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A NOTE ON THE RELATION OF Z_2-GRADED COMPLEX COBORDISM TO COMPLEX K-THEORY

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Let $MU^*(\ )$ and $K^*(\ )$ denote the Z_2-graded complex cobordism theory and the complex K-theory respectively. The Thom homomorphism μ_{c*}: $\pi_0(MU)\to \pi_0(K)$ on coefficient groups is identified (up to sign) with the classical Todd genus Td: $\Lambda\to Z$. We denote by I the ideal of Λ to be the kernel of Td: $\Lambda\to Z$. Wolff [7] proved that the decreasing filtration $\{I^q MU^*(\ )\}$ of $MU^*(\ )$ consists of cohomology theories defined on the category of based finite CW-complexes, and the associated quotients $I^q MU^*(\ )/I^{q+1}MU^*(\ )$ are determined by the complex K-theories $KG_q^*(\ )$ with coefficients $G_q=I^q/I^{q+1}$.

The purpose of this note is to extend the Wolff's result to the category of based CW-complexes. Let F_qMU be the CW-spectrum associated with the cohomology theory $I^qMU^*(\ )$, i.e., $\{Y, F_qMU\}^*\cong I^qMU^*(Y)$ for any based finite CW-complex (or finite CW-spectrum). We show that $\{F_qMU^*(\ )\}$ is a decreasing filtration of $MU^*(\ )$ consisting of Λ-modules so that the associated quotients are equal to $KG_q^*(\ )$, and in addition that $F_{q+1}MU^*(\ )$ is a direct summand of $F_qMU^*(\ )$.

Moreover we give a tower

$$MU\to\cdots\to Q_qMU\to Q_{q-1}MU\to\cdots\to Q_0MU=K$$

of MU-module spectra such that $KG_q\to Q_qMU\to Q_{q-1}MU$ is a cofiber sequence of MU-module spectra, which factorizes the Thom map μ_c: $MU\to K$.

Baas [3] constructed a tower of CW-spectra

$$MU\to\cdots\to MU\langle n\rangle\to MU\langle n-1\rangle\to\cdots\to MU\langle 0\rangle=H$$

factorizing the Thom map μ: $MU\to H$. In appendix we show that the tower is of MU-module spectra and the sequence $\Sigma^{2n}MU\langle n\rangle\xrightarrow{m_{x_n}}MU\langle n\rangle\to MU\langle n-1\rangle$ is a cofiber sequence where m_{x_n} is the multiplication by x_n a ring generator of Λ with degree $2n$.

1. Decreasing filtration of $MU_*(\ )$

1.1. A pair (E, ρ) is called a *Z_2-graded CW-spectrum* if E is a CW-spectrum and $\rho\colon \Sigma^2 E \to E$ is a homotopy equivalence. Such a pair (E, ρ) gives rise to natural isomorphisms

$$\rho_*\colon E_*(X) \to E_{*+2}(X)\,, \quad \rho_*\colon E^{*+2}(X) \to E^*(X)$$

for any CW-spectrum X. So we can define Z_2-graded homology and cohomology theories $E_\sharp(\ )$, $E^\sharp(\ )$ by putting

$$E_\sharp(X) = E_0(X) \oplus E_1(X)\,, \quad E^\sharp(X) = E^0(X) \oplus E^1(X)\,.$$

For a CW-spectrum E we put

$$\boldsymbol{E} = \bigvee_n \Sigma^{2n} E\,, \quad \bar{\boldsymbol{E}} = \prod_n \Sigma^{2n} E\,.$$

Taking the canonical identifications $\rho\colon \Sigma^2 \boldsymbol{E} \to \boldsymbol{E}$ and $\bar{\rho}\colon \Sigma^2 \bar{\boldsymbol{E}} \to \bar{\boldsymbol{E}}$ as structure morphisms $\boldsymbol{E}$ and $\bar{\boldsymbol{E}}$ admit structures of Z_2-graded CW-spectra respectively. From definition it follows that

$$\boldsymbol{E}_0(X) \cong \sum_n E_{2n}(X)\,, \quad \boldsymbol{E}_1(X) \cong \sum_n E_{2n+1}(X)\,,$$

$$\bar{\boldsymbol{E}}^0(X) \cong \prod_n E^{2n}(X)\,, \quad \bar{\boldsymbol{E}}^1(X) \cong \prod_n E^{2n+1}(X)$$

for all CW-spectra X. In particular, the canonical morphism $\boldsymbol{H} \to \bar{\boldsymbol{H}}$ becomes a homotopy equivalence for the Eilenberg-MacLane spectrum H.

The BU-spectrum K may be regarded as a Z_2-graded CW-spectrum because it possesses the Bott map $\beta\colon \Sigma^2 K \to K$ which is a homotopy equivalence.

Denote by F_n the direct sum of n-copies of the integers Z and by F the direct limit of F_n, i.e., F is a free abelian group with countably many factors. Putting

$$BU_{F_n} = BU \times \cdots \times BU, \text{ the product of } n\text{-copies of } BU,$$
$$BU_F = \bigcup_n BU_{F_n}\,, \text{ the union of } BU_{F_n},$$

we obtain

Proposition 1. *There exists a natural isomorphism*

$$[X, BU_F] \to KF^0(X)$$

for any based connected CW-complex X.

Proof. Let Y be a based connected finite CW-complex. Then we have a sequence of natural isomorphisms

$$[Y, BU_F] \leftarrow \varinjlim [Y, BU_{F_n}] \leftarrow \varinjlim [Y, BU] \otimes F_n \to \varinjlim K^0(Y) \otimes F_n$$
$$\to K^0(Y) \otimes F \to KF^0(Y)\,.$$

Therefore the contravariant functor KF^0 defined on the category of based connected CW-complexes is represented by BU_F (use [1, Addendum 1.5]).

Proposition 1 implies that BU_F is homotopy equivalent to $\Omega_0^2 BU_F$ where Ω_0^2 means the component of the base point in the double loop space. Hence we have

(1.1) *in the BU-spectrum KF with the coefficients F every even term is the based CW-complex* BU_F.

1.2. Let us denote by MU the unitary Thom spectrum and by μ_c: $MU \to K$ the Thom map which is a ring morphism. The composition

$$\boldsymbol{\mu}_c\colon \boldsymbol{MU} \to \boldsymbol{K} \to K$$

of $\vee \Sigma^{2n}\mu_c$ and $\vee \beta^n$ is a morphism of Z_2-graded ring-spectra, called the *Z_2-graded Thom map*. As is well known, it is characterized by the coefficient homomorphism $\boldsymbol{\mu}_{c\sharp}$: $\pi_\sharp(\boldsymbol{MU}) \to \pi_\sharp(K)$ which coincides (up to sign) with the classical Todd genus Td. Putting $\Lambda = \pi_\sharp(MU)$, $\pi_\sharp(K) = Z$ is viewed as a Z_2-graded Λ-module via $\boldsymbol{\mu}_{c\sharp} = Td$ and it is written Z_{Td} for emphasis.

Using the kernel I of Td: $\Lambda \to Z$ we define a decreasing filtration $\{I^q\}_{q\geq 0}$ consisting of ideals of Λ. Denoting by G_q the associated Z_2-graded Λ-module I^q/I^{q+1}, we see easily [7, Satz 3.8] that

(1.2) $G_0 \cong Z_{Td}$ *and* G_q *is a free abelian group with countably many factors for* $q \geqq 1$.

For a Z_2-graded Λ-module A we have a decreasing filtration $\{I^q A\}_{q\geq 0}$ consisting of submodules of A, whose associated Z_2-graded Λ-module $I^q A/I^{q+1}A$ is written $G_q(A)$. Applying the commutative diagram

$$\begin{array}{ccccccccc} 0 \to \mathrm{Tor}_1^\Lambda(Z_{Td}, A) & \to & I \underset{\Lambda}{\otimes} A & \to & A & \to & Z_{Td} \underset{\Lambda}{\otimes} A & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ & 0 \to & IA & \to & A & \to & G_0(A) & \to & 0 \end{array}$$

with exact rows, we get an isomorphism

$$G_q \underset{\Lambda}{\otimes} A \overset{\simeq}{\to} G_q \underset{Z}{\otimes} (Z_{Td} \underset{\Lambda}{\otimes} A) \overset{\simeq}{\to} G_q \underset{Z}{\otimes} G_0(A) \tag{1.3}$$

by means of "4 lemma".

Proposition 2. *Let A be a Λ-module with* $\mathrm{Tor}_k^\Lambda(Z_{Td}, A) = 0$ *for all* $k \geqq 1$. *Then, for every* $q \geqq 0$ *both* $I^q \underset{\Lambda}{\otimes} A \to I^q A$ *and* $G_q \underset{\Lambda}{\otimes} A \to G_q(A)$ *are isomorphisms and* $\mathrm{Tor}_k^\Lambda(I^q, A) = \mathrm{Tor}_k^\Lambda(G_q, A) = 0$ *for all* $k \geqq 1$.

Proof. Choose a free Λ-module F such that A is isomorphic to a quotient F/B. By induction on q we shall show that the sequences

$$0 \to I^q B \to I^q F \to I^q A \to 0, \quad 0 \to G_q(B) \to G_q(F) \to G_q(A) \to 0$$

are exact. The $q=0$ case is evident because of (1.3). Applying induction hypotesis and "3×3 lemma" we find easily that $0\to I^qB\to I^qF\to I^qA\to 0$ is exact. So we have a commutative diagram

$$\begin{array}{ccccccc} 0 \to & G_q\underset{Z}{\otimes}G_0(B) & \to & G_q\underset{Z}{\otimes}G_0(F) & \to & G_q\underset{Z}{\otimes}G_0(A) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & G_0(I^qB) & \to & G_0(I^qF) & \to & G_0(I^qA) & \to 0 \end{array}$$

with exact rows. Since all vertical arrows are epimorphisms and in particular the central one is an isomorphism, all vertical arrows become isomorphisms. Consequently we get that $0\to G_q(B)\to G_q(F)\to G_q(A)\to 0$ is exact.

Next, we consider the commutative diagrams

$$\begin{array}{cccccccc} 0 \to \mathrm{Tor}_1^\Lambda(I^q, A) \to & I^q\underset{\Lambda}{\otimes}B & \to & I^q\underset{\Lambda}{\otimes}F & \to & I^q\underset{\Lambda}{\otimes}A & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & I^qB & \to & I^qF & \to & I^qA & \to 0 \end{array}$$

$$\begin{array}{cccccccc} 0 \to \mathrm{Tor}_1^\Lambda(G_q, A) \to & G_q\underset{\Lambda}{\otimes}B & \to & G_q\underset{\Lambda}{\otimes}F & \to & G_q\underset{\Lambda}{\otimes}A & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & G_q\underset{Z}{\otimes}G_0(B) & \to & G_q\underset{Z}{\otimes}G_0(F) & \to & G_q\underset{Z}{\otimes}G_0(A) & \to 0 \end{array}$$

with exact rows. Remark that $\mathrm{Tor}_k^\Lambda(Z_{Td}, B)=0$ for all $k\geqq 1$. By use of "4 lemma" and (1.3) we see that all vertical arrows are isomorphisms, and hence we obtain the required results.

For a Λ-module A we put $J_q(A)=A/I^{q+1}A$ and abbreviate $J_q=J_q(\Lambda)$ when $A=\Lambda$. As an immediate corollary of Proposition 2 we have

Corollary 3. *Let A be a Λ-module with $\mathrm{Tor}_k^\Lambda(Z_{Td}, A)=0$ for all $k\geqq 1$. Then $J_q\otimes A\to J_q(A)$ is an isomorphism and $\mathrm{Tor}_k^\Lambda(J_q, A)=0$ for all $k\geqq 1$.*

1.3. Let $\mathcal{MU}$ denote the category of comodules over $MU_*(MU)$ which are finitely presented as Λ-modules. Notice that $\mathcal{MU}$ is an abelian category which has enough projectives, and also that $MU_*(Y)$ lies in the category $\mathcal{MU}$ whenever Y is a finite CW-spectrum. Since the functor $M\to Z_{Td}\underset{\Lambda}{\otimes}M$ is exact on $\mathcal{MU}$ [5, Example 3.3] it follows immediately that

(1.4) $\qquad \mathrm{Tor}_k^\Lambda(Z_{Td}, M)=0 \quad$ *for all $k\geqq 1$ if M lies in $\mathcal{MU}$.*

Proposition 1 and Corollary 2 combined with (1.4) say that

(1.5) *the functors $M\to I^qM\simeq I^q\underset{\Lambda}{\otimes}M$, $M\to G_q(M)\simeq G_q\underset{\Lambda}{\otimes}M$ and $M\to J_q(M)\simeq J_q\underset{\Lambda}{\otimes}M$ on $\mathcal{MU}$ are exact.*

Theorem 1 (Wolff [7]). i) *Both $I^qMU_*(\ )$ and $MU_*(\ )/I^{q+1}MU_*(\ )$ are* *homology theories defined on the category of CW-spectra, so that* $I^q \underset{\Lambda}{\otimes} MU_*(X) \to I^q MU_*(X)$ *and* $\Lambda/I^{q+1} \underset{\Lambda}{\otimes} MU_*(X) \to MU_*(X)/I^{q+1}MU_*(X)$ *are natural isomorphisms for all CW-spectra X.*

ii) $I^q MU_*(\ )/I^{q+1}MU_*(\ )$ *is a homology theory defined on the category of CW-spectra such that there exists a natural isomorphism* $I^q MU_*(X)/I^{q+1}MU_*(X) \to KG_{q*}(X)$ *for any CW-spectrum X which is induced by the* Z_2*-graded Thom map* μ_c.

Proof. i) and the first half of ii) are immediate from (1.5). The latter half of ii) is also valid because we have a natural isomorphism

$$G_q(MU_*(X)) \overset{\simeq}{\leftarrow} G_q \underset{\Lambda}{\otimes} MU_*(X) \overset{\simeq}{\rightarrow} G_q \underset{Z}{\otimes} (Z_{Td} \underset{\Lambda}{\otimes} MU_*(X))$$
$$\overset{\simeq}{\rightarrow} G_q \underset{Z}{\otimes} K_*(X) \overset{\simeq}{\rightarrow} KG_{q*}(X).$$

Let ϕ: $E_*(X) \to F_*(X)$ be a natural transformation for any CW-spectrum X. According to [1, Addendum 1.5] there exists a morphism f: $E \to F$ inducing ϕ, and it is unique up to weak homotopy. The proof in [1] is actually given for the category of based connected CW-complexes, but it is easily extended to that of CW-spectra. Such a morphism f is uniquely chosen (up to homotopy) under the assumption that $F^0(E)$ is Hausdorff.

Let $E_*(\ )$ be a Z_2-graded homology theory defined on the category of CW-spectra, i.e., a homology theory equipped with a natural isomorphism $E_*(X) \to E_{*+2}(X)$ for any CW-spectrum X. Then it gives E a structure of Z_2-graded CW-spectrum. In particular, the induced structure is uniqe if $E^0(E)$ is Hausdorff.

Recall that the Z_2-graded CW-spectrum MU is equipped with the canonical identification ρ: $\Sigma^2 MU \to MU$ as structure morphism. Since $MU^\sharp(MU)$ is Hausdorff (use Proposition 6 below), the Z_2-graded CW-spectrum (MU, ρ) is characterized only by the Z_2-graded homology theory $MU_*(\ )$.

Denote by F_qMU and Q_qMU the representing spectra of the new homology theories $I^qMU_*(\ )$ and $MU_*(\ )/I^{q+1}MU_*(\ )$ respectively, i.e.,

$$I^qMU_*(X) \cong \{\Sigma^*, F_qMU \wedge X\}, \quad MU_*(X)/I^{q+1}MU_*(X) \cong \{\Sigma^*, Q_qMU \wedge X\}$$

for any CW-spectrum X. Of course, they are both Z_2-graded CW-spectra. Then there exist morphisms

$$i_q\colon F_{q+1}MU \to F_qMU, \quad j_q\colon Q_qMU \to Q_{q-1}MU,$$
$$\iota_q\colon F_{q+1}MU \to MU, \quad \pi_q\colon MU \to Q_qMU$$

which induce the canonical morphisms in homology groups, and moreover we have morphisms

$$\mu_q\colon F_qMU \to KG_q\,,\quad \nu_q\colon KG_q \to Q_qMU$$

such that $\mu_{q*}\colon F_qMU_*(X)\to KG_{q*}(X)$ and $\nu_{q*}\colon KG_{q*}(X)\to Q_qMU_*(X)$ in homology groups are natural homomorphisms induced by the Z_2-graded Thom map μ_c.

Lemma 4. *Let $E\xrightarrow{f}F\xrightarrow{q}G$ be a sequence which satisfies the property that $0\to E_*(X)\to F_*(X)\to G_*(X)\to 0$ is a short exact sequence for every CW-spectrum X. Then it is a cofiber sequence.*

Proof. Let C_f be the mapping cone of f, i.e., $E\to F\to C_f$ a cofiber sequence. Then for any CW-spectrum X we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E_*(X) & \to & F_*(X) & \to & C_{f*}(X) & \to & 0 \\ & & \| & & \| & & \downarrow{\scriptstyle \phi=h_*} & & \\ 0 & \to & E_*(X) & \to & F_*(X) & \to & G_*(X) & \to & 0 \end{array}$$

with exact rows. Clearly $h\colon C_f\to G$ which induces ϕ is a homotopy equivalence.

By virtue of Lemma 4 we verify that

(1.6) *$F_{q+1}MU\to MU\to Q_qMU$, $F_{q+1}MU\to F_qMU\to KG_q$ and $KG_q\to Q_qMU\to Q_{q-1}MU$ are all cofiber sequences.*

2. Z_2-graded MU-module spectra

2.1. The inclusion $Z\subset Q$ induces a natural transformation $ch\colon E^*(X)\to EQ^*(X)$ for any CW-spectrum X, called the Chern-Dold character.

Proposition 5. *If $ch\colon E^*(X)\to EQ^*(X)$ is a monomorphism, then $E^*(X)$ is Hausdorff.*

Proof. Since $EQ^*(X)$ is always Hausdorff [8, Proposition 4], the result is immediate.

Let W be a connective CW-spectrum with $H_*(W)$ free and assume that $\pi_*(E)$ is torsion free. Then $H^*(W;\pi_*(E))\to H^*(W;\pi_*(E)\otimes Q)$ is a monomorphism, and hence the Atiyah-Hirzebruch spectral sequences for $E^*(W)$ and $EQ^*(W)$ collapse. Therefore we get that

(2.1) *$ch\colon E^*(W)\to EQ^*(W)$ is a monomorphism.* (Cf., [8, Lemma 11]).

Applying Proposition 5 we obtain

Proposition 6. *Let W be a connective CW-spectrum with $H_*(W)$ free. If $\pi_*(E)$ is torsion free, then $E^*(W)$ is Hausdorff.*

By means of Proposition 5 we get the following lemmas.

Lemma 7. *Assume that* $\pi_0(E)$ *is torsion free and* $\pi_1(E)=0$. *Then* $E^0(KG_q \wedge MU \wedge \cdots \wedge MU)$ *and* $E^0(Q_q MU \wedge MU \wedge \cdots \wedge MU)$ *are Hausdorff and* $E^1(F_q MU \wedge MU \wedge \cdots \wedge MU)=0$.

Proof. Since $E^{2n-1}(BU_{G_q} \wedge MU \wedge \cdots \wedge MU)=EQ^{2n-1}(BU_{G_q} \wedge MU \wedge \cdots \wedge MU)=0$ we have a commutative square

$$\begin{array}{ccc} E^0(KG_q \wedge MU \wedge \cdots \wedge MU) & \rightarrow & \varprojlim E^{2n}(BU_{G_q} \wedge MU \wedge \cdots \wedge MU) \\ \downarrow & & \downarrow \\ EQ^0(KG_q \wedge MU \wedge \cdots \wedge MU) & \rightarrow & \varprojlim EQ^{2n}(BU_{G_q} \wedge MU \wedge \cdots \wedge MU) \end{array}$$

such that the horizontal arrows are isomorphisms. The left arrow is a monomorphism because so is the right one by use of (2.1). Since Theorem 1 implies that $0 \rightarrow EQ^*(Q_{q-1}MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^*(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^*(KG_q \wedge MU \wedge \cdots \wedge MU) \rightarrow 0$ is exact, an induction on q involving "4 lemma" shows that ch: $E^0(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^0(Q_q MU \wedge MU \wedge \cdots \wedge MU)$ is a monomorphism. Then we find that $E^0(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow E^0(MU \wedge \cdots \wedge MU)$ is a monomorphism because so is $EQ^0(Q_q MU \wedge MU \wedge \cdots \wedge MU) \rightarrow EQ^0(MU \wedge \cdots \wedge MU)$. Therefore $E^1(MU \wedge \cdots \wedge MU) \rightarrow E^1(F_{q+1}MU \wedge MU \wedge \cdots \wedge MU)$ is an epimorphism, and hence $E^1(F_{q+1}MU \wedge MU \wedge \cdots \wedge MU)=0$.

Lemma 8. $KG_p^0(F_q MU \wedge MU \wedge \cdots \wedge MU)$ *and* $Q_p MU^0(F_q MU \wedge MU \wedge \cdots \wedge MU)$ *are Hausdorff and* $KG_p^1(F_q MU \wedge MU \wedge \cdots \wedge MU)=Q_p MU^1(F_q MU \wedge MU \wedge \cdots \wedge MU)=0$.

Proof. Putting $X=F_q MU \wedge MU \wedge \cdots \wedge MU$, we note by Theorem 1 i) that $K_0(X)$ is free and $K_1(X)=0$. Applying the universal coefficient sequence

$$0 \rightarrow \mathrm{Ext}(K_{\sharp-1}(X), G_p) \rightarrow KG_p^\sharp(X) \rightarrow \mathrm{Hom}(K_\sharp(X), G_p) \rightarrow 0$$

for K (see [9, (3.1)]) we get immediately that ch: $KG_p^0(X) \rightarrow KG_p \otimes Q^0(X)$ is a monomorphism and $KG_p^1(X)=0$. By induction on p we obtain that ch: $Q_p MU^0(X) \rightarrow Q_p MUQ^0(X)$ is a monomorphism and $Q_p MU^1(X)=0$ because $0 \rightarrow KG_p \otimes Q^*(X) \rightarrow Q_p MUQ^*(X) \rightarrow Q_{p-1}MUQ^*(X) \rightarrow 0$ is exact.

Assume that $\pi_*(E)$ is free and of finite type and put again $X=F_q MU \wedge MU \wedge \cdots \wedge MU$. Using the universal coefficient sequence for E [9, (1.8)] we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathrm{Ext}(\hat{E}_{*-1}(X), Z) & \rightarrow & E^*(X) & \rightarrow & \mathrm{Hom}(\hat{E}_*(X), Z) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathrm{Ext}(\hat{E}_{*-1}(X), Q) & \rightarrow & EQ^*(X) & \rightarrow & \mathrm{Hom}(\hat{E}_*(X), Q) & \rightarrow & 0 \end{array}$$

with exact rows where $\hat{E}$ is the dual of E constructed in [9]. Note that $\pi_*(\hat{E})$ is free and hence so is $\hat{E}_*(X)$. Then the central arrow becomes a monomorphism. Considering the commutative square

$$\begin{array}{ccc} \bar{E}^{*}(X) & \rightarrow & \prod E^{n}(X) \\ \downarrow & & \downarrow \\ \bar{E}Q^{*}(X) & \rightarrow & \prod EQ^{n}(X) \end{array}$$

in which the upper arrow is an isomorphism, we find that the left one is a monomorphism. Thus we get that

(2.2) $\bar{E}^{*}(F_q MU \wedge MU \wedge \cdots \wedge MU)$ *is Hausdorff.*

Let $\widetilde{MU}$ denote the mapping cone of the canonical morphism $MU \rightarrow \overline{MU}$. Since $\prod Z/\sum Z \rightarrow \prod Q/\sum Q$ is a monomorphism we remark that

(2.3) $\pi_0(\widetilde{MU}) = \prod_n \pi_{2n}(MU)/\sum_n \pi_{2n}(MU)$ *is torsion free and* $\pi_1(\widetilde{MU}) = 0$,

(see [4, Exercise IV 20]). Then $\widetilde{MU}$ has a unique structure of Z_2-graded CW-spectrum so that the cofiber sequence $MU \rightarrow \overline{MU} \rightarrow \widetilde{MU}$ is of Z_2-graded CW-spectra.

Lemma 9. *$F_p MU^0(F_q MU \wedge MU \wedge \cdots \wedge MU)$ is Hausdorff and $F_p MU^1(F_q MU \wedge MU \wedge \cdots \wedge MU) = 0$*

Proof. We put $X = F_q MU \wedge MU \wedge \cdots \wedge MU$. From Lemma 7 it follows that $F_p MU^1(X) = 0$. In the sequence

$$F_p MU^0(X) \rightarrow MU^0(X) \rightarrow \overline{MU}^0(X)$$

the former arrow is a monomorphism because of Lemma 8 and the latter one is so by means of (2.3) and Lemma 7. Thus the above composition is a monomorphism. On the other hand, (2.2) says that $\overline{MU}^0(X)$ is Hausdorff. So we get the remaining result.

2.2. Since $F_q MU^0(F_q MU)$ and $Q_q MU^0(Q_q MU)$ are both Hausdorff, we verify that

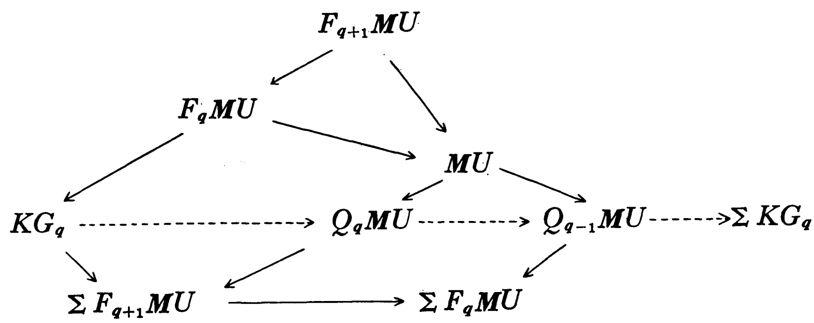
(2.4) *the Z_2-graded homology theories $I^q MU_*(\ )$ and $MU_*(\ )/I^{q+1} MU_*(\ )$ give $F_q MU$ and $Q_q MU$ unique structures of Z_2-graded CW-spectra respectively.*

Moreover, by virtue of Lemmas 7, 8 and 9 we see that

$$(2.5) \quad \begin{array}{lll} i_q\colon F_{q+1} MU \rightarrow F_q MU, & \iota_q\colon F_{q+1} MU \rightarrow MU, & \mu_q\colon F_q MU \rightarrow KG_q \\ j_q\colon Q_q MU \rightarrow Q_{q-1} MU, & \pi_q\colon MU \rightarrow Q_q MU, & \nu_q\colon KG_q \rightarrow Q_q MU \end{array}$$

are uniquely determined (up to homotopy), which induce the canonical morphisms in homology groups. In particular, the composition $\iota_{q-1} \cdot i_q$ is homotopic to ι_q and $j_q \cdot \pi_q$ is so to π_{q-1}.

Consider the diagram



consisting of cofiber sequences. With an application of Verdier's lemma (see [2, Lemma 6.8]) we get a cofiber sequence $KG_q \to Q_qMU \to Q_{q-1}MU \to \Sigma KG_q$ (denoted by dotted arrows in the above diagram) which makes the diagram homotopy commutative. Clearly this yields the canonical exact sequence $0 \to KG_{q*}(X) \to Q_qMU_*(X) \to Q_{q-1}MU_*(X) \to 0$. By uniqueness of ν_q, j_q the above cofiber sequence coincides with $KG_q \xrightarrow{\nu_q} Q_qMU \xrightarrow{j_q} Q_{q-1}MU \to \Sigma KG_q$.

The multiplication ϕ: $MU \wedge MU \to MU$ gives rise to natural Z_2-graded homomorphisms

$$m_q\colon\ F_qMU_\#(X) \otimes MU_\#(Y) \to F_qMU_\#(X \wedge Y)$$
$$\bar{m}_q\colon\ Q_qMU_\#(X) \otimes MU_\#(Y) \to Q_qMU_\#(X \wedge Y)$$

for all CW-spectra X and Y. By use of Lemmas 7 and 9 there exist unique pairings

$$\phi_q\colon\ F_qMU \wedge MU \to F_qMU\,, \quad \bar{\phi}_q\colon\ Q_qMU \wedge MU \to Q_qMU$$

which induce the above m_q and $\bar{m}_q$ respectively. Then it follows that

(2.6) *both F_qMU and Q_qMU are (associative) Z_2-graded MU-module spectra.*

Proposition 10. *Let M be a Z_2-graded ring spectrum, E, F and G Z_2-graded M-module spectra and $E \to F \to G$ a cofiber sequence. Assume that $E^0(E)$, $E^0(E \wedge M)$, $G^0(F)$ and $G^0(F \wedge M)$ are Hausdorff, or that $F^0(E)$, $F^0(E \wedge M)$, $G^0(G)$ and $G^0(G \wedge M)$ are Hausdorff. If for any CW-spectrum X $0 \to E_\#(X) \to F_\#(X) \to G_\#(X) \to 0$ is a short exact sequence of Z_2-graded $M_\#(\ )$-modules, then the cofiber sequence $E \to F \to G$ is of Z_2-graded M-module spectra.*

Proof. Assuming that $F^0(E)$, $F^0(E \wedge M)$, $G^0(G)$ and $G^0(G \wedge M)$ are Hausdorff, we consider the diagrams

$$\begin{array}{ccccccc} \Sigma^2 E & \to & \Sigma^2 F & \to & \Sigma^2 G & \to & \Sigma^3 E \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ E & \to & F & \to & G & \to & \Sigma E \end{array} \qquad \begin{array}{ccccccc} E \wedge M & \to & F \wedge M & \to & G \wedge M & \to & \Sigma E \wedge M \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ E & \to & F & \to & G & \to & \Sigma E \end{array}$$

with cofiber sequences. Under the first two assumptions two left squares become homotopy commutative because they induce the Z_2-graded homomorphism $E_\sharp(\ )\to F_\sharp(\ )$ of $M_\sharp(\ )$-modules. Therefore there exist morphisms $\Sigma^2 G\to G$ and $G\wedge M\to G$ which make the above diagrams into morphisms of cofiber sequences. As is easily cheked, they give $G_\sharp(\ )$ a structre of Z_2-graded $M_\sharp(\ )$-module, which coincides with the original one. So, using the remaining assumptions again we see that the above morphisms are homotopic to the given ones respectively. Consequently the cofiber sequence $E\to F\to G$ becomes the required one.

Another case is similarly proved.

The ring spectrum K may be regarded as a Z_2-graded MU-module spectrum via the Z_2-graded Thom map μ_c: $MU\to K$.

Applying Proposition 10 to three cofiber sequences of (1.6) we get

Theorem 2. *The sequences $F_{q+1}MU\to MU\to Q_qMU$, $F_{q+1}MU\to F_qMU\to KG_q$ and $KG_q\to Q_qMU\to Q_{q-1}MU$ are cofiber sequences of Z_2-graded MU-module spectra.*

Proof. The assumptions needed in Proposition 10 are satisfied by Lemmas 7,8 and 9.

As a result we have a tower

$$MU\to\cdots\to Q_qMU\to Q_{q-1}MU\to\cdots\to Q_0MU=K \tag{2.7}$$

of Z_2-graded MU-module spectra such that $KG_q\to Q_qMU\to Q_{q-1}MU$ is a cofiber sequence, which factorizes the Z_2-graded Thom map μ_c: $MU\to K$.

2.3. Here we extend the Wolff's result to the case of based CW-complexes.

Proposition 11. *There exists an (unstable) natural homomorphism*

$$\Phi_q\colon KG_q^*(X)\to F_qMU^*(X)$$

for any based CW-complex X, which satisfies the equality that $\mu_{q*}\cdot\Phi_q=\mathrm{id}$.

Proof. We may assume that X is connected. Let i: $BU_{G_q}\to KG_q$ be the inclusion. Then we can choose a morphism c_q: $BU_{G_q}\to F_qMU$ such that i is homotopic to the composition $\mu_q\cdot c_q$, because $F_{q+1}MU^1(BU_{G_q})=0$. In the commutative diagram

$$\begin{array}{ccccc} [X, BU_{G_q}] \xrightarrow{J_0} \{X, BU_{G_q}\} & \xrightarrow{i_*} & \{X, KG_q\} & = & KG_q^0(X) \\ & {\scriptstyle c_{q*}}\searrow & \uparrow {\scriptstyle \mu_{q*}} & & \\ & & \{X, F_qMU\} & = & F_qMU^0(X) \end{array}$$

the composition $i_* \cdot J_0$ is an isomorphism because of Proposition 2 (see [6, Theorem 14.5]). So we put that $\Phi_q = c_{q*} \cdot J_0 \cdot (i_* \cdot J_0)^{-1}$.

REMARK. If Φ_q is stable, then we have a natural split exact sequence

$$0 \to F_{q+1}MU^\sharp(X) \to F_q MU^\sharp(X) \to KG_q^\sharp(X) \to 0$$

for every CW-spectrum X. Therefore $F_q MU$ becomes homotopy equivalent to the wedge $F_{q+1}MU \vee KG_q$. However $H_*(F_q MU)$ is a free abelian group and $H_*(KG_q)$ is a Q-module. This is a contradiction.

We now obtain our main result.

Theorem 3. *For any based CW-complex X the natural sequences*

$$0 \to F_{q+1}MU^\sharp(X) \to F_q MU^\sharp(X) \to KG_q^\sharp(X) \to 0$$
$$0 \to KG_q^\sharp(X) \to Q_q MU^\sharp(X) \to Q_{q-1}MU^\sharp(X) \to 0$$

of Z_2-graded Λ-modules are split exact.

Proof. The first case is immediate from Proposition 11. On the other hand, a diagram chase shows that the second sequence is exact for any based CW-complex X and hence it is split.

Appendix

Recall that MU is a ring spectrum with coefficients $\Lambda_* = Z[x_1, \cdots, x_n, \cdots]$ where deg $x_n = 2n$. By killing certain bordism classes Baas [3] constructed homology theories $MU\langle n\rangle_*(\ )$ with coefficient $\pi_*(MU\langle n\rangle) = \Lambda_*/(x_{n+1}, \cdots)$, whose representing spectrum we denote by $MU\langle n\rangle$. $MU\langle n\rangle_*(\ )$ is an (associative) $MU_*(\ )$-module, thus there exists a natural homomorphism

$$m_n:\ MU_*(X) \otimes MU\langle n\rangle_*(Y) \to MU\langle n\rangle_*(X \wedge Y)$$

for any CW-spectra X and Y. This gives us a pairing

$$\phi\langle n\rangle:\ MU \wedge MU\langle n\rangle \to MU\langle n\rangle$$

by which the above m_n is induced.

An easy computation shows that $MU\langle n\rangle \hat{Z}/Z^{2k-1}(MU \wedge \cdots \wedge MU \wedge MU\langle n\rangle) = 0$ because $\pi_{2l+1}(MU\langle n\rangle) = 0$ for all l. Then [8, Theorem 1] says that

(A.1) *$MU\langle n\rangle^{2k}(MU \wedge \cdots \wedge MU \wedge MU\langle m\rangle)$ is Hausdorff* (see also [9, Corollary 13]).

Hence $\phi\langle n\rangle$ is uniquely determined (up to homotopy) and moreover

(A.2) *$MU\langle n\rangle$ is an (associative) MU-module spectrum.*

An important relationship between $MU\langle n\rangle_*(\ )$ and $MU\langle n-1\rangle_*(\ )$ is given in the form of a natural exact sequence

$$\to MU\langle n\rangle_{*-2n}(X) \xrightarrow{\cdot x_n} MU\langle n\rangle_*(X) \xrightarrow{t_n} MU\langle n-1\rangle_*(X) \to MU\langle n\rangle_{*-2n-1}(X) \to$$

of $MU_*(\ )$-modules where $\cdot x_n$ denotes the multiplication by x_n. Because of (A.1) there exists a unique morphism τ_n: $MU\langle n\rangle \to MU\langle n-1\rangle$ of MU-module spectra whose induced homomorphism is the above t_n. On the other hand, the composition

$$m_{x_n}\colon \Sigma^{2n} MU\langle n\rangle \xrightarrow{x_n \wedge 1} MU \wedge MU\langle n\rangle \xrightarrow{\phi\langle n\rangle} MU\langle n\rangle$$

is characterized by the above multiplication $\cdot x_n$.

Lemma A. *Let* $E \xrightarrow{f} F \xrightarrow{g} G$ *be a sequence of CW-spectra such that the composition* $g\cdot f$ *is homotopic to the zero. If* $0\to\pi_*(E)\to\pi_*(F)\to\pi_*(G)\to 0$ *is exact, then* $E\to F\to G$ *is a cofiber sequence.* (Cf., Lemma 4).

Proof. Let C_f be the mapping cone of f: $E\to F$. Then g: $F\to G$ admits a factorization $F\to C_f \xrightarrow{h} G$. Considering the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \pi_*(E) & \to & \pi_*(F) & \to & \pi_*(C_f) & \to & 0 \\ & & \| & & \| & & \downarrow h_* & & \\ 0 & \to & \pi_*(E) & \to & \pi_*(F) & \to & \pi_*(G) & \to & 0 \end{array}$$

with exact rows, we see easily that h: $C_f\to G$ is a homotopy equivalence.

Using (A.1) the composition $\tau_n\cdot m_{x_n}$ becomes homotopic to the zero. We get therefore that

(A.3) $\Sigma^{2n} MU\langle n\rangle \xrightarrow{m_{x_n}} MU\langle n\rangle \xrightarrow{\tau_n} MU\langle n-1\rangle$ *is a cofiber sequence.*

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References

[1] J.F. Adams: *A variant of E.H. Brown's representability theorem,* Topology **10** (1971), 185–198.

[2] J.F. Adams: Stable Homotopy and Generalized Homology, Chicago Lecture in Math., Univ. of Chicago Press.

[3] N.A. Baas: *On bordism theory of manifolds with singularities,* Math. Scand. **33** (1973), 279–302.

[4] P.A. Griffith: Infinite Abelian Group Theory, Chicago Lecture in Math., Univ. of Chicago Press.

[5] P.S. Landweber: *Homological properties of comodules over* $MU_*(MU)$ *and* $BP_*(BP)$, to appear.

[6] R. Vogt: Boardman's Stable Homotopy Category, Lecture notes series 21, Aarhus Univ. (1970).

[7] G. Wolff: *Der einfluss von* $K^*(-)$ *auf* $U^*(-)$, Manuscripta Math. **10** (1973), 141–161.

[8] Z. Yosimura: *On cohomology theories of infinite CW-complexes,* III, Publ. Res. Inst. Math. Sci. **9** (1974), 683–706.

[9] Z. Yosimura: *Universal coefficient sequences for cohomology theories of CW-spectra,* Osaka J. Math. **12** (1975), 305–323.