{m}$-homomorphism $\alpha_q\otimes 1$: $S^{q+1}\otimes R/\mathfrak{m}\to\mathrm{Hom}(H^q, S)\otimes R/\mathfrak{m}$ is an isomorphism, then that α_q is an isomorphism will follow immediately since S^{q+1} and $\mathrm{Hom}(H^q, S)$ are finitely generated projective R-modules. For this purpose we may assume that R itself is a field. Let $x=\sum_i a_i\otimes x_i$ be a non-zero element of S^{q+1} where $\{a_i\}$ is an R-basis of S and x_i's are the elements of S^q. Since $x\neq 0$, some x_i, say x_1, is non-zero. So there exists $h'\in H^{q-1}$ with the property $(\alpha_{q-1}(x_1))(h')\neq 0$. α_1 is an isomorphism and $\{a_i\}$ is an R-basis, hence there exists $h\in H$ such that $(\alpha_1(\sum_i a_i\otimes(\alpha_{q-1}(x_i))(h')))(h)\neq 0$. Since $(\alpha_1(\sum_i a_i\otimes (\alpha_{q-1}(x_i))(h')))(h)=(\alpha_q(\sum_i a_i\otimes x_i))(h\otimes h')$, α_q is a monomorphism. Hence comparing dimensions gives that it is an isomorphism. By easy computations, we can show that $\{\alpha_q\}$ gives an isomorphism between two complexes.

Let σ be a normal 2-cocycle with respect to the units functor U. We make a (general) smash product algebra $S\#_\sigma H$ as follows; As an R-module, $S\#_\sigma H= S\otimes H$, except that we write $s\#_\sigma h$ rather then $s\otimes h$, $s\in S$, $h\in H$. Multiplication in $S\#_\sigma H$ is defined by the formula

$$(x\#_\sigma g)(y\#_\sigma h)=\sum_{(g),(h)} x(g_{(1)}\cdot y)\sigma(g_{(2)}\otimes h_{(1)})\#_\sigma g_{(3)}h_{(2)},\ x, y\in S,\ g, h\in H\,.$$

We remark that a trivial smash product algebra $S\# H$ coincides with $S\#_{\varepsilon'} H$, where ε' is the trivial 2-cocycle ε': $H\otimes H\to S$, defined by $\varepsilon'(g\otimes h)=\varepsilon(gh)$.

Proposition 1.2 (cf. Sweedler [12] 9.1). *Let S/R be an H-Hopf Galois extension and σ a normal 2-cocycle, then the smash product algebra $S\#_\sigma H$ is an S/R-Azumaya algebra.*

Proof. We shall show that $S\otimes(S\# H)$ is S-algebra isomorphic to $\mathrm{Hom}_{S\otimes R}(S^2, S^2)$, then the other properties will follow easily. We put $\alpha_2^{-1}(\sigma)=\sum_i x_i\otimes y_i\otimes z_i$. And we consider an $S\otimes H$-Hopf Galois extension S^2 of S. Define an S-homomorphism ρ: $S\otimes H\to S^2$ and a normal 2-cocycle $\tilde{\sigma}$: $(S\otimes H)\otimes_S(S\otimes H)\to S^2$ by setting $\rho(s\otimes g)=\sum_i sx_i\otimes y_i g\cdot z_i$ and $\tilde{\sigma}((s\otimes g)\otimes_S(t\otimes h))=st\otimes\sigma(g\otimes h)$, s, $t\in S$, g, $h\in H$. Then $D^1(\rho)=\tilde{\sigma}$, i.e. $\tilde{\sigma}$ is cohomologous to the trivial 2-cocycle ε_S'. So $S^2\#_{\tilde{\sigma}}(S\otimes H)$ is isomorphic to $S^2\#_{\varepsilon_S'}(S\otimes H)$ as is easily verified. We have a chain of S-algebra isomorphisms;

$$S\otimes(S\#_\sigma H)\cong S^2\#_\sigma(S\otimes H)\cong S^2\#_{\varepsilon_S'}(S\otimes H)\cong \mathrm{Hom}_{S\otimes R}(S^2, S^2)$$

Thus we get the proposition.

An isomorphism between S/R-Azumaya algebras is called *S/R-isomorphism* if it is compatible with the maximal commutative imbeddings.

Proposition 1.3 (cf. Sweedler [12] 9.4). *Let σ, τ be normal 2-cocycles. Then two smash product algebras $S \#_\sigma H$ and $S \#_\tau H$ are S/R-isomorphic, if and only if, σ and τ are cohomologous 2-cocycles.*

Proof. We define the homomorphisms v_σ, v_σ': $H \to S \#_\sigma H$, v_τ, v_τ', w, w': $H \to S \#_\tau H$, by setting for $h \in H$

$$v_\sigma(h) = 1 \#_\sigma h,\ v_\sigma'(h) = \sum_{(h)} (h_{(1)} \cdot \sigma^{-1}(h_{(2)} \otimes h_{(3)}^{-1})) \#_\sigma h_{(4)}^{-1},\ v_\tau(h) = 1 \#_\tau h\,,$$
$$v_\tau'(h) = \sum_{(h)} (h_{(1)} \cdot \tau^{-1}(h_{(2)} \otimes h_{(3)}^{-1})) \#_\tau h_{(4)}^{-1},\ w = (Vv_\sigma) * v_\tau',\ w' = v_\tau * (Vv_\sigma')$$

where V is the given S/R-isomorphism $S \#_\sigma H \cong S \#_\tau H$.

Since $sw(h) = w(h)s$ and $sw'(h) = w'(h)s$ for all $s \in S$, $h \in H$, w and w' are contained in the convolution algebra Hom (H, S) and are inverse to each other. From $V(1 \#_\sigma h) = \sum_{(h)} w(h_{(1)}) \#_\tau h_{(2)}$, we have

$$\sum_{(g,h)} \sigma(g_{(1)} \otimes h_{(1)}) w(g_{(2)} h_{(2)}) \#_\tau g_{(3)} h_{(3)} = V((1 \#_\sigma g)(1 \#_\sigma h))$$
$$= V(1 \#_\tau g) V(1 \#_\sigma h) = \sum_{(g,h)} w(g_{(1)})(g_{(2)} \cdot w(h_{(1)}) \tau(g_{(3)} \otimes h_{(2)}) \#_\tau g_{(4)} h_{(3)}\,.$$

Applying $1 \otimes \varepsilon$ on both sides, we get

$$\sigma * (w m_H) = (w \otimes \varepsilon) * \psi (1 \otimes w) * \tau\,,$$

where m_H is the multiplication in H and ψ is the measuring. This proves that σ and τ are cohomologous.

Conversely if σ and τ are cohomologous, then there exists $\rho \in \text{Hom}\,(H, S)$ such that $\sigma * \tau^{-1} = D^1(\rho)$, $\rho(1_H) = 1_S$. Define the homomorphism V': $S \#_\sigma H \to S \#_\tau H$ by $V'(s \#_\sigma h) = \sum_{(h)} s\rho(h_{(1)}) \#_\tau h_{(2)}$, then V' is a desired S/R-isomorphism.

2. $S \otimes S$-module structure of S/R-Azumaya algebras

Let S be a commutative R-algebra, which is finitely generated faithful projective as an R-module, and A be an S/R-Azumaya algebra. Since A contains S as a maximal commutative subalgebra and contains R as a center, we can regard A as a left S^2-module by setting for $x \otimes y \in S^2$, $a \in A$,

$$(x \otimes y) a = xay\,,$$

As to S^2-projectivity of S/R-Azumaya algebras, we have

Theorem 2.1. *Let S be a commutative R-algebra, which is a finitely generated faithful projective R-module. Then the following conditions are equivalent:*

(i) *S/R is a quasi-Frobenius extension.*

(ii) Hom (S, R) *is a finitely generated faithful projective S-module.*

(iii) Hom (S, S) *is a finitely generated faithful projective S^2-module.*

(iv) *Any S/R-Azumaya algebra is a finitely generated faithful projective S^2-module.*

Proof. The equivalence (i)⇔(ii) follows from the definition of quasi-Frobenius extensions.

(ii)⇔(iii). By the Morita theory, Hom $(S, S) \cong S \otimes$ Hom (S, R) as Hom (S, S)-Hom (S, S)-bimodules, hence as S^2-modules. In this case the S^2-module structure of $S \otimes$ Hom (S, R) is given by $(x \otimes y)(s \otimes f) = xs \otimes yf$, where yf is the homomorphism defined by $(yf)(t) = f(yt)$, $x, y, s, t \in S$, $f \in$ Hom (S, R). Hence, that Hom (S, S) is a finitely generated faithful projective S^2-module is equivalent to that $S \otimes$ Hom (S, R) is a finitely generated faithful projective S^2-module, which is equivalent to that Hom (S, R) is a finitely generated faithful projective S-module.

(iii)⇔(iv). Let A be any S/R-Azumaya algebra, then $S \otimes A \cong \mathrm{Hom}_S(P, P)$ for some finitely generated faithful projective S-module P. By the same arguments in Chase-Rosenberg [5] 2.13, P is a finitely generated faithful projective S^2-module. If P is isomorphic to S^2 as S^2-modules, then we have S^3-isomorphisms $\mathrm{Hom}_S(P, P) \cong \mathrm{Hom}_{S \otimes R}(S^2, S^2) \cong S \otimes$ Hom (S, S). So $\mathrm{Hom}_S(P, P)$ is a finitely generated faithful projective S^3-module by (iii). The general case follows by usual direct summand arguments. Thus A is a finitely generated faithful projective S^2-module. The converse is trivial.

Theorem 2.2. *If S/R is an H-Hopf Galois extension, then any S/R-Azumaya algebra is a finitely generated faithful projective S^2-module.*

Proof. Larson-Sweedler [11] ensures that a Hopf algebra $S \otimes H$ over S is a finitely generated faithful projective left $\mathrm{Hom}_S(S \otimes H, S)$-module (the assumption that S is a principal ideal domain is unnecessary). And we have isomorphisms $\mathrm{Hom}_S(S \otimes H, S) \cong$ Hom $(H, S) \cong S^2$ and Hom $(S, S) \cong S \# H = S \otimes H$. The S^2-module structure on $S \otimes H$ given by Larson-Sweedler and our structure are same. Thus Hom (S, S) is a finitely generated faithful projective S^2-module. By Theorem 2.1, we get the theorem.

Corollary 2.3. *If S/R is an H-Hopf Galois extension, then S/R is a quasi-Frobenius extension.*

From now on, we always assume that S/R is an H-Hopf Galois extension. By theorem 2.2, any S/R-Azumaya algebra A, especially $\mathrm{Hom}(S, S) \cong S \# H$ is a finitely generated projective S^2-module of rank one. So we can put $A = \theta(A) \otimes_{S^2} (S \# H)$ as an S^2-module, where $\theta(A)$ is a finitely generated projective S^2-module of rank one.

We shall investigate $\theta(A)$. First we have

Proposition 2.4. *Let P be a finitely generated projective S-module of rank one. Then we have an S^2-isomorphism:*

$$\mathrm{Hom}(P, P) \cong (P \otimes S) \otimes_{S^2} (S \otimes P^*) \otimes_{S^2} (S \# H),$$

where $P^* = \mathrm{Hom}_S(P, S)$. Thus

$$\theta(\mathrm{Hom}(P, P)) = (P \otimes S) \otimes_{S^2} (S \otimes P^*)$$

Proof. Define an S^2-homomorphism β: $(P \otimes S) \otimes_{S^2} (S \otimes P^*) \otimes_{S^2} (S \# H) \to \mathrm{Hom}(P, P)$ by $\beta((p \otimes s) \otimes (t \otimes q^*) \otimes (u \# h))(x) = tuh \cdot (sq^*(x))p$, $s, t, u \in S$, $p, x \in P$, $q^* \in P^*$, $h \in H$. By localization, we get easily that β is an isomorphism.

3. Characterization of smash product algebras as $S \otimes S$-modules

In this section we shall prove

Theorem 3.1. *Let $A = \theta(A) \otimes_{S^2} (S \# H)$ be an S/R-Azumaya algebra, then the following conditions are equivalent:*

(i) *A is a smash product algebra.*

(ii) *$\theta(A) \cong S^2$ as S^2-modules, i.e. $A \cong S \# H$ as S^2-modules.*

Lemma 3.2. *Let $A = \theta(A) \otimes_{S^2} (A \# H)$ be an S/R-Azumaya algebra, then the subalgebra $\theta(A) \otimes_{S^2} (S \# R) = \theta(A) \otimes_{S^2} S$ coincides with the maximal commutative subalgebra S.*

Proof. Since any element in $\theta(A) \otimes_{S^2} S$ commutes with any element in S, $\theta(A) \otimes_{S^2} S$ is contained in S. Passing to an arbitrary residue class field of R, we see $\theta(A) \otimes_{S^2} S$ and S are in fact equal by comparing dimensions.

Lemma 3.3. *If an S/R-Azumaya algebra A is S^2-isomorphic to $S \# H$, then its opposite algebra A^0 is also S^2-isomorphic to $S \# H$.*

Proof. We define a new S^2-module $\widetilde{S \# H}$ as follows; As an R-module $\widetilde{S \# H} = S \otimes H$, except that we write $\widetilde{s \# h}$ rather than $s \otimes h$. The S^2-action is defined by $(x\otimes y)(\widetilde{s\# h})=\sum_{(h)}\widetilde{ysh_{(1)}\cdot x\otimes h_{(2)}}$, $x, y\in S$, $s\# h\in S\# H$, i.e. the twisted S^2-module of $S\# H$. Consider an S^2-isomorphism $\gamma\colon S\# H\to\widetilde{S\# H}$ defined by $\gamma(s\# h)=\sum_{(h)}\widetilde{h_{(1)}^{-1}\cdot s\# h_{(2)}^{-1}}$, the inverse of γ is given by $\gamma^{-1}(s\# h)=\sum_{(h)} h_{(1)}^{-1}\cdot s\# h_{(2)}^{-1}$. Then, since the S^2-module structure of A^0 is the twisted one of A, we get the lemma.

Let B be an arbitrary algebra containing S as a subalgebra. Then following to Sweedler [12], we say that the action of H on S is *B-inner* if there exists an invertible element $v\in\mathrm{Hom}\,(H, B)$ such that $v(h)s=\sum_{(h)}(h_{(1)}\cdot s)v(h_{(2)})$ or equivalently $h\cdot s=\sum_{(h)}v(h_{(1)})sv^{-1}(h_{(2)})$, and $v(1_H)=1_B$ for all $h\in H$, $s\in S$. We say such v gives the B-inner action.

Proposition 3.4. *Let $P\in Pic(S)$ have the S^2-isomorphism $\pi\colon P\otimes S\cong S\otimes P$. Then the action of H on S is* Hom (P, P)*-inner, where we regard that S is contained in* Hom (P, P) *as usual.*

Proof. We define $v(h)$ and $V(h)$, $h\in H$, by the following commutative diagram;

$$\begin{array}{ccccc} P & \xrightarrow{\ inc\ } & P\otimes S & \overset{\pi}{\cong} & S\otimes P \\ \downarrow{\scriptstyle v(h)} & & \downarrow{\scriptstyle V(h)} & & \downarrow{\scriptstyle V_1(h)} \\ P & \xleftarrow{\ con\ } & P\otimes S & \overset{\pi}{\cong} & S\otimes P \end{array}$$

where *inc* is the canonical inclusion map, *con* is the contraction map and $V_1(h)$ is defined by setting $V_1(h)(s\otimes p)=h\cdot s\otimes p$, $s\otimes p\in S\otimes P$.

Then v is an element of Hom $(H$, Hom $(P, P))$. For $s\in S$, $p\in P$.

$$V(h)(inc\,(sp)) = V(h)(sp\otimes 1) = \sum_{(h)}(h_{(1)}\cdot s\otimes 1)V(h_{(2)})(p\otimes 1)\,.$$

Applying the contraction map on both sides, we get

$$v(h)(sp) = \sum (h_{(1)}\cdot s)v(h_{(2)})(p),\ \text{i.e.}\, v(h)s = \sum (h_{(1)}\cdot s)v(h_{(2)})\,.$$

And the identity $v(1_H)=1$ is clear.

Next we must show that v is invertible. For this purpose, we define a homomorphism $V'(h)\colon P\otimes S\to P\otimes S$ by $V'(h)=V(h^{-1})$, $h\in H$. Then V and V' are contained in the convolution algebra Hom $(H, \mathrm{Hom}_{R\otimes S}(P\otimes S, P\otimes S))$, and for any $p\in P$ we have

$$(V*V')(h)(p\otimes 1) = \sum V(h_{(1)})V'(h_{(2)})(p\otimes 1) = \varepsilon(h)p\otimes 1\,.$$

Since $V(h)$ and $V'(h)$ are contained in $\mathrm{Hom}_{R\otimes S}(P\otimes S, P\otimes S)\cong\mathrm{Hom}\,(P, P)\otimes S$, we identify this isomorphism and write $V(h)=\sum_i f_i^h\otimes s_i^h$, $V'(h)=\sum_j f_j'^h\otimes s_j'^h$, $f_i^h, f_j'^h\in \mathrm{Hom}(P, P)$, s_i^h, $s_j'^h\in S$. Then $v(h)=\sum_i s_i^h f_i^h$. Define $v'\in \mathrm{Hom}(H, \mathrm{Hom}(P, P))$ by setting $v'(h)=\sum_{j,(h)}(h_{(1)}^{-1}\cdot s_j'^{h_{(2)}})f_j'^{h_{(2)}}$. By the identities $(V*V')(h)=\varepsilon(h)$ and $v(h)s=\sum_{(h)}(h_{(1)}\cdot s)v(h_{(2)})$, we can easily see that v' is the inverse of v.

Proposition 3.5 (Sweedler [12] 9.6). *Let A be an S/R-Azumaya algebra and assume that v gives the A-inner action. If we puts $\sigma=(m_A(v\otimes v))*(v^{-1}m_H)$: $H\otimes H\to A$, where m_A means the multiplication in A and m_H the multiplication in H. Then*

(i) *The image of σ is contained in S.*

(ii) *σ is a normal 2-cocycle (with respect to units functor U).*

(iii) *ω: $S\underset{\sigma}{\#}H\to A$ given by $\omega(s\underset{\sigma}{\#}h)=sv(h)$, $s\underset{\sigma}{\#}h\in S\underset{\sigma}{\#}H$, is an S/R-isomorphism.*

Proof. (i), (ii) can be proved in the same manner as Sweedler [12] 9.6.

(iii). ω is an algebra homomorphism by direct computations. Since ω restricted to R is a monomorphism and $S\underset{\sigma}{\#}H$ is an Azumaya algebra, ω itself is a monomorphism. By the usual arguments of passing to residue class fields of R,that ω is an isomorphism will follow easily.

Combining above propositions we get ,

Corollary 3.6. *Let P as in Proposition* 3.4, *then its endomorphism ring* Hom (P, P) *is a smash product algebra.*

Proof of Theorem 3.1. The implication (i)⇒(ii) is clear. (ii)⇒(i). We may assume that the S^2-isomorphism $A\cong S\# H$ carries 1 to $1\# 1$, because the image of 1 is an invertible element of S^2 by Lemma 3.2. Let A^0 be an opposite algebra of A, then we have $A\otimes A^0=\mathrm{Hom}(A, A)$. If we regard the extension S^2/R as an H^2-Hopf Galois extension, then from Proposition 2.4 and Lemma 3.3 we have a chain of S^4-isomrphisms

$$S^2\# H^2\cong A\otimes A^0=\mathrm{Hom}(A, A)\cong(A\otimes S^2)\underset{S^4}{\otimes}(S\otimes A^*)\underset{S^4}{\otimes}(S^2\# H^2).$$

Hence by Corollary 3.6, there exists a normal 2-cocycle τ: $H^2\otimes H^2\to S^2$ such that W: $S^2\underset{\tau}{\#}H^2\cong \mathrm{Hom}(A, A)$. We denote the S^2-isomorphisms $S\# H\cong A$ and $S\# H\cong A^0$ by V and V^0, their restrictions to H by v and v^0. $W^{-1}(V\otimes V^0)$ is an S^4-automorphism of $S^2\# H^2$, so there exists an invertible element $u=\sum_i p_i\otimes q_i\otimes r_i\otimes s_i\in S^4$ such that $W^{-1}(V\otimes V^0)=u$. We have for $g\otimes h\in H^2$

$$(v\otimes v^0)(g\otimes h) = W(u((1\otimes 1)\underset{\tau}{\#}(g\otimes h)))$$
$$=W(\sum_{(g)(h)}\sum_i (p_i g_{(1)}\cdot r_i\otimes q_i h_{(1)}\cdot s_i)\underset{\tau}{\#}(g_{(2)}\otimes h_{(2)}))\ .$$

Define the homomorphism $(v\otimes v^0)'$: $H\otimes H\to A\otimes A^0=\mathrm{Hom}(A, A)$ as follows; $(v\otimes v^0)'(g\otimes h)=W(\sum_{(g,h)}\sum_i (r_j' g_{(1)}\cdot p_j'\otimes s_j' h_{(1)}\cdot q_j')((g_{(2)}^{-1}\otimes h_{(2)}^{-1})\cdot\tau^{-1}(g_{(3)}\otimes h_{(3)}\otimes g_{(4)}\otimes h_{(4)}))\underset{\tau}{\#}g_{(5)}\otimes h_{(5)})$, where $g\otimes h\in H\otimes H$. Easy computations show that $(v\otimes v^0)'$ is the inverse of $v\otimes v^0$ in the convolution algebra $\mathrm{Hom}(H\otimes H, A\otimes A^0)$, hence v itself is invertible. Since V is an S^2-isomorphism, v gives the A-inner action.

Let v: $H\to S\underset{\sigma}{\#}H$ be the canonical imbedding of H to the smash product algebra $S\underset{\sigma}{\#}H$, then the homomorphism v': $H\to S\underset{\sigma}{\#}H$ defined by $v'(h)=\sum_{(h)}(h_{(1)}\cdot\sigma^{-1}(h_{(2)}\otimes h_{(3)}^{-1}))\underset{\sigma}{\#}h_{(4)}^{-1}$, $h\in H$, is the inverse of $v\in\mathrm{Hom}(H, S\underset{\sigma}{\#}H)$. And v gives the $S\underset{\sigma}{\#}H$-inner action as is easily verified. So we have

Corollary 3.7. *Let A be an S/R-Azumaya algebra. Then the action of H on S is A-inner, if and only if, A is a smash product algebra.*

Corollary 3.8. *If Pic (S^2) is trivial, then for any S/R-Azumaya algebra A, the action of H on S in A can be extended innerly to the action on A.*

4. Properties of θ

We shall denote the S/R-isomorphism classes of S/R-algebras by $A(S/R)$, and we shall not distinguish an algebra from an algebra isomorphism class. Chase-Rosenberg [5] 2.14 showed that $A(S/R)$ forms an abelian group by an abstract manner. In this section, we first show that the inverse of A in $A(S/R)$ is given by its opposite algebra A^0.

Let A, $B\in A(S/R)$, then the product $A\cdot B$ is defined by

$$A\cdot B = \mathrm{Hom}_{A\otimes B}(S\underset{S^2}{\otimes}(A\otimes B), S\underset{S^2}{\otimes}(A\otimes B)) = \mathrm{Hom}_{A\otimes B}(A\underset{S}{\dot\otimes}B, A\underset{S}{\dot\otimes}B)$$

where S is an S^2-module via the contraction map $S^2\to S$, and $\underset{S}{\dot\otimes}$ denotes the tensor product regarding A and B as left S-modules.

By the monomorphism from $A\cdot B$ to $A\underset{S}{\dot\otimes}B$ which carries $f\in A\cdot B$ to $f(1\underset{S}{\dot\otimes}1)$, we consider $A\cdot B$ is contained in $A\underset{S}{\dot\otimes}B$. Thus

$$A\cdot B = \{x\in A\underset{S}{\dot\otimes}B\,|\,x(1\otimes s) = x(s\otimes 1) \qquad \text{for all} \quad s\in S\}\ .$$

Let Δ' be the monomorphism from $\theta(A)\underset{S^2}{\otimes}\theta(B)\underset{S^2}{\otimes}(S\#H)$ to $\theta(A)\underset{S^2}{\otimes}\theta(B)\underset{S^2}{\otimes}((S\#H)\underset{S}{\dot\otimes}(S\#H))=A\underset{S}{\dot\otimes}B$ induced from the diagonalization of H. Then $Im(\Delta')$ is contained in $A\cdot B$. By usual arguments, $Im(\Delta')=A\cdot B$. Thus we get

Proposition 4.1. *Let $A, B \in A(S/R)$, then $\theta(A\cdot B)$ is S^2-isomorphic to $\theta(A)\underset{S^2}{\otimes}\theta(B)$.*

Now let $A \in A(S/R)$, then since A is a finitely generated faithful projective S^2-module, an opposite algebra A^0 is also an element of $A(S/R)$.

Theorem 4.2. *A^0 is the inverse of A in $A(S/R)$.*

Corollary 4.3. $\theta(A^0) \cong \mathrm{Hom}_{S^2}(\theta(A), S^2) = \theta(A)^*$.

Proofs. For $x=\sum_i a_i \otimes b_i^0 \in A\cdot A^0$, we define $\eta(x) \in \mathrm{Hom}(S, A)$ by $(\eta(x))(s)=\sum_i a_i s b_i$, $s \in S$. To see $\eta(x)$ is contained in $\mathrm{Hom}(S, S)$, we may assume that R is a local ring. Then $A=S\# H$ and $A^0=S\# H$ as S^2-modules. Since $x \in Im(\Delta')$, we put $x=\sum_i \sum_{(h_i)} (s_i \# h_{i_{(1)}}) \dot{\otimes} (t_i \# h_{i_{(2)}})$, $s_i \# h_{i_{(1)}} \in A$, $t_i \# h_{i_{(2)}} \in A^0$. Define the isomorphisms γ_1, γ_2 and γ_2 as follows;

$$\gamma\colon \widetilde{S\# H} \text{ (twisted } S^2\text{-module)} \to A^0 = S\# H,\ \gamma(\widetilde{s\# h})= \sum_{(h)} h_{(1)}^{-1}\cdot s \# h_{(2)}^{-1},\ \widetilde{s\# h} \in \widetilde{S\# H}\,.$$

$$\gamma_1\colon A = S\# H \to A^0 = S\# H, \text{ anti-isomorphism.}$$

$$\gamma_2\colon A = S\# H \to \widetilde{S\# H},\ \gamma_2(s\# h) = \widetilde{s\# h},\ s\# h \in S\# H\,.$$

Since $A^0 \in Pic(S^2)$ and $\gamma_1\gamma_2^{-1}\gamma^{-1}\colon A^0 \to A^0$ is an S^2-isomorphism, there exists an invertible element $u \in S^2$ such that $\gamma_1\gamma_2^{-1}\gamma^{-1}=u$. We put $u^{-1}=\sum_j u_j \otimes v_j$, then for $t\# h \in A^0$

$$\gamma_1^{-1}(t\# h) = \gamma_2^{-1}\gamma^{-1}u^{-1}(t\# h) = \sum_j \sum_{(h)} v_j h_{(1)}^{-1}{}_j(tu_j) \# h_{(2)}^{-1}\,.$$

Hence

$$x = \sum_i \sum_{(h_i)} (s_i \# h_{i_{(1)}}) \dot{\otimes} (t_i \# h_{i_{(2)}}) = (\sum_i \sum_j \sum_{(h_i)} (s_i \# h_{i_{(1)}}) \dot{\otimes} (v_j h_{i_{(2)}}^{-1}\cdot (t_i u_j) \# h_{i_{(3)}}^{-1})\,.$$

Further, we may assume that A is a smash product algebra $S \underset{\sigma}{\#} H$ for some normal 2-cocycle σ by Theorem 3.1, then for any $s \in S$, we have

$$\begin{aligned}(\eta(x))(s) &= \sum_i \sum_j \sum_{(h_i)} (s_i \underset{\sigma}{\#} h_{i_{(1)}})(s \underset{\sigma}{\#} 1)(v_j h_{i_{(2)}}^{-1}\cdot (t_i u_j) \underset{\sigma}{\#} h_{i_{(3)}}^{-1}) \\ &= \sum_i \sum_j \sum_{(h_i)} s_i t_i u_j (h_{i_{(1)}}\cdot s v_j)\, \sigma(h_{i_{(2)}} \otimes h_{i_{(3)}}^{-1}) \underset{\sigma}{\#} 1\,,\end{aligned}$$

which is contained in S. Thus η is a homomorphism from $A\cdot A^0$ to $\mathrm{Hom}(S, S)$. By usual arguments, η is in fact an S/R-isomorphism. This completes the proof.

Now we shall consider some cohomological properties of $\theta(A)$.

Lemma 4.4. *For* $\mathrm{Hom}(S, S)=S\# H$, *we have an S^3-isomorphism*

$(S\# H)\overset{d_0'}{\underset{S^2}{\otimes}}S^3\simeq(S\# H)\overset{d_1'}{\underset{S^2}{\otimes}}S^3$, *where* $\overset{d_i'}{\underset{S^2}{\otimes}}(i=0, 1)$ *means a tensor product regarding* S^3 *as an* S^3*-module by the homomorphisms* $d_i'\colon S^2\to S^3$ *given by* $d_0'(x\otimes y)=1\otimes x\otimes y$, $d_1'(x\otimes y)=x\otimes 1\otimes y$, $x\otimes y\in S^2$.

Proof. Consider the S^2/S-isomorphism $\phi\colon \mathrm{Hom}\,(S, S)\otimes S\simeq \mathrm{Hom}_{R\otimes S}(\mathrm{Hom}\,(S, S), \mathrm{Hom}\,(S, S))$ induced by left homotheties of an algebra $\mathrm{Hom}\,(S, S)$, i.e. $(\phi(g\otimes 1))(1)=gf$, $g, f\in \mathrm{Hom}\,(S, S)$. Then from Proposition 2.4, the lemma follows easily.

Proposition 4.5 (Cocycle condition of θ(A)). *Let A be an S/R-Azumaya algebra, then we have an S^3-isomorphism*:

$$(\theta(A)\overset{d_0'}{\underset{S^2}{\otimes}}S^3)\underset{S^3}{\otimes}(\theta(A)\overset{d_2'}{\underset{S^2}{\otimes}}S^3)\simeq\theta(A)\overset{d_1'}{\underset{S^2}{\otimes}}S^3\,,$$

where $d_2'\colon S^2\to S^3$ *is given by* $d_2'(x\otimes y)=x\otimes y\otimes 1$, $x\otimes y\in S^2$.

Proof. Consider the S^2/S-isomorphism $A\otimes S\simeq \mathrm{Hom}_{R\otimes S}(A, A)$ induced by left homotheties of an algebra A. Then we get our conclusion from Proposition 2.4 and Lemma 4.4.

Next, we shall determine the condition that an element in *Pic* (S^2) can be expressed in the form $\theta(A)$ for some A in $A(S/R)$. For this purpose, let M be in *Pic* (S^2) satisfying the cocycle condition of Proposition 4.5, i.e. $(M\overset{d_0'}{\underset{S^2}{\otimes}}S^3)\underset{S^2}{\otimes}(M\overset{d_2'}{\underset{S^2}{\otimes}}S^3)\simeq(M\overset{d_1'}{\underset{S^2}{\otimes}}S^3)$. We set $A=M\underset{S^2}{\otimes}(S\# H)$ as an S^2-module, then the above isomorphism gives an S^3-isomorphism $\phi\colon A\otimes S\simeq\mathrm{Hom}_{R\otimes S}(A, A)$. Define the homomorphisms Φ_1, $\Phi_2\colon A\otimes A\to\mathrm{Hom}\,(A, A)$ by

$$(\Phi_1(a\otimes b))(x)=(\phi((\phi(a\otimes 1)(x))\otimes 1))(b)$$
$$(\Phi_2(a\otimes b))(x)=(\phi(a\otimes 1))((\phi(x\otimes 1))(b)),\ a,\ b,\ x\in A\,.$$

We regard $A\otimes A$ and $\mathrm{Hom}\,(A, A)$ as S^4-modules as follows;

$$((p\otimes q\otimes r\otimes s)(f))(x)=(p\otimes q)(f((r\otimes s)x))$$
$$(p\otimes q\otimes r\otimes s)(a\otimes b)=((p\otimes r)a)\otimes((s\otimes q)b)\,,$$

where p, q, r, $s\in S$, $f\in\mathrm{Hom}\,(A, A)$, a, b, $x\in A$.

Then, Φ_1 and Φ_2 are S^4-homomorphisms.

REMARK. If A is an S/R-Azumaya algebra and $\phi\colon A\otimes S\simeq\mathrm{Hom}_{R\otimes S}(A, A)$ is the isomorphism induced by left homotheties of A. Then Φ_1 and Φ_2 coincide and are S^4-isomorphisms.

Easily we get for $A=M\underset{S^2}{\otimes}(S\# H)$

Lemma 4.6. *Let $\phi'=\phi u$: $A\otimes S\cong \mathrm{Hom}_{R\otimes S}(A, A)$ be another S^3-isomorphism, where $u=\sum_i p_i\otimes q_i\otimes r_i$ is an invertible element of S^3. Then*

$$\Phi_1' = \Phi_1(\sum_i\sum_j p_ip_j\otimes r_i\otimes q_j\otimes q_ir_j) \text{ and } \Phi_2' = \Phi_2(\sum_i\sum_j p_j\otimes r_ir_j\otimes p_iq_j\otimes q_i),$$

where Φ_1' and Φ_2' are the homomorphisms defined from ϕ' in similar manners.

By localization, we get from Remark and Lemma 4.6 that Φ_1 and Φ_2 are isomorphisms. So, $\Phi_1^{-1}\Phi_2$ is an S^4-automorphism of $A\otimes A\in Pic(S^4)$. We define an element $\mu(M, \phi)\in S^4$ by $\mu(M, \phi)=(perm\ (243))(\Phi_1^{-1}\Phi_2)$, where $(perm\,(243))(p\otimes q\otimes r\otimes s)=(p\otimes r\otimes s\otimes q)$, $p, q, r, s\in S$. Lemma 4.6 asserts that $\mu(M, \phi)$ and $\mu(M, \phi')$ differ only by a coboundary in the Amitsur complex with respect to U. Also by localization techniques, we get easily from Remark and Lemma 4.6 that $\mu(M, \phi)$ is a 3-cocycle.

Theorem 4.7. *Let $M\in Pic(S^2)$ satisfying the cocycle condition of Proposition 4.5. Then, $A=M\underset{S^2}{\otimes}(S\# H)$ has an S/R-Azumaya algebra structure compatible with the original S^2-module structure, if and only if, $\mu(M, \phi)$ is a 2-coboundary in Amitsur complex with respect to U.*

Proof. The only if part follows from Remark and Lemma 4.6. If part: Let $\mu(M, \phi)=D'^2(v)$, where D'^2 is the coboundary operator of Amitsur complex, v is a unit of S^3. We consider a new S^3-isomorphism $\phi'=\phi v^{-1}$: $A\otimes S\cong \mathrm{Hom}_{R\otimes S}(A, A)$ and define the multiplication in A by $a\cdot b=(\phi'(a\otimes 1))(b)$, $a, b\in A$. Then this product is associative and gives an S/R-Azumaya algebra structure compatible with the original S^2-module structure.

Appendix. On seven terms exact sequence

From the exact sequence of Amitsur cohomology (Chase-Rosenberg [5]), we get also an exact sequence related to a Hopf Galois extension by Theorem 1.1. We shall give a rough sketch of a concrete construction of an exact sequence, the details of proofs are omitted but they follow straightforward. We always assume that S/R is an H-Hopf Galois extension, and we often identify $\mathrm{Hom}(H^q, S)$ with S^{q+1} by the isomorphism α_q of Theorem 1.1.

θ_1: $H^1(H, S/R, U)\to Pic(R)$

For $\bar{\rho}\in H^1(H, S/R, U)$, we take a normal 1-cocycle ρ as a representative. We make a new $S\# H$-module ${}_\rho S$ as follows; ${}_\rho S=S$ as S-modules with the $S\# H$-action defined by $(s\# h)x=\sum_{(h)} s\rho(h_{(1)})h_{(2)}\cdot x$, $s\# h\in S\# H$, $x\in S$. We set

$${}_\rho S^H = \{x\in {}_\rho S\,|\,(1\# h)x = \varepsilon(h)x \quad \text{for all} \quad h\in H\}.$$

Since S is a finitely generated faithful projective $S\#H$-module, we get from the Morita theory

$$ {}_\rho S \cong \mathrm{Hom}_{S\#H}(S, {}_\rho S)\otimes S \cong {}_\rho S^H \otimes S . $$

Hence ${}_\rho S^H \in Pic\,(R)$.

Next ρ' be another representative of $\bar{\rho}$, then there exists a unit element $u\in \mathrm{Hom}\,(R, S)=S$ such that $\rho'=\rho_0 * \rho$ where $\rho_0(h)=u^{-1}h\cdot u$, $h\in H$. Then the homomorphism ${}_{\rho'}S^H \to {}_\rho S^H$ which carries $x\in {}_{\rho'}S^H$ to $u^{-1}x\in {}_\rho S^H$ is an isomorphism. We define θ_1: $H^1(H, S/R, U)\to Pic\,(R)$ by $\theta_1(\bar{\rho})=$*isomorphism class of* ${}_\rho S^H$. We have

Lemma A.1. *θ_1 is a monomorphism.*

Proof. Let $\theta_1(\rho)={}_\rho S^H=Ru$ be a free R-module with a free base u. Since ${}_\rho S^H\otimes S\cong S$, u is a unit element of S. Let ρ^{-1} be the inverse of ρ, then since $u\in {}_\rho S^H$ we have

$$ \sum_{(h)} \rho(h_{(1)})\rho^{-1}(h_{(2)}) = (\rho * \rho^{-1})(h) = \varepsilon(h) = (\sum_{(h)} \rho(h_{(1)})h_{(2)}\cdot u)u^{-1},\ h\in H . $$

Thus $\rho(h)=(h\cdot u^{-1})u$ and $\rho^{-1}(h)=(h\cdot u)u^{-1}$. This gives that θ_1 is injective.

Next ρ_1 and ρ_2 be 1-cocycles, we define the homomorphism

$$ \nu\colon {}_{\rho_1}S^H\otimes {}_{\rho_2}S^H \to {}_{\rho_1 * \rho_2}S^H \text{ by } \nu(x\otimes y) = xy,\ x\otimes y\in {}_{\rho_1}S^H\otimes {}_{\rho_2}S^H , $$

xy is the product of x and y in S. To see ν is an isomorphism, we may assume that R is a local ring. Then by the above arguments of free case, we get easily that ν is an isomorphism. So, θ_1 is a monomorphism.

θ_2: $Pic\,(R) \to H^0(H, S/R, Pic)$

We define θ_2 by $\theta_2(P) =$ *class of* $P\otimes\mathrm{Hom}\,(R, S)=P\otimes S$, $P\in Pic\,(R)$. θ_2 is a well defined homomorphism and we have

Lemma A.2. *$H^1(H, S/R, U) \xrightarrow{\theta_1} Pic\,(R) \xrightarrow{\theta_2} H^0(H, S/R, Pic)$ is an exact sequence of abelian groups.*

Proof. Let ρ be a 1-cocycle then as S-modules $(\theta_2\theta_1)(\rho)={}_\rho S^H\otimes S\cong {}_\rho S\cong S$. Thus $\theta_2\theta_1=0$

Conversely, let P be in Pic (R) such that $P\otimes S$ is S-isomorphic to S. Define the homomorphism g_h for $h\in H$ by the following commutative diagram;

$$ \begin{array}{ccc} P\otimes S & \xrightarrow{G_h} & P\otimes S \\ \wr\Vert\pi & & \wr\Vert\pi \\ S & \xrightarrow{g_h} & S \end{array} $$

where $G_h(p\otimes s)=p\otimes h\cdot s$, $p\otimes s\in P\otimes S$, and π is the given isomorphism. And define $\rho\in \mathrm{Hom}\,(H, S)$ by $\rho(h)=g_h(1_S)$. Then ρ is invertible (the inverse of ρ is given from $P^*=\mathrm{Hom}\,(P, R)$ in the same manner) and ρ is a 1-cocycle with respect to U. Further $\pi(P\otimes R)$ is equal to ${}_\rho S^H$. Thus we get the lemma.

θ_3: $H^0(H, S/R, Pic) \rightarrow H^2(H, S/R, U)$

For $\bar{P}\in H^0(H, S/R, Pic)$ let P be its representative. Then we have an S^2-isomorphism $P\otimes S\cong S\otimes P$. By Proposition 3. 4, 3. 5, we get a 2-cocycle σ_P such that $\mathrm{Hom}\,(P, P)\cong S\underset{\sigma_P}{\#}H$. We define θ_3 by $\theta_3(\bar{P})=$*class of* σ_P.

Lemma A.3. $Pic\,(R) \overset{\theta_2}{\rightarrow} H^0(H, S/R, Pic) \overset{\theta_3}{\rightarrow} H^2(H, S/R, U)$ *is an exact sequence of abelian groups.*

Proof. By direct computations, we get easily that θ_3 is a well-defined homomorphism and $\theta_3\theta_2(\bar{P})=0$.

For $\bar{P}\in Ker\,(\theta_3)$ let P be its representative. Then we have an isomorphism $\mathrm{Hom}\,(P, P)\cong S\underset{\sigma_P}{\#}H$, which is isomorphic to $\mathrm{Hom}\,(S, S)=S\# H$ since σ_P is a coboundary. By the above isomorphisms, we regard P as an $S\# H$-module, then from Morita theory we get an isomorphism $P\cong \mathrm{Hom}_{S\# H}(S, P)\otimes S$, and $\mathrm{Hom}_{S\# H}(S, P)$ is a finitely generated faithful projective R-module of rank one. Thus we get the lemma.

θ_4: $H^2(H, S/R, U) \rightarrow Br(S/R)$

For $\bar{\sigma}\in H^2(H, S, U)$, we take a normal 2-cocycle σ as a representative. By Proposition 1.2, $S\underset{\sigma}{\#}H$ is an S/R-Azumaya algebra. We define θ_4 by $\theta_4(\bar{\sigma})=$ *class of* $S\underset{\sigma}{\#}H$.

Lemma A.4. $H^0(H, S/R, Pic) \overset{\theta_3}{\rightarrow} H^2(H, S/R, U) \overset{\theta_4}{\rightarrow} Br(S/R)$ *is an exact sequence of abelian groups.*

Proof. That θ_4 is well-defined follows from Proposition 1.3. Next, let σ, τ be normal 2-cocycles, we put $\alpha_2^{-1}(\tau)=\sum_i x_i\otimes y_i\otimes z_i$ and $\alpha_2^{-1}(\tau^{-1})=\sum_j x_j'\otimes y_j'\otimes z_j'$. We consider an H^2-Hopf Galois extension S^2/R and define the maps ρ, ρ': $H^2\rightarrow S^2$ and 2-cocycles $\sigma\otimes\tau$, $\sigma*\tau\otimes\varepsilon$: $H^4\rightarrow S^2$ as follows;

$$\rho(g\otimes h)=\sum_i g\cdot y_i\otimes x_i h\cdot z_i,\ \rho'(g\otimes h)=\sum_j x_j' g\cdot z_j'\otimes y_j'\varepsilon(h),$$
$$(\sigma\otimes\tau)(g\otimes g'\otimes h\otimes h')=\sigma(g\otimes g')\otimes\tau(h\otimes h') \quad\text{and}\quad (\sigma*\tau\otimes\varepsilon)$$
$$(g\otimes g'\otimes h\otimes h')=\sum_{(g),(g')}\sigma(g_{(1)}\otimes g_{(1)}')\tau(g_{(2)}\otimes g_{(2)}')\otimes\varepsilon(hh'),\ g, g', h, h'\in H.$$

Then $D^1(\rho)*D^1(\rho')*(\sigma\otimes\tau)=\sigma*\tau\otimes\varepsilon$, where D^1 is the coboundary operator.

Hence we have a chain of R-algebra isomorphisms;

$$(S\underset{\sigma}{\#}H)\otimes(S\underset{\tau}{\#}H) \cong S^2\underset{\sigma\otimes\tau}{\#}H^2 \cong S^2\underset{\sigma*\tau\otimes\varepsilon}{\#}H^2$$
$$\cong (S\underset{\sigma*\tau}{\#}H)\otimes(S\underset{\varepsilon}{\#}H) \cong (S\underset{\sigma*\tau}{\#}H)\otimes\mathrm{Hom}(S, S)$$

This proves that θ_4 is a group homomorphism.

By Proposition 3.4, 3.5, $\theta_4\theta_3=0$. Conversely, let σ be a normal 2-cocycle such that $S\underset{\sigma}{\#}H\cong\mathrm{Hom}(P, P)$ for some finitely generated faithful projective R-module P. By this isomorphism, P has an S-module structure and as an S-module P is contained in *Pic* (S).

We must show that $P\overset{d_0}{\underset{S}{\otimes}}\mathrm{Hom}(H, S)$ is $\mathrm{Hom}(H, S)$-isomorphic to $P\overset{d_1}{\underset{S}{\otimes}}\mathrm{Hom}(H, S)$, where $\overset{d_i}{\underset{S}{\otimes}}$ $(i=1, 2)$ means a tensor product regarding $\mathrm{Hom}(H, S)$ as an S-module by the homomorphisms $d_i\colon S\to\mathrm{Hom}(H, S)$ given by $(d_0(s))(h)=h\cdot s$, $(d_1(s))(h)=\varepsilon(h)s$, $s\in S$, $h\in H$. And that σ is cohomologous to σ_P. For this purpose, we shall consider a Hopf algebra $S\otimes H$ over S, then its diagonalization induces an S-algebra structure on $\mathrm{Hom}_S(S\otimes H, S)=\mathrm{Hom}(H, S)$. We denote its multiplication by p. By Larson-Sweedler [11] §3, $\mathrm{Hom}(H, S)$ has a left $S\otimes H$-comodule structure and its structure map $q\colon \mathrm{Hom}(H, S)\to H\otimes \mathrm{Hom}(H, S)$ is defined uniquely to make the following diagram commutative;

$$\begin{array}{ccc}
\mathrm{Hom}(H, S)\underset{S}{\otimes}\mathrm{Hom}(H, S) & \xrightarrow{\quad p\quad} & \mathrm{Hom}(H, S)\\
\Big\downarrow q\otimes 1 & & \Big\downarrow \langle\,\rangle\otimes 1\\
(H\otimes\mathrm{Hom}(H, S))\underset{S}{\otimes}\mathrm{Hom}(H, S) & \xrightarrow{1\otimes t} & H\otimes\mathrm{Hom}(H, S)\underset{S}{\otimes}\mathrm{Hom}(H, S)
\end{array}$$

where $t(f\otimes g)=g\otimes f$, $f, g\in\mathrm{Hom}(H, S)$ and $\langle\,\rangle(h\otimes f)=f(h)$, $h\otimes f\in H\otimes\mathrm{Hom}(H, S)$.

Let v be the restriction of $V\colon S\underset{\sigma}{\#}H\cong\mathrm{Hom}(P, P)$ to H, then v has the inverse v^{-1} by Corollary 3.7. We define

$\pi_1\colon P\overset{d_0}{\underset{S}{\otimes}}\mathrm{Hom}(H, S)\to P\overset{d_1}{\underset{S}{\otimes}}\mathrm{Hom}(H, S)$ and $\pi_2\colon P\overset{d_1}{\underset{S}{\otimes}}\mathrm{Hom}(H, S)$
$\to P\overset{d_0}{\underset{S}{\otimes}}\mathrm{Hom}(H, S)$ as follows;

$$\pi_1(p\otimes f)=\sum_i (v^{-1}(h_i))(p)\otimes f_i,\ \pi_2(p\otimes f)=\sum_i (v(h_i))(p)\otimes f_i,$$

where $p\in P$, $f\in\mathrm{Hom}(H, S)$ and $q(f)=\sum_i h_i\otimes f_i\in H\otimes\mathrm{Hom}(H, S)$.

Then we get easily that π_1 and π_2 are $\mathrm{Hom}(H, S)$-homomorphisms and π_1 is the inverse of π_2. From Proposition 1.3, 3.4, 3.5, we get the lemma.

$\theta_5\colon Br(S/R)\to H^1(H, S/R, Pic)$

For $\bar{A}\in Br\,(S/R)$ we can take an S/R-Azumaya algebra A as a representative (cf. Chase-Rosenberg [5]). We define θ_5 by $\theta_5(\bar{A})=$*class of* $\theta(A)$. From Proposition 2.4, 4.1, θ_5 is a well-defined homomorphism, and from Theorem 3.1 we get

Lemma A.5. $H^2(H, S/R, U) \xrightarrow{\theta_4} Br(S/R) \xrightarrow{\theta_5} H^1(H, S/R, Pic)$ *is an exact sequence of abelian groups.*

θ_6: $H^1(H, S/R, Pic) \rightarrow H^3(H, S/R, U)$

For $\bar{P}\in H^1(H, S/R, Pic)$, let P be its representative. Then by Theorem 4.7, $\mu(\phi, P)$ is a 3-cocycle in Amitsur complex. We define θ_6 by $\theta_6(\bar{P})=$*class of* $\alpha_3\,(\mu(\phi, P))$.

From Lemma 4.6 and Theorem 4.7, we get

Lemma A.6. $Br(S/R) \xrightarrow{\theta_5} H^1(H, S/R, Pic) \xrightarrow{\theta_6} H^3(H, S/R, U)$ *is an exact sequence of abelian groups.*

Summing up lemmas, we get

Theorem A.7.

$$0 \rightarrow H^1(H, S/R, U) \xrightarrow{\theta_1} Pic(R) \xrightarrow{\theta_2} H^0(H, S/R, Pic) \xrightarrow{\theta_3} H^2(H. S/R, U)$$
$$\xrightarrow{\theta_4} Br(S/R) \xrightarrow{\theta_5} H^1(H, S/R, Pic) \xrightarrow{\theta_6} H^3(H, S/R. U)$$

is an exact sequence of abelian groups.

REMARK. If S/R is a separable Galois extension, then above homomorphisms coincide with those of Kanzaki [10].

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