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ON THE LOOP-ORDER OF A FIBRE SPACE

Dedicated to Professor Ryoji Shizuma on his 60-th birthday

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Introduction

Let ΩX denote the space of loops on a based topological space X. M. Sugawara [8] called the order of the identity class $1_{\Omega X}$ of ΩX in the group $[\Omega X, \Omega X]$ the loop-order of X, denoted by $l(X)$, and proved ([8], Theorem 3) that, for a Hurewicz fibration $F \rightarrow E \rightarrow B$, $l(E)$ is a divisor of the multiple $l(B) \cdot l(F)$.

The aim in this note is to determine, using a technique of Larmore and Thomas [2], the loop-order of a total space obtained as a 2-stage Postnikov tower and to discuss that of a space obtained as a 3-stage Postnikov tower.

In this note, let p denote a fixed prime. Let $\mathcal{A}(p)$ denote the mod p Steenrod algebra, and let $\mathcal{E}\colon \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ denote the Kristensen map of degree -1, which is a derivation and is given by

$$\begin{aligned} &\mathcal{E}(Sq^n) = Sq^{n-1} \ (n \geqq 1) && \text{if } p = 2, \\ &\mathcal{E}(\Delta) = 1, \quad \mathcal{E}(P^k) = 0 \quad (k \geqq 0) && \text{if } p > 2, \end{aligned}$$

(cf. [2], Proposition 3.5; [5]). We shall write $\mathcal{E}(\alpha) = \tilde{\alpha}$.

Also denote by $K_n = K(Z_p, n)$ the Eilenberg-MacLane complex of type (Z_p, n). Let E_1 and E_2 be principal fibre spaces with classifying classes

$$\{\theta_1, \theta_2, \cdots, \theta_m\}\colon K_n \rightarrow \mathop{\times}_{j=1}^{m} K_{n+r_j}, \qquad 0 < r_1 \leqq r_2 \leqq \cdots \leqq r_m \leqq n-3$$

and

$$\sum_{i=1}^{k} \pi_i^* \gamma_i\colon \mathop{\times}_{i=1}^{k} K_{n+s_i} \rightarrow K_{n+r}, \qquad s_1 = 0 \leqq s_2 \leqq \cdots \leqq s_k < r \leqq n-3$$

respectively, where θ_j and γ_i are cohomology operations of degree r_j and $r - s_i$, regarded as elements of $\mathcal{A}(p)$, and $\pi_i\colon \mathop{\times}_{t=1}^{k} K_{n+s_t} \rightarrow K_{n+s_i}$ is the projection on the i-th factor. We then obtain

Theorem A. *$l(E_1) = p^2$ if, and only if, there exists j, $1 \leqq j \leqq m$, such that $\tilde{\theta}_j$ does not belong to the left $\mathcal{A}(p)$-module, $\sum_{t=1}^{j-1} \mathcal{A}(p)\theta_t$, of $\mathcal{A}(p)$ generated by*

$\theta_1, \cdots, \theta_{j-1}$.

Theorem B. *$l(E_2)=p^2$ if, and only if, there exists i, $1 \leqq i \leqq k$, such that $\tilde{\gamma}_i$ does not belong to the right $\mathcal{A}(p)$-module, $\sum_{t=i+1}^{k} \gamma_t \mathcal{A}(p)$, of $\mathcal{A}(p)$ generated by $\gamma_{i+1}, \cdots, \gamma_k$.*

The following corollary is a restatement of Theorem 1.3 of L. Smith [5].

Corollary 1. *Let E be a fibre space induced from the path-fibration on K_{n+r} by $\theta=\theta\iota_n$: $K_n \to K_{n+r}$, where $0<r\leqq n-3$ and ι_n denotes the fundamental class. Then $l(E)$ is p^2 if, and only if, $\tilde{\theta}\neq 0$.*

We next consider the situation shown in the diagram below:

$$(*)\qquad \begin{array}{ccccc} & & \Omega L & \xrightarrow{j} & E \\ & & & & \downarrow{\scriptstyle \pi} \\ & & \Omega B & \xrightarrow{l} & K & \xrightarrow{\theta} & L \\ & & & & \downarrow{\scriptstyle \rho} \\ & & & & A & \xrightarrow{\alpha} & B \end{array}$$

where we set

$$A = K_n, \quad B = \mathop{\times}_{i=1}^{m} K_{n+r_i}, \quad L = K_{n+s}, \quad 0<r_1\leqq r_2 \leqq \cdots \leqq r_m \leqq s \leqq n-3\,,$$

$$\alpha = \{\alpha_1, \cdots, \alpha_m\}, \quad \alpha_i \in \mathcal{A}(p), \quad \deg \alpha_i = r_i\,,$$

$$\beta = \theta l = \sum_{i=1}^{m} (\Omega\pi_i)^* \beta_i, \quad \beta_i \in \mathcal{A}(p), \quad \deg \beta_i = s - r_i + 1\,,$$

and where K and E are principal fibre spaces with classifying classes α and θ. Let

$$\psi\colon \bigcap_{i=1}^{m} (\operatorname{Ker} \alpha_i \cap \operatorname{Ker} \tilde{\alpha}_i) \to \operatorname{Coker} \sum_{i=1}^{m} (\beta_i + \tilde{\beta}_i)$$

denote a secondary operation associated with the relation $\sum_{i=1}^{m} [\tilde{\beta}_i \alpha_i + (-1)^{s-r_i+1} \beta_i \tilde{\alpha}_i] = 0$, which is deduced from $\sum_{i=1}^{m} \beta_i \alpha_i = 0$ by taking the map ε.

Theorem C. *Suppose that, for all $i=1, \cdots, m$, $\tilde{\alpha}_i \in \sum_{k=1}^{i-1} \mathcal{A}(p)\alpha_k$.*

1) *If there exists j such that $\tilde{\beta}_j \notin \sum_{k=j+1}^{m} \beta_k \mathcal{A}(p)$, then $l(E)=p^2$.*

2) *If* $\deg \beta_m > 1$ *(i.e., $s > r_m$) and if*

$$\psi(\Omega\rho)\not\equiv 0 \bmod \sum_{i=1}^{m} [\beta_i H^{n+r_i-3}(\Omega K; Z_p)+\tilde{\beta}_i H^{n+r_i-2}(\Omega K; Z_p)] + (\Omega\rho)^* H^{n+s-2}(\Omega A; Z_p),$$

then $l(E)=p^2$.

3) *If for all* $i=1, \cdots, m$, $\tilde{\beta}_i \in \sum_{k=j+1}^{m} \beta_k \mathcal{A}(p)$, *and if* $\deg \beta_m > 1$ *and*

$$(\Omega\rho)^* H^{n+s-2}(\Omega A; Z_p) \subset \sum_{i=1}^{m} \beta_i H^{n+r_i-3}(\Omega K; Z_p),$$

$$\psi(\Omega\rho) \equiv 0 \bmod \sum_{i=1}^{m} \beta_i H^{n+r_i-3}(\Omega K; Z_p),$$

then $l(E)=p$.

Corollary 2. *Suppose that, for all* i, $\tilde{\alpha}_i \in \sum_{k=1}^{i-1} \mathcal{A}(p)\alpha_k$ *and* $\tilde{\beta}_i \in \sum_{k=j+1}^{m} \beta_k \mathcal{A}(p)$ *and that the homogeneous part* $\mathcal{A}(p)$ *of degree* $s-1$ *is contained in* $\sum_{k=1}^{m} \beta_k \mathcal{A}(p) + \sum_{k=1}^{m} \mathcal{A}(p)\alpha_k$. *If* $\deg \beta_m > 1$ *and the homogeneous part of* $\mathcal{A}(p)$ *of degree* $s-r_i$ *is trivial for all* i, *then* $l(E)=p$.

Theorem D. *Suppose that there exists* i *such that* $\tilde{\alpha}_i \notin \sum_{k=1}^{i-1} \mathcal{A}(p)\alpha_k$. *If* $(\Omega\rho)^*[\sum_{i=1}^{m} (-1)^{r_i} \tilde{\beta}_i \tilde{\alpha}_i] \not\equiv 0 \bmod \sum_{i=1}^{m} \beta_i H^{n+r_i-3}(\Omega K; Z_p)$, *then* $l(E)=p^3$; *otherwise* $l(E)=p^2$.

Corollary 3. *Suppose that there exists* i *such that* $\tilde{\alpha}_i \notin \sum_{k=1}^{i-1} \mathcal{A}(p)\alpha_k$.

1) *If* $\sum_{i=1}^{m} (-1)^{r_i} \tilde{\beta}_i \tilde{\alpha}_i \notin \sum_{k=1}^{m} \{\beta_k \mathcal{A}(p) + \mathcal{A}(p)\alpha_k\}$ *and if*

$$\sum_{i=1}^{m} \beta_i \colon \bigoplus_{i=1}^{m} H^{n+r_i-3}(\Omega^2 B) \to H^{n+s-2}(\Omega^2 B)$$

is monic, then $l(E)=p^3$.

2) *If* $\sum_{i=1}^{m} (-1)^{r_i} \tilde{\beta}_i \tilde{\alpha}_i \in \sum_{i=1}^{m} \{\beta_k \mathcal{A}(p) + \mathcal{A}(p)\alpha_k\}$, *then* $l(E)=p^2$.

REMARK. $\sum_{i=1}^{m} \beta_i$ is monic in each of the following cases:

i) $\beta_i = Sq^{a_i}$, $a_1 > a_2 > \cdots > a_m$, $a_i \geqq 2(r_i - r_1 - 1)$ for $p=2$;

ii) β_i are of the form P^{a_i} or ΔP^{a_i} and are all distinct, and $(2p-2)a_i \geqq p(r_i - r_1 - 1)$ for $p>2$.

1. A basic theorem

In this note we work in the category of based spaces having the homotopy types of CW complexes and based continuous maps, and we don't distinguish between a map and the homotopy class it represents. Let $\pi: E \to K$ be the principal fibre space with $\theta: K \to L$ as classifying map and let $j: \Omega L \to E$ denote the fibre inclusion. Let p denote a fixed prime. A map of degree $p^k (k>0)$ of $S=S^1$ yields the Puppe sequence

$$S \xrightarrow{p^k} S \xrightarrow{i} P \xrightarrow{q} S^2 \xrightarrow{p^k} S^2 \longrightarrow \cdots .$$

Form the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & & & L^{S^2} \\
 & & & & & & & & \downarrow p^{k\#} \\
 & & & & & & K^{S^2} & \xrightarrow{\theta^{S^2}} & L^{S^2} \\
 & & & & & & \downarrow q^{\#} & & \downarrow q^{\#} \\
 & & & & & & K^{P} & \xrightarrow{\theta^{P}} & L^{P} \\
 & & & & & & \downarrow i^{\#} & & \downarrow i^{\#} \\
 & & \Omega L^S & \xrightarrow{j^S} & E^S & \xrightarrow{\pi^S} & K^S & \xrightarrow{\theta^S} & L^S \\
 & & \downarrow p^{k\#} & & \downarrow p^{k\#} & & \downarrow p^{k\#} & & \\
\Omega K^S & \xrightarrow{(\Omega\theta)^S} & \Omega L^S & \xrightarrow{j^S} & E^S & \xrightarrow{\pi^S} & K^S & &
\end{array}$$

where rows and columns are fibration sequences and $\#$ indicates induced maps of function spaces.

We now assume that K and L are loop spaces. Larmore and Thomas [2] have defined a sort of functional operation

$$\Phi_k: [X, K^S] \cap \operatorname{Ker} (p^{k\#})_* \cap \operatorname{Ker} \theta^S_* \to [X, L^{S^2}]/\theta^{S^2}_*[X, K^{S^2}] + (p^{k\#})_*[X, L^{S^2}]$$

by setting $\Phi_k = (q^\#)^{-1}_* \theta^P_* (i^\#)^{-1}_*$, with the property that, for $x \in [X, E^S]$ such that $(p^{k\#})_* \pi^S_* x = 0$,

$$p^k x \equiv -j^S_* \Phi_k(\pi^S_* x) \mod j^S_* p^k [X, \Omega L^S]\,, \tag{1.1}$$

where we have made the adjoint identification $[X, L^{S^2}] = [X, \Omega L^S]$ (cf. Theorem 3.2 of [3]).

In what follows we assume that

(1.2) $l(K)$ and $l(L)$ are divisors of p^k;

(1.3) $[\Omega^2 L, \Omega^2 K] = 0$;

(1.4) $[\Omega^2 L, Y] \xleftarrow{(\Omega j)^*} [\Omega E, Y] \xleftarrow{(\Omega\pi)^*} [\Omega K, Y] \xleftarrow{(\Omega\theta)^*} [\Omega L, Y]$
is exact for $Y = \Omega^2 L$ and $\Omega^2 K$, (this condition may be verified using Theorem 6.5 of Sugawara [7]).

Taking $X = \Omega E$, $x = 1_{\Omega E}$ in (1.1), we then have

Theorem 1.5. *With the hypotheses* (1.2), (1.3) *and* (1.4), *we have*

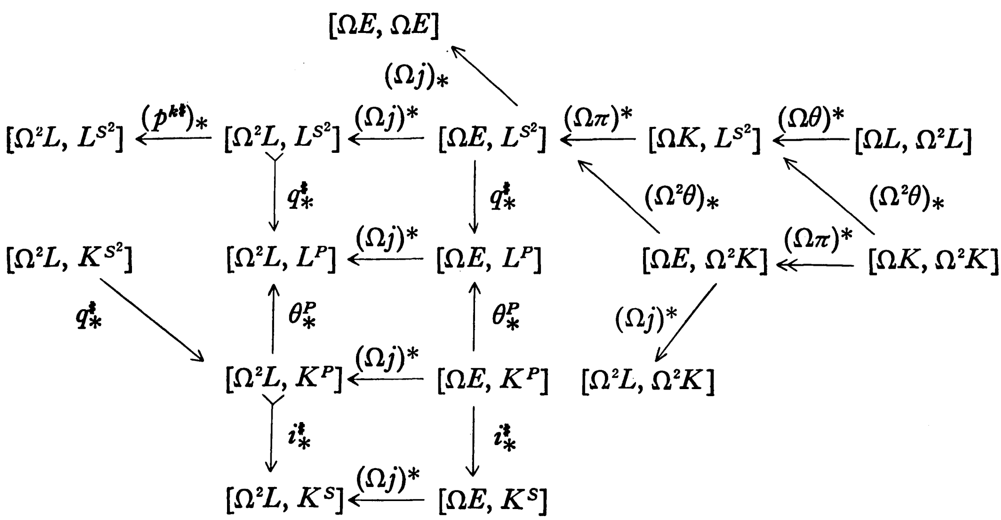
1) $$p^k 1_{\Omega E} = -(\Omega j)_* \Phi_k(\Omega\pi)\,.$$

2) *Write* $\Psi_k(E)$ *for the subset* $(\Omega\pi)^{*^{-1}}\Phi_k(\Omega\pi)$ *of* $[\Omega K, \Omega^2 L]$.

Then $\Psi_k(E)$ *is non-empty and is a coset of* $(\Omega\theta)^*[\Omega L, \Omega^2 L]+(\Omega^2\theta)_*[\Omega K, \Omega^2 K]$ *such that* $p^k 1_{\Omega E}=0$ *if, and only if,*

$$\Psi_k(E) \equiv 0 \bmod (\Omega\theta)^*[\Omega L, \Omega^2 L]+(\Omega^2\theta)_*[\Omega K, \Omega^2 K]\,.$$

Proof. 1) is obvious by (1.1) and (1.2). Consider the commutative diagram



Since $(\Omega j)^*\Omega\pi=0=i^\#_*(\Omega j)^*(i^\#_*)^{-1}\Omega\pi$ and $q^\#_*$ and the left $i^\#_*$ are monic by virtue of (1.2) and (1.3), we see that $(\Omega j)^*\Phi_k(\Omega\pi)=0$, and hence there exists $y\in[\Omega K, \Omega^2 L]$ with $(\Omega\pi)^* y\in\Phi_k(\Omega\pi)$, which shows that $\Psi_k(E)$ is non-empty. By diagram-chasing we may easily verify that $(\Omega\pi)^{*^{-1}}\operatorname{Ker}(\Omega j)_*=(\Omega\pi)^{*^{-1}}(\Omega^2\theta)_*[\Omega E, \Omega^2 K]$ coincides with $(\Omega\theta)^*[\Omega L, \Omega^2 L]+(\Omega^2\theta)_*[\Omega K, \Omega^2 K]$. The last assertion follows from 1), since $p^k 1_{\Omega E}=0$ iff $\Phi_k(\Omega\pi)=\operatorname{Ker}(\Omega j)_*$.

We note that the assignment $\theta\to\Psi_1(E)$ is dual to Toda's derivative θ ([9], *p*. 209).

2. Proofs of Theorems A and B

We may prove Corollary 1 in the introduction as follows. Let $\theta\colon K_n\to K_{n+r}$. Then, by Corollary 3.7 of [2], $\Psi_1(E)=(-1)^{n+r+1}\tilde{\theta}\iota_{n-1}$. Hence our assertion follows from 2) of Theorem 1.5.

We now consider more general situation. Let

$$
\begin{aligned}
&K=\mathop{\times}_{i=1}^{k} K_{n+s_i}\,, \quad L=\mathop{\times}_{j=1}^{m} K_{n+r_j}\,,\\
&0=s_1\leqq s_2\leqq\cdots\leqq s_k<r_1\leqq r_2\leqq\cdots\leqq r_m\leqq n-3\,,\\
&\theta=\{\theta_1,\cdots,\theta_m\}\,,\\
&\theta_j=\sum_{i=1}^{k}\pi_i^*\theta_{ji}\,, \quad \theta_{ji}\in\mathcal{A}(p)\,, \quad \deg\theta_{ji}=r_j-s_i\,,
\end{aligned}
$$

where $\pi_i\colon K\to K_{n+s_i}$ is the projection on the i-th factor. Then Theorems A and B are consequences of the following

Theorem 2.1. *Let E be the principal fibre space with the above θ as classifying class. Then $l(E)=p^2$ if, and only if, there exist j and i, $1\leqq j\leqq m$, $1\leqq i\leqq k$, such that*

$$
\tilde{\theta}_{ji}\notin\sum_{t=1}^{j-1}\mathcal{A}(p)\theta_{ti}+\sum_{t=i+1}^{k}\theta_{jt}\mathcal{A}(p)\,.
$$

Proof. Introduce the diagram

$$
\begin{array}{ccccccc}
K_{n+r_j-2} & \xrightarrow{l_j} & K_{n+r_j-2}\times K_{n+r_j-1} & \xleftarrow{\varphi} & \mathop{\times}\limits_{i=1}^{k}(K_{n+s_i-2}\times K_{n+s_i-1}) & \xrightarrow{\mathop{\times}\limits_{i=1}^{k} p_i} & \mathop{\times}\limits_{i=1}^{k} K_{n+s_i-1}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
K_{n+r_j}^{S^2} & \xrightarrow{q^\sharp} & K_{n+r_j}^{P} & \xleftarrow{\quad\theta_j^P\quad} & \mathop{\times}\limits_{i=1}^{k} K_{n+s_i}^{P} & \xrightarrow{\quad i^\sharp\quad} & K^S=\mathop{\times}\limits_{i=1}^{k} K_{n+s_i}^{S}
\end{array}
$$

where p_i denotes the projection on the second factor, l_j the injection and vertical maps are homotopy equivalences as given in Proposition 3.3 of [2]. Here we take the cofibre of $p\colon S\to S$ for P. φ is defined by

$$
\begin{aligned}
&\varphi^*(\iota_{n+r_j-2}\times 1)=\sum_{i=1}^{k}\pi_i^*[\theta_{ji}\iota_{n+s_i-2}\times 1+(-1)^{n+r_j}1\times\tilde{\theta}_{ji}\iota_{n+s_i-1}]\,,\\
&\varphi^*(1\times\iota_{n+r_j-1})=\sum_{i=1}^{k}\pi_i^*(1\times\theta_{ji}\iota_{n+s_i-1})\,.
\end{aligned}
$$

We see from Theorem 3.6 of [2] that the above diagram homotopy-commutes.

Apply $[\Omega E,\ ]$ to the above diagram. Since $\theta_j^P=\sum_{i=1}^{k}\theta_{ji}^P\pi_i^P$, $\theta^P=\{\theta_1^P,\cdots,\theta_m^P\}$ and since

$$
\begin{aligned}
&l_{j*}(\sum_{i=1}(-1)^{n+r_j}(\Omega\pi)^*(\Omega\pi_i)^*\tilde{\theta}_{ji}\iota_{n+s_i-1})\\
&\qquad=(\sum_{i=1}^{k}(-1)^{n+r_j}(\Omega\pi)^*(\Omega\pi_i)^*\tilde{\theta}_{ji}\iota_{n+s_i-1},\,0)\,,
\end{aligned}
$$

$$(\mathop{\times}_{i=1}^{k} p_i)_*(0,\ \Omega(\pi_1\pi);\ \cdots;\ 0,\ \Omega(\pi_k\pi)) = \Omega\pi\ ,$$

$$\varphi_*(0,\ \Omega(\pi_1\pi);\ \cdots;\ 0,\ \Omega(\pi_k\pi)) = (\sum_{i=1}^{k}(-1)^{n+r_j}(\Omega\pi)^*(\Omega\pi_i)^*\tilde{\theta}_{ji}\iota_{n+s_i-1},\ 0)$$

by $\theta_j(\Omega\pi)=0$, it follows that the j-th component of $\Phi_1(\Omega\pi)$ has a representative $(\Omega\pi)^*\sum_{i=1}^{k}(-1)^{n+r_j}(\Omega\pi_i)^*\tilde{\theta}_{ji}\iota_{n+s_i-1}$. Hence

$$\sum_{i=1}^{k}(-1)^{n+r_j}(\Omega\pi_i)^*\tilde{\theta}_{ji}\iota_{n+s_i-1}$$

represents the j-th component of $\Psi_1(E)$.

Now, by the Künneth theorem, we compute $(\Omega^2\theta_j)_*[\Omega K,\ \Omega^2K]+(\Omega^2\pi_j)_*(\Omega\theta)^*[\Omega L,\ \Omega^2L]$ as follows:

$$\begin{aligned}(\Omega^2\theta_j)_*[\Omega K,\ \Omega^2K] &= \sum_{t=1}^{k}(\Omega^2\theta_{jt})_*H^{n+s_t-2}(\Omega K;\ Z_p)\\ &= \{\sum_{t=1}^{k}\theta_{jt}\sum_{i=1}^{k}(\Omega\pi_i)^*\alpha_{ti}\iota_{n+s_i-1};\ \alpha_{ti}\in\mathcal{A}(p),\\ &\qquad\qquad \deg\alpha_{ti}=s_t-s_i-1\}\ ,\\ (\Omega^2\pi_j)_*(\Omega\theta)^*[\Omega L,\ \Omega^2L] &= H^{n+r_j-2}(\Omega L;\ Z_p)(\Omega\theta_1,\ \cdots,\ \Omega\theta_m)\\ &= \sum_{t=1}^{m}H^{n+r_j-2}(Z_p,\ n+r_t-1;\ Z_p)(\Omega\theta_t)\\ &= \sum_{t=1}^{m}\sum_{i=1}^{k}H^{n+r_j-2}(Z_p,\ n+r_t-1;\ Z_p)(\theta_{ti})(\Omega\pi_i)\ .\end{aligned}$$

These complete the proof of Theorem 2.1.

In connection with Corollary 1 we examine some elements in the kernel of the Kristensen map $\varepsilon\colon\mathcal{A}(2)\to\mathcal{A}(2)$. Let $Sq(i_1,\cdots,i_k)$ denote $Sq^{i_1}\cdots Sq^{i_k}$. Then, using the Adem relation $Sq(2m-1,\ m)=0$ $(m\geqq 1)$, we may easily verify

Proposition 2.2. *The following elements are in the kernel of* ε:

$$Sq(3k)+\sum_{i=1}^{k}Sq(3k-i,\ i)\ ,\quad k\geqq 1;$$

$$Sq(6k+1)+Sq(6k,\ 1)+\sum_{i=1}^{k}Sq(6k+1-2i,\ 2i)+\sum_{j=2}^{2k}Sq(6k-j,\ j,\ 1)\ ,\quad k\geqq 1;$$

$$\sum_{i=1}^{k}Sq(6k+3-2i,\ 2i+1)+\sum_{j=2}^{2k+1}Sq(6k+3-j,\ j,\ 1)\ ,\quad k\geqq 1;$$

$$Q+Sq(6k-1,2,1)+Sq(6k-2,3,1)+\sum_{j=2}^{k}Sq(6k-2j+1,2j,1)+\sum_{r=4}^{2k}Sq(6k-r,r,2),$$

where

$$Q=\begin{cases} Sq(6k+2)+Sq(6k+1,\,1)+Sq(6k,\,2)+Sq(6k-2,\,4) \\ \qquad +\sum_{i=2}^{k/2}[Sq(6k-4i+3,\,4i-1)+Sq(6k-4i+2,\,4i)] \quad \textit{for } k \textit{ even}, \\ Sq(6k-1,\,3)+\sum_{i=1}^{(k-1)/2}[Sq(6k-4i+1,\,4i+1)+Sq(6k-4i,\,4i+2] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } k \textit{ odd}; \end{cases}$$

$$R+\sum_{j=2}^{k}Sq(6k-2j+3,\,2j+1,\,1)+\sum_{r=4}^{2k+1}Sq(6k-r+3,\,r,\,2)\,,$$

where

$$R=\begin{cases} Sq(6k+5)+\sum_{i=1}^{3}Sq(6k+5-i,\,i)+\sum_{i=1}^{k/2}[Sq(6k+5-4i,\,4i) \\ \qquad +Sq(6k+4-4i,\,4i+1)] \quad \textit{for } k \textit{ even}, \\ \sum_{i=1}^{(k-1)/2}[Sq(6k-4i+3,\,4i+2)+Sq(6k-4i+2,\,4i+3)] \quad \textit{for } k \textit{ odd}. \end{cases}$$

We mention some examples. The loop-order of the fibre space with classifying class $\{Sq^{i_1}, \cdots, Sq^{i_k}\}$, $0<i_1\leqq i_2\leqq\cdots\leqq i_k$, is 4, but those of fibre spaces with classifying classes $Sq^3+Sq^2Sq^1$, $Sq^4Sq^2+Sq^2Sq^4$, $Sq^7+Sq^6Sq^1+Sq^5Sq^2+Sq^4Sq^2Sq^1$ are 2. The loop-order of the fibre space with classifying class $\{P^k, \Delta P^k\}$ $(k\geqq 1)$ is p.

3. Proof of Theorem C

First we prove 1). Introduce the commutative diagram

$$\begin{array}{ccccc} E_0 & \xrightarrow{l_0} & E & & \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ \Omega B & \xrightarrow{l} & K & \xrightarrow{\theta} & L \end{array}$$

where the square is a pull-back. Observe that the fibre of l_0 is homotopy-equivalent to that of l, i.e., ΩA. Since $\pi_0\colon E_0\to\Omega B$ is a principal fibration with $\beta=\theta l$ as classifying map, we have $l(E_0)=p^2$ by Theorem B, and hence it follows from the exact sequence

$$[\Omega E_0,\,\Omega^2 A]\longrightarrow[\Omega E_0,\,\Omega E_0]\xrightarrow{(\Omega l_0)_*}[\Omega E_0,\,\Omega E]$$

and from the $(n+r_1-2)$-connectedness of E_0 that the order of Ωl_0 is p^2 and $l(E)$ is a multiple of p^2. Also, since $l(K)=p$ by Theorem A, we see that $l(E)=p^2$.

We now proceed to prove 2) and 3). Note that, in the situation $(*)$, θ determines a secondary operation $\varphi\colon\bigcap_{i=1}^{m}\operatorname{Ker}\alpha_i\to\operatorname{Coker}\sum_{i=1}^{m}\beta_i$ associated with the relation $\sum_{i=1}^{m}\beta_i\alpha_i=0$ (cf. Adams [1], Spanier [6]). Take the cofibre P of $p^k\colon S\to S$

($k=1, 2$). Applying the functor $(\quad)^P$ to the diagram (*), we see similarly that θ^P determines a secondary operation

$$\bar{\varphi}\colon [X, A^P] \cap \operatorname{Ker} \alpha^P \to [X, L^P]/\operatorname{Im} \beta^P$$

associated with $\beta^P(\Omega\alpha)^P=0$, where

$$(\Omega\alpha)^P = \mathop{\times}_{i=1}^{m} (\alpha_i \times 1 + (-1)^{n+r_i-1}\lambda_k(1\times\tilde{\alpha}_i),\ 1\times\alpha_i)\,,$$
$$\beta^P = \sum_{i=1}^{m} (\Omega\pi_i^P)^*\{\beta_i\times 1+(-1)^{n+s}\lambda_k(1\times\tilde{\beta}_i),\ 1\times\beta_i\}\,, \quad (\lambda_1=1,\ \lambda_2=0)$$

Let $t\colon L^P \to \Omega^2 L$ denote a projection with $tq^\sharp \simeq 1$ and let $e\colon \Omega A \to A^P$, $e\colon \Omega B \to B^P$ denote injections with $i^\sharp e \simeq 1$. Then

$$\alpha^P e = \{(-1)^{n+r_1}\lambda_k\tilde{\alpha}_1, \alpha_1; \cdots; (-1)^{n+r_m}\lambda_k\tilde{\alpha}_m, \alpha_m\}\,,$$
$$t\beta^P = \sum_{i=1}^{m} (\Omega\pi_i^P)^*(\beta_i\times 1+(-1)^{n+s}\lambda_k(1\times\tilde{\beta}_i))\,.$$

Consider the following commutative diagram

$$\text{(3.1)}\qquad \begin{array}{ccccccccc}
\Omega^2 B & \xrightarrow{\Omega e} & \Omega B^P & = & \Omega B^P & & & & \\
\Omega l\downarrow & & \downarrow \bar{l} & & \downarrow l^P & & & & \\
\Omega K & \overset{f}{\dashrightarrow} & \bar{K} & \xrightarrow{\varepsilon} & K^P & \xrightarrow{\theta^P} & L^P & \xrightarrow{t} & \Omega^2 L \\
\Omega\rho\downarrow & & \downarrow \bar{\rho} & & \downarrow \rho^P & & & & \\
\Omega A & = & \Omega A & \xrightarrow{e} & A^P & \xrightarrow{\alpha^P} & B^P & &
\end{array}$$

where $\bar{\rho}$ is the pull-back of ρ^P by e, hence the principal fibration with classifying map $\alpha^P e$. We denote by $\psi_k(\theta)$ the secondary operation determined by $t\theta^P\varepsilon$, which is associated with $(t\beta^P)\Omega(\alpha^P e)=0$. Since $\alpha_i(\Omega\rho)=0$ yields $\tilde{\alpha}_i(\Omega\rho)=0$ for $k=1$ with $\tilde{\alpha}_i \in \sum_{j=1}^{i-1} \mathcal{A}(p)\alpha_j$ and since $\lambda_2=0$, we may define $\bar{\varphi}(0, \Omega\rho)$ and $\psi_k(\theta)(\Omega\rho)$. Note that $\psi_k(\theta)(\Omega\rho)$ is the first component of $\bar{\varphi}(0, \Omega\rho)$.

Lemma 3.2. *Let $k=1$ or 2. Suppose* $\deg \beta_m > 1$ *for $k=1$. Then there exists $f\colon \Omega K \to \bar{K}$ such that $\bar{\rho} f = \Omega\rho$ and $t\theta^P\varepsilon f$ represents both $\psi_k(\theta)(\Omega\rho)$ and $\Psi_k(E)$. Moreover, if $k=2$, $i^\sharp\varepsilon f \simeq 1$ and $f(\Omega l) \simeq \bar{l}(\Omega e)$.*

Proof. Assume first $k=1$ and $\deg \beta_m > 1$. Take $x\colon \Omega E \to K^P$ with $i^\sharp_* x = \pi^S$. Since $[\Omega^2 L, K^P]=0$ by $s>r_m$, we have $(\Omega j)^* x=0$, and hence we may pick $y\in[K^S, K^P]$ with $x=(\Omega\pi)^* y$. Further, since $[\Omega K, A^{S^2}]=0$, we may set $\rho^P y=(0, z)$ for $z=i^\sharp_*\rho^P y=(\Omega\rho)i^\sharp_* y$. We have

$$(0, z(\Omega\pi)) = (0, z)(\Omega\pi) = \rho^P y(\Omega\pi) = \rho^P x = (0, (\rho\pi)^S)$$

by $i^{\sharp}_{*}\rho^P x = (\rho\pi)^S$ and $[\Omega E, \Omega^2 A] = 0$. Therefore,

$$z - \Omega\rho \in \mathrm{Ker}\,(\Omega\pi)^* = (\Omega\theta)^*[\Omega L, \Omega A] = 0\,.$$

This gives rise to $\rho^P y = (0, \Omega\rho) = e(\Omega\rho)$, which yields $f\colon \Omega K \to \bar{K}$ with $\bar{\rho} f = \Omega\rho$, $\varepsilon f = y$. Now $\Phi_1(\Omega\pi)$ has, by definition, a representative $(q^{\sharp}_{*})^{-1}\theta^P(x)$. Thus

$$\Phi_1(\Omega\pi) = t_* q^{\sharp}_{*}\Phi_1(\Omega\pi) \ni t_*\theta^P(x) = t_*\theta^P y(\Omega\pi)\,.$$

This shows that $t\theta^P y = t\theta^P\varepsilon f$ represents $\Psi_1(E)$ and $\psi_1(\theta)(\Omega\rho)$.

Next let $k=2$; then, $\alpha^P e \simeq e(\Omega\alpha)$ by virtue of the expression of $\alpha^P e$, and hence one gets an induced map $\bar{e}\colon \Omega K \to K^P$ which makes the following diagram homotopy-commute:

$$\begin{array}{ccccccc}
\Omega^2 B & \xrightarrow{\Omega l} & \Omega K & \xrightarrow{\Omega\rho} & \Omega A & \xrightarrow{\Omega\alpha} & \Omega B \\
\downarrow{\scriptstyle \Omega e} & & \downarrow{\scriptstyle \bar{e}} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\
\Omega B^P & \xrightarrow{l^P} & K^P & \xrightarrow{\rho^P} & A^P & \xrightarrow{\alpha^P} & B^P \\
\downarrow{\scriptstyle i^{\sharp}} & & \downarrow{\scriptstyle i^{\sharp}} & & \downarrow{\scriptstyle i^{\sharp}} & & \downarrow{\scriptstyle i^{\sharp}} \\
\Omega^2 B & \xrightarrow{\Omega l} & \Omega K & \xrightarrow{\Omega\rho} & \Omega A & \xrightarrow{\Omega\alpha} & \Omega B
\end{array}$$

Since $i^{\sharp}e \simeq 1$, it follows from the five lemma that $i^{\sharp}\bar{e}$ is a homotopy equivalence with a homotopy inverse $\xi\colon \Omega K \to \Omega K$. Thus, by factoring $\bar{e}$, we may find $f\colon \Omega K \to \bar{K}$ such that $\bar{e}\xi = \varepsilon f$, $\bar{\rho} f = \Omega\rho$, $i^{\sharp}\varepsilon f \simeq 1$ and $\varepsilon f(\Omega l) \simeq \varepsilon\bar{l}(\Omega e)$. Since the fibre of $e\colon \Omega A \to A^P$ is homotopy-equivalent to the loop space of that of $i^{\sharp}$ by inspection of the relative mapping sequence for $i^{\sharp}e \simeq 1$ (cf. [4], Lemma 2.1 (ii)), and since the fibre of $i^{\sharp}$ is $\Omega^2 A$, we see from $[\Omega^2 B, \Omega^3 A] = 0$ that $\varepsilon_*\colon [\Omega^2 B, \bar{K}] \to [\Omega^2 B, K^P]$ is monic. This implies that $f(\Omega l) \simeq \bar{l}(\Omega e)$. $i^{\sharp}\varepsilon f \simeq 1$ implies $i^{\sharp}(\varepsilon f(\Omega\pi)) \simeq \Omega\pi$, hence $tq^{\sharp}_{*}(q^{\sharp}_{*})^{-1}\theta^P\varepsilon f(\Omega\pi)$ represents $\Phi_2(\Omega\pi)$. q.e.d.

Now let $k=1$. We observe that

$$\begin{aligned}
t_*\beta^P[\Omega K, \Omega B^P] &\supset t_*\beta^P q^{\sharp}_{*}[\Omega K, \Omega^3 B] \\
&= (\Omega^2\beta)_*[\Omega K, \Omega^3 B] \\
&= (\Omega^2\theta)_*[\Omega K, \Omega^2 K] \qquad \text{by } [\Omega K, \Omega^2 A] = 0\,,
\end{aligned}$$

and that, if $\tilde{\beta}_i \in \sum_{j=i+1}^{m} \beta_j \mathcal{A}(p)$ then

$$t_*\beta^P[\Omega K, \Omega B^P] = (\Omega^2\beta)_*[\Omega K, \Omega^3 B]\,.$$

Thus we may infer from Theorem 1.5, 2) that $\psi_1(\theta)(\Omega\rho) \not\equiv 0 \bmod t_*\beta^P[\Omega K, \Omega B^P]$

implies $p1_{\Omega E}\neq 0$. Since $\psi(\Omega\rho)$ differs from $\psi_1(\theta)(\Omega\rho)$ by an element of $(\bar{p}f)^*[\Omega A, \Omega^2 L]=(\Omega\rho)^*[\Omega A, \Omega^2 L]$, the assertions 2) and 3) of Theorem C are obtained.

Corollary 2 is obtained from 3) of Theorem C, by noting that the sequence $H^{n+s-2}(\Omega^2 B)=H^{n+s-2}(\overset{m}{\underset{i=1}{\times}} K_{n+r_i-2})\leftarrow H^{n+s-2}(\Omega K)\xleftarrow{(\Omega\rho)^*} H^{n+s-2}(\Omega A)$ is exact and $H^{n+s-2}(\Omega A)$ is contained in $\sum_{j=i+1}^{m}\beta_j\mathcal{A}(p)+\mathrm{Ker}\,(\Omega\rho)^*$.

By the way, we examine the extent to which $\psi_k(\theta)(\Omega\rho)$ may be altered with θ being a universal example of a secondary operation associated with $\beta(\Omega\alpha)=0$.

Proposition 3.3. $\psi_1(\theta+\rho^*\gamma)(\Omega\rho)=\psi_1(\theta)(\Omega\rho)\pm(\Omega\rho)^*\widetilde{\Omega\gamma}$, $\psi_2(\theta+\rho^*\gamma)(\Omega\rho)=\psi_2(\theta)(\Omega\rho)$ *for* $\gamma\in[A, L]$.

Proof. Since t can be delooped, we have

$$\begin{aligned} t(\theta^P+\gamma^P\rho^P)\varepsilon f &= t\theta^P\varepsilon f+t\gamma^P\rho^P\varepsilon f \\ &= t\theta^P\varepsilon f+t\gamma^P e(\Omega\rho) \\ &= t\theta^P\varepsilon f+(\Omega^2\gamma\times 1)e(\Omega\rho)\pm\lambda_k(1\times\widetilde{\Omega\gamma})e(\Omega\rho) \\ &= t\theta^P\varepsilon f\pm\lambda_k(\widetilde{\Omega\gamma})(\Omega\rho)\,. \end{aligned}$$

4. Proof of Theorem D

In this section let P and P' be cofibres of $p^2\colon S\to S$ and of $p\colon S\to S$ respectively. Given a generalized Eilenberg-MacLane space Z, let

$$Z^{S^2}\underset{t}{\overset{q^\#}{\rightleftarrows}} Z^P \underset{e}{\overset{i^\#}{\rightleftarrows}} Z^S$$

and

$$Z^{S^2}\underset{t'}{\overset{q'^\#}{\rightleftarrows}} Z^{P'} \underset{e'}{\overset{i'^\#}{\rightleftarrows}} Z^S$$

denote product representations.

Introduce the following commutative diagram

$$\begin{array}{ccccccc} S & \xrightarrow{p} & S & \xrightarrow{i'} & P' & \xrightarrow{q'} & S^2 \\ \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle (1,p)} & & \| \\ S & \xrightarrow{p^2} & S & \xrightarrow{i} & P & \xrightarrow{q} & S^2 \\ \downarrow & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle (0,i')} & & \downarrow \\ * & \longrightarrow & P' & = & P' & \longrightarrow & * \\ & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle (0,q')} & & \\ & & S^2 & \xrightarrow{Si'} & SP' & & \end{array} \tag{4.1}$$

in which rows and columns are Puppe sequences by the 3×3 lemma (cf. Nomura [4], Lemma 1.2) and $(1,p)$ and $(0,i')$ are induced maps.

Lemma 4.2. (4.1) *induces a fibration sequence*

$$K_n^{P'} \xleftarrow{(1,p)^\#} K_n^{P} \xleftarrow{(0,i')^\#} K_n^{P'} \xleftarrow{(0,q')^\#} K_n^{SP'}$$

which is homotopically equivalent to

$$K_{n-2}\times K_{n-1} \xleftarrow{1\times 0} K_{n-2}\times K_{n-1} \xleftarrow{0\times 1} K_{n-2}\times K_{n-1} \xleftarrow{T(0\times 1)} K_{n-3}\times K_{n-2}$$

where $T\colon K_{n-1}\times K_{n-2}\to K_{n-2}\times K_{n-1}$ *denotes the switching map.*

Proof. From the diagram (4.1) one can form the homotopy-commutative diagram

$$\begin{array}{ccccccc}
K_n^{S^2} & = & K_n^{S^2} & & & & \\
{\scriptstyle t'}\uparrow\downarrow{\scriptstyle q'^\#} & & {\scriptstyle t}\uparrow\downarrow{\scriptstyle q^\#} & & & & \\
K_n^{P'} & \xleftarrow{(1,p)^\#} & K_n^{P} & \xleftarrow{(0,i')^\#} & K_n^{P'} & \xleftarrow{(0,q')^\#} & K_n^{SP'} \\
\downarrow{\scriptstyle i'^\#} & & \downarrow{\scriptstyle i^\#} & & \| & & \downarrow{\scriptstyle (Si')^\#} \\
K_n^{S} & \xleftarrow{p^\#=0} & K_n^{S} & \xleftarrow{i'^\#} & K_n^{P'} & \xleftarrow{q'^\#} & K_n^{S^2}
\end{array}$$

Then $t'(1,p)^\#\in H^{n-2}(K_n^P; Z_p)\cong H^{n-2}(K_{n-2}\times K_{n-1}; Z_p)$ is a multiple of the projection $t\colon K_{n-2}\times K_{n-1}\to K_{n-2}$. Since $t'(1,p)^\# q^\#\simeq t'q'^\#\simeq 1$, it follows that $t'(1,p)^\#\simeq t$. This shows that $(1,p)^\#$ is essentially 1×0 and that $t(0,i')^\#\simeq t'(1,p)^\#(0,i')^\#\simeq 0$. Hence $(0,i')^\#$ is essentially 0×1 and, by $i'^\#(0,q')^\#\simeq 0$ and $t'(0,q')^\#\simeq (Si')^\#$, we see that $(0,q')^\#$ is homotopy-equivalent to $T(0\times 1)$.

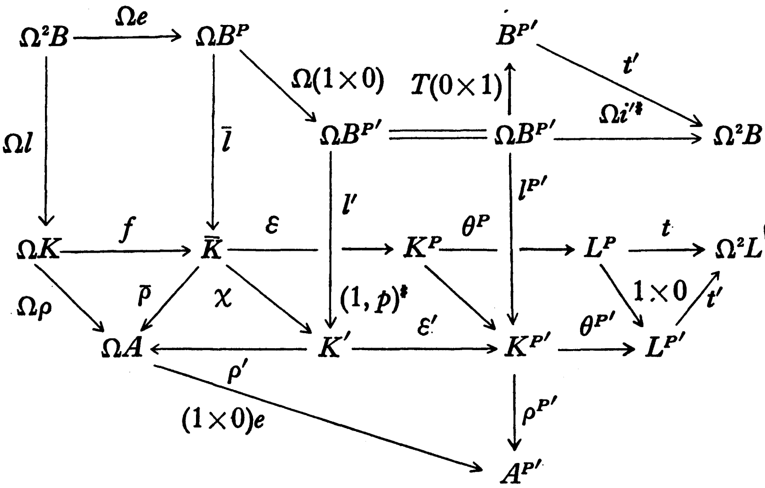
Consider now the homotopy-commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & B^{SP'} & \searrow{\scriptstyle (Si')^\#} & \\
 & & & & & & \downarrow{\scriptstyle T(0\times 1)} & & \\
 & & & & A^{P'} & \xrightarrow{\alpha^{P'}} & B^{P'} & \xrightarrow{t'} & B^{S^2} \\
 & & & & \downarrow{\scriptstyle 0\times 1} & & \downarrow{\scriptstyle 0\times 1} & & \\
\bar{K} & \xrightarrow{\bar{\rho}} & \Omega A & \xrightarrow{e} & A^{P} & \xrightarrow{\alpha^{P}} & B^{P} & & \\
{\scriptstyle \chi}\downarrow & & \| & & \downarrow{\scriptstyle 1\times 0} & & \downarrow{\scriptstyle 1\times 0} & & \\
K' & \xrightarrow{\rho'} & \Omega A & \xrightarrow{(1\times 0)e} & A^{P'} & \xrightarrow{\alpha^{P'}} & B^{P'} & &
\end{array}$$

where $\bar{K}$ is, as in (3.1), the fibre of $\alpha^P e$ and K' is the fibre of $\alpha^{P'}(1\times 0)e$. Note that K' is homotopy-equivalent to $\Omega A\times \Omega B^{P'}$ because of $(1\times 0)e\simeq 0$. The maps

1×0 induce a map $\chi\colon \bar{K}\to K'$.

Let $f\colon \Omega K\to \bar{K}$ be a map constructed in Lemma 3.2 for $k=2$. Then one gets the homotopy-commutative diagram



Since $\chi f(\Omega l)\simeq \chi\bar{l}(\Omega e)\simeq l'\Omega(1\times 0)(\Omega e)\simeq 0$ by $\Omega(1\times 0)(\Omega e)\simeq 0$, we may find $f'\colon \Omega A\to K'$ such that

$$f'(\Omega\rho)\simeq \chi f \tag{4.3}$$

and so

$$t\theta^P\varepsilon f\simeq t'\theta^{P'}\varepsilon' f'(\Omega\rho)\,. \tag{4.4}$$

Further, since $\rho^{P'}\varepsilon' f'\simeq 0$, there exists $g\colon \Omega A\to \Omega B^{P'}$ such that

$$l^{P'}g\simeq \varepsilon' f'\,. \tag{4.5}$$

Therefore, by (4.4) and $\theta l=\beta$,

$$t\theta^P\varepsilon f\simeq t'\beta^{P'}g(\Omega\rho)\,. \tag{4.6}$$

We next show that

$$T(0\times 1)g(\Omega\rho)\simeq -\alpha^{P'}e'(\Omega\rho)\,. \tag{4.7}$$

For this purpose, introduce the commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & & & & & B^{SP} \\
 & & & & & & & & \downarrow{\scriptstyle 1\times 0} \\
 & & & & & & A^{SP'} & \xrightarrow{\alpha^{SP'}} & B^{SP'} \\
 & & & & & & \downarrow & & \downarrow{\scriptstyle T(0\times 1)} \\
 & & & & & & A^{P'} & \xrightarrow{\alpha^{P'}} & B^{P'} \\
 & & & & & & \downarrow{\scriptstyle 0\times 1} & & \downarrow{\scriptstyle 0\times 1} \\
 & & \Omega B^{P} & \xrightarrow{l^{P}} & K^{P} & \xrightarrow{\rho^{P}} & A^{P} & \xrightarrow{\alpha^{P}} & B^{P} \\
 & & \downarrow{\scriptstyle \Omega(1\times 0)} & & \downarrow{\scriptstyle (1,p)^{\sharp}} & & \downarrow{\scriptstyle 1\times 0} & & \downarrow{\scriptstyle 1\times 0} \\
\Omega A^{P'} & \xrightarrow{\Omega\alpha^{P'}} & \Omega B^{P'} & \xrightarrow{l^{P'}} & K^{P'} & \longrightarrow & A^{P'} & \xrightarrow{\alpha^{P'}} & B^{P'}
\end{array}
$$

Apply the functor $[\Omega K, \quad]$ to the above diagram and observe that

$$\alpha^{P}e(\Omega\rho) \simeq \alpha^{P}e\bar{p}f \simeq \alpha^{P}\rho^{P}\varepsilon f \simeq 0\,, \quad (1\times 0)e(\Omega\rho) \simeq 0\,.$$

Since, by Lemma 3.2, (4.3) and (4.5),

$$\rho^{P}\varepsilon f = e(\Omega\rho)\,, \quad (1, p)^{\sharp}\varepsilon f \simeq l^{P'}g(\Omega\rho)\,, \quad (0\times 1)e'(\Omega\rho) \simeq e(\Omega\rho)\,,$$

we can apply two kinds of functional operations to $e(\Omega\rho)\in[\Omega K, A^{P}]$ to yield $g(\Omega\rho)\in[\Omega K, \Omega B^{P'}]$ and $[T_*(0\times 1)_*]^{-1}\alpha^{P'}e'(\Omega\rho)\in[\Omega K, B^{SP'}]$. Thus, according to Spanier [6],

$$-g(\Omega\rho) \equiv [T_*(0\times 1)_*]^{-1}\alpha^{P'}e'(\Omega\rho) \bmod \alpha^{SP'}_*[\Omega K, A^{SP'}]+(1\times 0)_*[\Omega K, B^{SP}]$$

under the adjoint isomorphism. Hence (4.7) follows from the fact that $[\Omega K, \Omega A^{P'}]=0$ and $[T(0\times 1)]_*(1\times 0)_*=0$.

We now compute, by the expression for $t'\beta^{P'}$ and $\alpha^{P'}e'$ in §3,

$$
\begin{aligned}
t'\beta^{P'}g(\Omega\rho) &= (\Omega\rho)^*g^*\sum_{j=1}^{m}\pi_j^{P'*}(\beta_j\times 1+(-1)^{n+s}1\times\tilde{\beta}_j) \\
&\equiv (\Omega\rho)^*g^*\sum_{j=1}^{m}(-1)^{n+s}\pi_j^{P'*}(1\times\tilde{\beta}_j) \bmod (\Omega\rho)^*(\Omega^2\beta)_*[\Omega A, \Omega^3 B] \\
&= (-1)^{n+s}(\Omega\rho)^*g^*\sum_{j=1}^{m}\pi_j^{P'*}(\Omega i'^{\sharp})^*\tilde{\beta}_j \\
&= (-1)^{n+s}(\Omega\rho)^*g^*(\Omega i'^{\sharp})^*\sum_{j=1}^{m}\pi_j^*\tilde{\beta}_j \\
&= (-1)^{n+s}(\Omega\rho)^*g^*(t'T(0\times 1))^*\sum_{j=1}^{m}\pi_j^*\tilde{\beta}_j \\
&= (-1)^{n+s+1}(\Omega\rho)^*(t'\alpha^{P'}e')^*\sum_{j=1}^{m}\pi_j^*\tilde{\beta}_j \\
&= (-1)^{s+1}(\Omega\rho)^*\sum_{j=1}^{m}(-1)^{r_j}\tilde{\beta}_j\tilde{\alpha}_j\,.
\end{aligned}
$$

This reveals that $(-1)^{s+1}\sum_{j=1}^{m}(-1)^{r_j}\tilde{\beta}_j\tilde{\alpha}_j$ represents $\Psi_2(E)$ by Lemma 3.2, since $(\Omega\rho)^*(\Omega^2\beta)_*[\Omega A, \Omega^2 B]$ is contained in the indeterminacy, $(\Omega^2\theta)_*[\Omega K, \Omega^2 K]=(\Omega^2\beta)_*[\Omega K, \Omega^3 B]$, of $\Psi_2(E)$. Therefore, Theorem D follows from Theorem 1.5 and from the fact $p^2 \mid l(E)$ is a consequence of the exact sequence

$$[\Omega E, \Omega K] \xleftarrow{(\Omega\pi)^*} [\Omega K, \Omega K] \xleftarrow{(\Omega\theta)^*} [\Omega L, \Omega K] = 0 .$$

Corollary 3, 1) follows from Theorem D by inspecting the exact ladder

$$\begin{array}{ccccccc}
[\Omega^2 B, \Omega^3 B] & \longleftarrow & [\Omega K, \Omega^3 B] & \xleftarrow{(\Omega\rho)^*} & [\Omega A, \Omega^3 B] & & \\
\downarrow{\scriptstyle(\Omega^2\beta)_*} & & \downarrow{\scriptstyle(\Omega^2\beta)_*} & & \downarrow{\scriptstyle(\Omega^2\beta)_*} & & \\
H^{n+s-2}(\Omega^2 B) & \longleftarrow & H^{n+s-2}(\Omega K) & \xleftarrow{(\Omega\rho)^*} & H^{n+s-2}(\Omega A) & \xleftarrow{(\Omega\alpha)^*} & H^{n+s-2}(\Omega B)
\end{array}$$

and by observing that the left hand $(\Omega^2\beta)_*$ may be identified with

$$\sum_{i=1}^{m}\beta_i\colon\ \oplus H^{n+r_i-3}(\Omega^2 B) \to H^{n+s-2}(\Omega^2 B) .$$

5. Some examples

As an illustration of Theorems C and D in the introduction, we list some relations in $\mathcal{A}(p)$ to which the theorems are applicable:

i) Relations to which Theorem C, 1), is applicable:

$$(P^k\Delta)P^{p-1} = 0 \quad (2 \leqq k < p),$$
$$(P^p\Delta)P^k + (k-1)\Delta P^{p+k} - (\Delta P^{p+k-1})P^1 = 0 \quad (1<k<p).$$

ii) Relations to which Theorem C, 2) is applicable:

$$(\Delta P^{kp})P^{k-1} - P^{kp}(\Delta P^{k-1}) - P^{kp-1}(\Delta P^k) = 0 \quad (k \geqq 2,\ k \not\equiv 0 \bmod p,$$
$$p>3,\ k<(p^{2p-4}+2p-3)(p^2-1)^{-1}).$$

iii) Relations to which Corollary 2 is applicable:

$$P^{p-1}P^1 = 0 \quad (p>3),$$
$$P^pP^{p+2} - P^{2p+1}P^1 = 0 .$$

iv) Relations to which Corollary 3, 2) is applicable:

$$Sq^{2k-1}Sq^{k-1} + Sq^{2k-2}Sq^k = 0 \quad (k \geqq 2),$$
$$Sq^{2k-1}Sq^{k-3} + Sq^{2k-2}Sq^{k-2} + Sq^{2k-4}Sq^k = 0 \quad (k \geqq 4),$$
$$Sq^{2k-1}Sq^{k-5} + Sq^{2k-2}Sq^{k-4} + Sq^{2k-3}Sq^{k-3} + Sq^{2k-6}Sq^k = 0 \quad (k \geqq 6).$$

v) Relations to which Corollary 3, 1) is applicable:

$$l(E)=8 \quad \text{iff} \quad Sq^{2k-2}Sq^{k-1} \notin \mathcal{A}(2)Sq^{k}+Sq^{2k-1}\mathcal{A}(2)$$
$$\text{for} \quad Sq^{2k-1}Sq^{k}=0 \quad (k \geqq 1),$$
$$l(E)=8 \quad \text{iff} \quad Sq^{2k-2}Sq^{k-7} \notin \mathcal{A}(2)Sq^{k-6}+\mathcal{A}(2)Sq^{k-4}+\mathcal{A}(2)Sq^{k}$$
$$+Sq^{2k-1}\mathcal{A}(2)+Sq^{2k-3}\mathcal{A}(2)+Sq^{2k-7}\mathcal{A}(2)$$
$$\text{for} \quad Sq^{2k-1}Sq^{k-6}+Sq^{2k-3}Sq^{k-4}+Sq^{2k-7}Sq^{k}=0 \quad (k \geqq 9).$$

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