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THE HEAT EQUATION ON COMPACT LIE GROUP

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Introduction

McKean and Singer [9] posed the problem of the existence of an analogue of the Poisson's summation formula for manifolds other than flat tori. Y. Colin de Verdiere [3] gave an answer to it in the case of a 2-dimensional compact Riemannian manifold with negative sectional curvature.

The purpose of this paper is to determine the Minakshisundaram's expansion (Theorem 3) related to the heat equation and to give an analogue of the Poisson's summation formula of the fundamental solution of this equation on a simply-connected compact semi-simple Lie group (Theorem 2).

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1. Preliminaries

Let (M, g) be an n-dimensional compact connected orientable Riemannian manifold with the fundamental tensor g, and Δ be the Laplace-Beltrami operator acting on differentiable functions of M. Consider the *fundamental solution* $Z(t, m, m')$ of the heat equation

$$\Delta_x u(t, x) = \frac{\partial}{\partial t} u(t, x).$$

Then it satisfies the following facts:

(i) $0 < Z(t, m, m') \in C^\infty((0, \infty) \times M \times M)$

(ii) $\dfrac{\partial Z}{\partial t} = \Delta_m Z = \Delta_{m'} Z$

(iii) $\lim_{t \to 0+} \int_M Z(t, m, m') f(m') dv_{m'} = f(m), \qquad m \in M$

for every continuous function f on M where $dv_{m'}$ is the volume element of (M,g). And also define the trace $Z(t)$ of $Z(t, m, m')$ as

$$Z(t) = \int_M Z(t, m, m) dv_m. \tag{1.1}$$

Then it is known (cf. Berger [2]) that there is an asymptotic expansion

$$Z(t) \underset{t\to 0+}{\sim} \frac{1}{(4\pi t)^{n/2}} \sum_{k=1}^{\infty} a_k t^k \tag{1.2}$$

and coefficients a_k's are determined as follows. Let $(y^1, \cdots, y^n)$ be the normal coordinate on a neighborhood V about $m \in M$. We put $g_{ij}(y) = g_y\left(\left(\frac{\partial}{\partial y^i}\right)_y, \left(\frac{\partial}{\partial y^j}\right)_y\right)$, and define c_0 on V by

$$c_0(y) = [\det(g_{ij}(y))]^{-1/4}, \tag{1.3}$$

and define $c_k(k>0)$ inductively by

$$c_k(y) = c_0(y) \int_0^1 \frac{t^{k-1} \Delta c_{k-1}(y_t)}{c_0(y_t)} dt, \tag{1.4}$$

where y_t is the point with normal coordinates $(ty^1, \cdots, ty^n)$. Now we define a function u_k on M by $u_k(m) = c_k(0)$. Then it is known that

$$Z(t, m, m) = \frac{1}{(4\pi t)^{n/2}} \left(\sum_{k=0}^{t} u_k(m) t^k + 0(t^{i+1})\right) \qquad (t \to 0+) \tag{1.5}$$

and also

$$a_k = \int_M u_k(m) dv_m. \tag{1.6}$$

Berger [2], and Sakai [11] have calculated a_0, a_1, a_2, a_3; more precisely,

$$a_0 = \text{volume } M, \tag{1.7}$$

$$a_1 = \frac{1}{6} \int_M \tau(m) dv_m, \tag{1.8}$$

where $\tau(m)$ is the mean curvature of M at m.

2. Statements of the main theorem

Now let M be a simply-connected compact semi-simple Lie group and T be a maximal torus in M. Let $\mathfrak{m}$ (resp. $\mathfrak{t}$) be the Lie algebra of M (resp. of T). Since the Killing form B is negative definite on $\mathfrak{m} \times \mathfrak{m}$, we define and $Ad(M)$-invariant positive definite inner product (,) on $\mathfrak{m} \times \mathfrak{m}$ by $(X, Y) = -B(X, Y)$, which induces a Riemannian metric g on M by

$$g_m(X_m, Y_m) = (X, Y), \qquad X, Y \in \mathfrak{m}, \tag{2.1}$$

where X_m, Y_m are the tangent vectors at m correspoinding to X, Y. Let Δ be the root system of the complexification $\mathfrak{m}^C$ of $\mathfrak{m}$ with respect to $\mathfrak{t}$, i.e. the set of non-zero elements α of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ such that $\mathfrak{m}_\alpha^C=\{E\in\mathfrak{m}^C; [H, E]=\sqrt{-1}\alpha(H)E$ for any $H\in\mathfrak{t}\}$ is not zero. We introduce a lexicographic order $>$ of Δ and fix it once and for all and let Δ^+ be the set of all positive roots. Let $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ ($l=\dim\mathfrak{t}$) be a fundamental system of Δ with respect to this order $>$. We identify $\mathfrak{t}$ with $\mathfrak{t}^*$ under the identification of $\lambda\in\mathfrak{t}^*$ with $H_\lambda\in\mathfrak{t}$, where H_λ is the unique element $\mathfrak{t}$ of such that $(H_\lambda, H)=\lambda(H)$ for every $H\in\mathfrak{t}$.

For $\lambda\in\mathfrak{t}^*$ ($\lambda\neq 0$), we define

$$\lambda^*=\frac{2}{(\lambda, \lambda)}\lambda\,, \quad H_\lambda^*=\frac{2}{(\lambda, \lambda)}H_\lambda\,.$$

Define a lattice Γ in $\mathfrak{t}$ to be the subgroup of $\mathfrak{t}$ generated by $\{2\pi H_\alpha^*; \alpha\in\Delta\}$. Then by the simply-connectedness of M,

$$\Gamma=2\pi\sum_{i=1}^{l}\boldsymbol{Z}H_{\alpha_i}^*=\{H\in\mathfrak{t}; \exp H=e\}\,.$$

Put

$$\begin{aligned} I &= \{\lambda\in\mathfrak{t}^*; (\lambda, \alpha^*)\in\boldsymbol{Z} \quad \text{for every } \alpha\in\Delta\} \\ &= \{\lambda\in\mathfrak{t}^*; (\lambda, \alpha_i^*)\in\boldsymbol{Z} \quad (1\leq i\leq l)\}\,. \end{aligned}$$

Then $I=\{\lambda\in\mathfrak{t}^*; \lambda(\Gamma)\subset 2\pi\boldsymbol{Z}\}$, i.e. $\frac{1}{2\pi}\Gamma=\sum_{i=1}^{l}\boldsymbol{Z}H_{\alpha_i}^*$ is a dual lattice of I. Put

$$\begin{aligned} D &= \{\lambda\in I; (\lambda, \alpha)\geqq 0 \quad \text{for every } \alpha\in\Delta^+\} \\ &= \{\lambda\in I; (\lambda, \alpha_i)\geqq 0 \quad (1\leq i\leq l)\}\,. \end{aligned}$$

An element of D is called a *dominant integral form* on $\mathfrak{t}$. Define $\lambda_i\in\mathfrak{t}^*$ ($1\leq i\leq l$) by

$$(\lambda_i, \alpha_j^*)=\delta_{ij} \qquad (1\leq i, j\leq l)\,.$$

It follows that

$$I=\sum_{i=1}^{l}\boldsymbol{Z}\lambda_i\,, \quad D=\{\sum_{i=1}^{l}m_i\lambda_i; m_i\in\boldsymbol{Z}, \quad m_i\geqq 0 \qquad (1\leq i\leq l)\}\,. \tag{2.2}$$

Since an irreducible representation of M is uniquely determined (cf. Serre [13]), up to equivalence, by its highest weight, there exists a bijection from D onto the set of equivalent classes of irreducible representations of M. For $\lambda\in D$, let χ_λ (resp. d_λ) be the trace (resp. the degree) of the irredducible representation with the highest weight λ. Then it follows (cf. Sugiura [16]) that the eigenvalues of the Laplace-Beltrami operator Δ with respect to the metric g (2.1) is given by $-(\lambda+2\delta, \lambda)=-|\lambda+\delta|^2+|\delta|^2$, $\lambda\in D$ and its multiplicities are d_λ^2 and

$$\Delta\chi_\lambda=-(\lambda+2\delta, \lambda)\chi_\lambda\,, \tag{2.3}$$

where $\delta=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

Then the following theorem is immediately obtained.

Theorem 1. *Let M be a simply-connected compact semi-simple Lie group. Put*

$$Z(t, m) = \sum_{\lambda \in D} d_\lambda e^{-(\lambda+2\delta,\lambda)t} \chi_\lambda(m), \qquad t>0, m\in M. \tag{2.4}$$

Then $Z(t, m)$ is the fundamental solution of the heat equation, that is, the above series is absolutely convergent and it satisfies the following conditions:

$$\text{(i)} \quad \frac{\partial}{\partial t} Z(t, xy^{-1}) = \Delta_x Z(t, xy^{-1})$$

$$\text{(ii)} \quad \lim_{t\to 0+} \int_M Z(t, xy^{-1}) f(y) dy = f(x)$$

for every continuous function f on M. Here dy is the Haar measure of M with total volume 1.

Proof. By means of the inequality $|\chi_\lambda(m)| \leqq d_\lambda (m\in M)$, we have

$$\sum_{\lambda\in D} d_\lambda e^{-(\lambda+2\delta,\lambda)t} |\chi_\lambda(m)| \leqq \sum_{\lambda\in D} d_\lambda^2 e^{-(\lambda+2\delta,\lambda)t} = Z(t).$$

The convergence of $Z(t)$ is shown in Berger [2] (Cor. E. II. 2.). The condition (i) is followed from (2.3). Peter-Weyl's theorem leads the condition (ii). Q.E.D.

REMARK. $\dfrac{1}{\mathrm{Vol}(M)} Z(t, xy^{-1})$ is the fundamental solution in the sense in §1.

Now consider a summation

$$\sum_{\gamma\in\Gamma} \pi(\lambda+\gamma) e^{-(1/4t)|\lambda+\gamma|^2} \qquad (\lambda\in \mathfrak{t}^*, t>0) \tag{2.5}$$

where Γ is identified with a lattice $2\pi \sum_{i=1}^{l} Z\alpha_i^*$ in $\mathfrak{t}^*$ and $\pi(\lambda) = \prod_{\alpha\in\Delta^+} (\lambda, \alpha)$, $\lambda \in \mathfrak{t}^*$. Then this is an absolutely convergent series which induces a function $K(t, h)$ on T via the identification $\mathfrak{t}$ with $\mathfrak{t}^*$.

Then we can state the main theorem as follows.

Theorem 2. *Under the assumption of Theorem* 1, *we have the following Poisson's summation formula*:

$$Z(t, h) = Cj(h)^{-1} e^{|\delta|^2 t} (4\pi t)^{-(n/2)} K(t, h) \tag{2.6}$$

for $t>0$, $h\in T_r = \exp \mathfrak{t}_r$, where is $\mathfrak{t}_r$ the set of all regular elements in $\mathfrak{t}$,

$$j(h) = \prod_{\alpha\in\Delta^+} \left(e^{(\sqrt{-1}/2)\alpha(H)} - e^{-(\sqrt{-1}/2)\alpha(H)}\right), \quad h = \exp H$$

and $C = \dfrac{c}{\pi(\delta)} (2\pi)^{l+m} (\sqrt{-1})^m$. Here c is the constant defined in Lemma 5 and m

is the number of the elements of Δ^+

Theorem 2 will be proved in §4.

3. Minakshisundaram's expansion

In this section we will calculate the coefficients a_k of the Minakshisundaram's expansion (1.2). The situation in §2 is preserved.

By means of the uniqueness of the fundamental solution of the heat equation, it follows that

$$Z(t, gm, gm') = Z(t, m, m'),\ g, m, m' \in M .$$

Together with (1.5), we obtain $u_k(m)=u_k(e)$, $m\in M$. Hence

$$a_k = \int_M u_k(m)dv_m = u_k(e)\mathrm{vol}(M) \tag{3.1}$$

where dv_m is the volume element with respect to the Riemannian metric g (2.1). So let $X_1, \cdots, X_n$ be an orthonormal basis in $\mathfrak{m}$ with respect to (,). Consider the canonical coordinate $(y^1, \cdots, y^n)$ of the first kind with respect to $\{X_i\}_{i=1}^n$. Then c_0 (1.3) is calculated as follows.

Lemma 1. *Let M_r be the set of all regular elements in M. Then c_0 is a C^∞ class function on M_r, that is,*

$$c_0(mm'm^{-1}) = c_0(m'), \qquad m\in M,\ m'\in M_r .$$

Moreover

$$c_0(\exp H) = \prod_{\alpha\in\Delta^+} \frac{\alpha\left(\frac{H}{2}\right)}{\sin\alpha\left(\frac{H}{2}\right)}, \qquad H\in\mathfrak{t}_r . \tag{3.2}$$

Proof. Let $m=\exp X$, $X\in\mathfrak{m}$ where $X=Ad(m_0)H$, $m_0\in M$, $H\in\mathfrak{t}_r$. Then

$$\left(\frac{\partial}{\partial y_i}\right)_m = \exp_{*X}(X_i)$$

where $\exp_{*X}$ is the differential of the exponential mapping at X. Hence from Helgason [7] (p. 256),

$$\det\left(g_m\left(\left(\frac{\partial}{\partial y_i}\right)_m, \left(\frac{\partial}{\partial y_j}\right)_m\right)\right) = \left(\prod_{\alpha\in\Delta}\frac{1-e^{-\sqrt{-1}\alpha(H)}}{\sqrt{-1}\alpha(H)}\right)^2$$

$$= \left(\prod_{\alpha\in\Delta^+}\frac{\sin\frac{1}{2}\alpha(H)}{\frac{1}{2}\alpha(H)}\right)^4. \qquad \text{Q.E.D.}$$

Let $\mathcal{L}(M)$ be the algebra of all two sided-invariant differential operators on M and let $C_W^\infty(T_r)$ be the set of all C^∞ functions invariant under the Weyl group W. Then for every element $\Delta \in \mathcal{L}(M)$ there exists a unique element $\mathring{\Delta} \in \mathrm{End}(C_W^\infty(T_r))$ such that $(\Delta f)^- = \mathring{\Delta}\bar{f}$ for every C^∞ *class* function f on M_r where $\bar{f}$ is the restriction of f to T_r. $\mathring{\Delta}$ is called the *radial part of* Δ.

Let $\{H_1, \cdots, H_l\}$ be an orthonormal basis of $\mathfrak{t}$ with respect to (,). For a function f on T, we put $f'(t_1, \cdots, t_l) = f(\exp(\sum_{i=1}^{l} t_i H_i))$. Then the following theorem is due to Berezin [1].

Theorem. *Let $I(\mathfrak{t})$ be the algebra of all polynomial functions on $\mathfrak{t}$ invariant under W. Then*

(i) *For $P \in I(\mathfrak{t})$, the operator*

$$\mathring{\Delta}(P) = \frac{1}{j} P\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_l}\right) \cdot j \tag{3.3}$$

is the radial part of an element in $\mathcal{L}(M)$. Conversely the radial part of any element in $\mathcal{L}(M)$ is given by (3.3) *for some $P \in I(\mathfrak{t})$.*

(ii) *In particular, for the Laplace-Beltrami operator Δ, it follows that*

$$\mathring{\Delta} = \frac{1}{j} P_0\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_l}\right) j \tag{3.4}$$

where $P_0\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_l}\right) = (\delta, \delta) + \Delta_0$, $\Delta_0 = \sum_{i=1}^{l} \frac{\partial^2}{\partial t_i^2}$.

Lemma 2. *Put $\pi(H) = \prod_{\alpha \in \Delta^+} \alpha(H)$, $H \in \mathfrak{t}$. Then*

(i) *π is a skew function on $\mathfrak{t}$ and each skew polynomial on $\mathfrak{t}$ is divisible by π.*

(ii) *$\Delta_0 \pi \equiv 0$ on $\mathfrak{t}$.*

Proof. (i) is due to Harish-Chandra [6]. For (ii), $\Delta_0 \pi$ is divisible by π since is skew and since (i) is valid. On the other hand, since the total degree of $\Delta_0 \pi$ is less than that of π, we obtain $\Delta_0 \pi \equiv 0$. Q.E.D.

Theorem 3. *Under the above assumption, we have*

$$Z(t) \underset{t \to 0+}{\sim} \frac{\mathrm{vol}(M)}{(4\pi t)^{n/2}} e^{|\delta|^2 t}.$$

Proof. For $h = \exp H$, $H \in \mathfrak{t}_r$, we have

$$(j \cdot c_0)(H) = (\sqrt{-1})^m \pi(H).$$

Hence $\Delta_0(j \cdot c_0) \equiv 0$. Therefore from the above theorem,

$$\Delta c_0(h) = \frac{1}{j(H)}((\delta, \delta)+\Delta_0)(jc_0)(h) = (\delta, \delta)c_0(h)\,.$$

Hence

$$c_1(h) = c_0(h)\int_0^1 \frac{\Delta c_0(\exp(tH))}{c_0(\exp(tH))}dt = (\delta, \delta)c_0(h)$$

and

$$c_2(h) = c_0(h)\int_0^1 \frac{t\Delta c_0(\exp tH)}{c_0(\exp tH)}dt = \frac{(\delta, \delta)^2}{2}c_0(h)\,.$$

Inductively we obtain

$$c_n(h) = \frac{(\delta, \delta)^n}{n!}c_0(h)\,.$$

Therefore $a_n = u_n(e)\mathrm{vol}(M) = \lim\limits_{\substack{H\to 0\\ \in\\ \mathfrak{t}_r}} c_n(\exp H)\mathrm{vol}(M) = \frac{(\delta, \delta)^n}{n!}\mathrm{vol}(M)$

Q.E.D.

REMARK. In the case of the compact semi-simple Lie group M, the Ricci tensor with respect to the Riemannian metric g (2.1) is given by $S(X, Y) = \frac{1}{4}(X, Y)$ for $X, Y \in \mathfrak{m}$, and the mean curvature $\tau(e)$ at e is given by $\tau(e) = \sum_{i=1}^{n} S(X_i, Y_i) = \frac{n}{4}$ where $\{X_i\}_{i=1}^{l}$ is an orthonormal basis of $\mathfrak{m}$. Then the coefficient a_1 (1.8) is given by $a_1 = \frac{n}{24}\mathrm{vol}(M)$. On the other hand, from Theorem 3, we get $a_1 = |\delta|^2\mathrm{vol}(M)$. Therefore we have

$$|\delta|^2 = \frac{n}{24}\,.$$

4. Poisson's summation formula

In this section we will prove Theorem 2. First of all, we notice the following facts:

(i) $\delta = \lambda_1 + \cdots + \lambda_l \in D$

(ii) $\bigcup\limits_{s\in W}(s(D+\delta)-\delta) \subset I$, (disjoint sum)

(iii) For every $\mu \in I$, there exists a unique element $s \in W$ such that

$$\mu + \delta \in s(D+\delta)$$

if $\mu+\delta$ is regular, i.e. $\pi(\mu+\delta) \neq 0$.

Using the Weyl's dimension formula $d_\lambda = \frac{\pi(\lambda+\delta)}{\pi(\delta)}$, his trcae formula, that is,

$$\chi_\lambda(h)=\frac{1}{j(h)}\sum_{s\in W} sgn(s)e^{\sqrt{-1}s(\lambda+\delta)(H)} \qquad (h=\exp H,\ H\in\mathfrak{t}_r)$$

and the above remark, we rewrite $Z(t, h)$ as follows:

$$\begin{aligned} Z(t, h) &= \frac{e^{|\delta|^2 t}}{\pi(\delta)j(h)}\sum_{\substack{\lambda\in D\\ s\in W}} \pi(s(\lambda+\delta))e^{-|\lambda+\delta|^2 t}e^{\sqrt{-1}s(\lambda+\delta)(H)} \\ &= \frac{e^{|\delta|^2 t}}{\pi(\delta)j(h)}\sum_{\lambda\in I} \pi(\lambda+\delta)e^{-|\lambda+\delta|^2 t}e^{\sqrt{-1}(\lambda+\delta)(H)} \\ &= \frac{e^{|\delta|^2 t}}{\pi(\delta)j(h)}\sum_{\lambda\in I} \pi(\lambda)e^{-|\lambda|^2 t}e^{\sqrt{-1}\lambda(H)} \qquad (4.1)\end{aligned}$$

Therefore we may prove Poisson's summation formula for the summation in (4.1).

Now for every rapidly decreasing function f on $\mathfrak{t}^*$, we define a *Fourier transform* $\tilde{f}$ (cf. Warner [18]) as follows:

$$\tilde{f}(\lambda)=\frac{1}{(2\pi)^{l/2}}\int_{\mathfrak{t}^*} f(\mu)e^{-\sqrt{-1}(\lambda,\mu)}d\mu\,, \qquad \lambda\in\mathfrak{t}^*$$

where the measure $d\mu=dx_1\cdots dx_l$ $(\mu=\sum_{i=1}^{l} x_i\lambda_i)$. For $\alpha\in\Delta$, we define a differential operator $\partial(\alpha)$ (also cf. Warner [18]) as follows:

$$\partial(\alpha)f(\lambda)=\left[\frac{d}{ds}f(\lambda+s\alpha)\right]_{s=0}$$

for every differentiable function f on $\mathfrak{t}^*$. Put

$$\partial(\pi)=\prod_{\alpha\in\Delta^+}\partial(\alpha)\,.$$

Lemma 3. *For every rapidly decreasing function f on $\mathfrak{t}^*$, we have*

$$(\partial(\pi)f)^\sim(\lambda)=(\sqrt{-1})^m\pi(\lambda)\tilde{f}(\lambda) \qquad (4.2)$$

and

$$\partial(\pi)\tilde{f}(\lambda)=(-\sqrt{-1})^m(\pi\cdot f)^\sim(\lambda)\,, \qquad (4.3)$$

for $\lambda\in\mathfrak{t}^*$.

Proof. For $\alpha\in\Delta^+$, using integration by parts, we have

$$\int_{\mathfrak{t}^*}\partial(\alpha)f(\mu)e^{-\sqrt{-1}(\lambda,\mu)}d\mu=\sqrt{-1}(\lambda,\mu)\int_{\mathfrak{t}^*}f(\mu)e^{-\sqrt{-1}(\lambda,\mu)}d\mu\,.$$

Using this repeatedly, we have the first claim. Differentiating $\tilde{f}$ under the integral sign and noticing the fact such that

$$\partial(\pi(\lambda))e^{-\sqrt{-1}(\lambda,\mu)}d\mu=(-\sqrt{-1})^m\pi(\mu)e^{-\sqrt{-1}(\lambda,\mu)},$$

the second assertion is obtained. Q.E.D.

Lemma 4. *Under the above assumption, we have*

$$\partial(\pi)e^{-(a/2)|\lambda|^2}=(-a)^m\pi(\lambda)e^{-(a/2)|\lambda|^2} \tag{4.4}$$

where a is an arbitrary constant.

Proof. For $\alpha\in\Delta^+$,

$$\partial(\alpha)e^{-(a/2)|\lambda|^2}=-a(\lambda,\alpha)e^{-(a/2)|\lambda|^2}.$$

Then we may write

$$\partial(\pi)e^{-(a/2)|\lambda|^2}=((-a)^m\pi(\lambda)+P(\lambda))e^{-(a/2)|\lambda|^2} \tag{4.5}$$

where $P(\lambda)$ is a polynomial in λ whose total degree is less than that of π. On the other hand since $(-\sqrt{-1}a)^{-m}\partial(\pi)e^{-(a/2)|\lambda|^2}$ is the Fourier transform of the function $\pi(\lambda)e^{-a/2|\lambda|^2}$ by Lemma 3, the left hand side of (4.5) is a skew function. Hence $P(\lambda)$ is skew since $e^{-(a/2)|\lambda|^2}$ is invariant under W. Therefore $P\equiv 0$ from Lemma 2. Q.E.D.

Lemma 5. *For an arbitrary constant a, put $h_a(\lambda)=e^{-(a/2)|\lambda|^2}$. Then we have*

$$\tilde{h}_a(\lambda)=\frac{c}{a^{l/2}}e^{-(|\lambda|^2/2a)}$$

where c is the following constant: Let $e_1,\cdots,e_l$ be an orthonormal basis of $\mathfrak{t}^*$ with respect to $(\ ,\ )$ and let $d\tilde{\mu}=dy_1\cdots dy_l$ $(\mu=\sum_{i=1}^{l}y_ie_i)$ be the Euclidean measure on $\mathfrak{t}^*$. Then there exists a unique constant c such that

$$d\mu=cd\tilde{\mu}.$$

Proof. For $\lambda=\sum_{j=1}^{l}x_ie_i\in\mathfrak{t}^*$, we have

$$\begin{aligned}\tilde{h}_a(\lambda)&=\frac{c}{(2\pi)^{l/2}a^{l/2}}\int_{\mathfrak{t}^*}e^{-(|\mu^2|/2)}e^{-\sqrt{-1}(\lambda/\sqrt{a},\mu)}d\mu\\&=\frac{c}{a^{l/2}}\prod_{j=1}^{l}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-1/2y_j^2}e^{-\sqrt{-1}(x_j/\sqrt{a})\cdot y_j}dy_j\\&=\frac{c}{a^{l/2}}e^{-(|\lambda|^2/2a)}\end{aligned}$$

Q.E.D.

Lemma 6. *Under the assumption of Lemma 5, we have*

$$(\pi\cdot h_a)^{\sim}(\lambda) = c(-\sqrt{-1})^m a^{-n/2}\pi(\lambda)e^{-(|\lambda|^2/2a)}. \tag{4.6}$$

Proof. $(\pi\cdot h_a)^{\sim}(\lambda) = (-a)^{-m}(\partial(\pi)h_a)^{\sim}(\lambda)$ (by Lemma 3)
$= (-a)^{-m}(\sqrt{-1})^m\pi(\lambda)\tilde{h}_a(\lambda)$ (by Lemma 4)
$= c(-\sqrt{-1})^m a^{-(n/2)}\pi(\lambda)e^{-(|\lambda|^2/2a)}$ (by Lemma 5).

Q.E.D.

Proof of Theorem 2. Since the lattice I is a dual lattice of $\frac{1}{2\pi}\Gamma$, each character of $\mathfrak{t}^*/I$ is the form

$$\lambda \to e^{\sqrt{-1}(\lambda,\gamma)},$$

where $\gamma\in\Gamma$.

Now let

$$\vartheta(t, \nu, \mu) = \sum_{\tau\in I}\pi(\tau-\mu)e^{-t|\tau-\mu|^2}e^{2\pi\sqrt{-1}(\tau,\nu)}e^{-\pi\sqrt{-1}(\nu,\mu)}. \tag{4.7}$$

Then

$$\vartheta\left(t, \frac{\nu}{2\pi}, 0\right) = \sum_{\tau\in I}\pi(\tau)e^{-t|\tau|^2}e^{\sqrt{-1}(\tau,\nu)}. \tag{4.8}$$

For $H=H_\nu$, $\nu\in\mathfrak{t}^*$, the right hand side of (4.8) is identical with the summation in (4.1).

On the other hand, consider

$$\vartheta(t, \nu, \mu)e^{-\pi t(\nu,\mu)} = \sum_{\tau\in I}\pi(\tau-\mu)e^{-t|\tau-\mu|^2}e^{2\pi\sqrt{-1}(\tau-\mu,\nu)} \tag{4.9}$$

which is periodic in μ mod I. Therefore

$$\text{the right hand side of (4.9)} = \sum_{\gamma\in\Gamma}c_\gamma e^{\sqrt{-1}(\mu,\gamma)}$$

where

$$\begin{aligned}
c_\gamma &= \int_{\mathfrak{t}^*/I}\sum_{\tau\in I}\pi(\tau-\mu)e^{-t|\tau-\mu|^2}e^{2\pi\sqrt{-1}(\tau-\mu,\nu)}e^{-\sqrt{-1}(\mu,\gamma)}d\mu\\
&= \int_{\mathfrak{t}^*}\pi(\mu)e^{-t|\mu|^2}e^{2\pi\sqrt{-1}(\mu,\gamma)}e^{-\sqrt{-1}(\mu,\nu)}d\mu \quad (\text{by } e^{\sqrt{-1}(\tau,\gamma)}=1,\ \tau\in I,\ \gamma\in\Gamma)\\
&= \int_{\mathfrak{t}^*}\pi(\mu)e^{-t|\mu|^2}e^{-\sqrt{-1}(\mu,\ -(2\pi\nu+\gamma))}d\mu\\
&= (2\pi)^{l/2}c(-\sqrt{-1})^m(2t)^{-n/2}\pi(-(2\pi\nu+\gamma))e^{-(|2\pi\nu+\gamma|^2)/4t} \quad (\text{by Lemma 6})\\
&= c(\sqrt{-1})^m(2\pi)^{l/2}(2t)^{-n/2}\pi(2\pi\nu+\gamma)e^{-(|2\pi\nu+\gamma|^2)/4t}.
\end{aligned}$$

Therefore we have

$$\vartheta(t, \nu, \mu) = \frac{(2\pi)^{l/2}c(\sqrt{-1})^m}{(2t)^{n/2}}\sum_{\gamma\in\Gamma}\pi(2\pi\nu+\gamma)e^{-(|2\pi\nu+\gamma|^2)/4t}e^{\pi\sqrt{-1}(\nu,\mu)}.$$

Together with (4.1), we obtain for $h_\nu = \exp H_\nu$,

$$Z(t, h_\nu) = \frac{e^{|\delta|^2 t}(2\pi)^{l/2} c(\sqrt{-1})^m}{j(h_\nu)\pi(\delta)(2t)^{n/2}} \sum_{\gamma\in\Gamma} \pi(\nu+\gamma) e^{-(|\nu+\gamma|^2)/4t}$$
$$= \frac{Ce^{|\delta|^2 t}}{j(h_\nu)(4\pi t)^{n/2}} \sum_{\gamma\in\Gamma} \pi(\nu+\gamma) e^{-(|\nu+\gamma|^2)/4t},$$

where $C = \frac{(\sqrt{-1})^m c(2\pi)^{l/2}(4\pi)^{n/2}}{\pi(\delta)2^{n/2}} = \frac{c}{\pi(\delta)}(2\pi)^{l+m}(\sqrt{-1})^m$. Q.E.D.

REMARK. From theorem 2, we obtain

$$Z(t) = \lim_{h_\nu \in T_r \to e} Z(t, h_\nu)$$
$$= C\frac{e^{|\delta|^2 t}}{(4\pi t)^{n/2}} \lim_{\nu\in t_r\to 0} \left(\frac{1}{j(h_\nu)} \sum_{\gamma\in\Gamma} \pi(\nu+\gamma) e^{-(|\nu+\gamma|^2)/4t}\right) \underset{t\to 0+}{\sim} C\frac{e^{|\delta|^2 t}}{(4\pi t)^{n/2}}(-\sqrt{-1})^m.$$

Together with theorem 1 and the equality $C = \frac{c}{\pi(\delta)}(2\pi)^{l+m}(\sqrt{-1})^m$, we have

$$\mathrm{vol}(M) = \frac{c}{\pi(\delta)}(2\pi)^{l+m},$$

where the constant c is calculated as follows:

$$c = \mathrm{vol}(\mathfrak{t}^*/I)^{-1} = \mathrm{vol}\left(\mathfrak{t}\Big/\frac{1}{2\pi}\Gamma\right) = \sqrt{\det((\alpha_i^*, \alpha_j^*))_{1\le i,j\le l}} = 2^{l/2}\frac{\sqrt{D}}{\prod_{i=1}^{l}|\alpha_i|}.$$

Here $D = \det(-a_{ij})$, $a_{ij} = -\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ (Cartan's integer).

Another proof of Theorem 2. Since the right hand side of (2.6) is a W-invariant function in h, it can be extended as a *class* function $\tilde{Z}(t, m)$ in m on $\boldsymbol{R}_+^* \times M_r$. Therefore we may compare both Fourier transforms on M.

From the orthogonal relation of characters, we have

$$\frac{1}{d_\lambda}\int_M Z(t, m)\overline{\chi_\lambda(m)}dm = e^{-(\lambda+2\delta,\lambda)t}, \qquad \lambda\in D. \tag{4.9}$$

On the other hand, from a Weyl's integral formula for a class function, we obtain

$$\frac{1}{d_\lambda}\int_M \tilde{Z}(t, m)\overline{\chi_\lambda(m)}dm = \frac{1}{d_\lambda w}\int_T |j(h)|^2\left(C\frac{e^{|\delta|^2}}{(4\pi t)^{n/2}}\frac{K(t, h)}{j(h)}\right)\overline{\chi_\lambda(h)}dh \tag{4.10}$$

where dh is the Haar measure on T with total volume 1 and w is the order of W. By means of Weyl's character formula,

$$(4.10) = \frac{Ce^{|\delta|^2 t}}{d_\lambda w(4\pi t)^{n/2}} \sum_{s \in W} \operatorname{sgn}(s) \int_T K(t, h) e^{\sqrt{-1} s(\lambda+\delta)(-H)} dh$$

$$= \frac{Ce^{|\delta|^2 t}}{d_\lambda (4\pi t)^{n/2}} \int_T K(t, h) e^{\sqrt{-1}(\lambda+\delta)(-H)} dh . \qquad (4.11)$$

Consider the measure $d\bar{\mu} = dx_1 \cdots dx_l$ ($\mu = 2\pi \sum_{i=1}^{l} x_i \alpha_i^*$) on $\mathfrak{t}^*$, then $d\bar{\mu}$ induces the above measure dh on T. Therefore the integral in (4.11) is given as follows:

$$\int_{\mathfrak{t}^*/\Gamma} \sum_{\gamma \in \Gamma} \pi(\mu+\gamma) e^{-(|\mu+\gamma|^2)/4t} e^{-\sqrt{-1}(\lambda+\delta, \mu)} d\bar{\mu}$$

$$= \int_{\mathfrak{t}^*} \pi(\mu) e^{-(|\mu|^2/4t)} e^{-\sqrt{-1}(\lambda+\delta, \mu)} d\bar{\mu} . \qquad (4.12)$$

Here let $d\mu$ be the measure on $\mathfrak{t}^*$ defined before Lemma 3, then from the above Remark, we have

$$\operatorname{vol}(\mathfrak{t}^*/I) d\mu = \operatorname{vol}(\mathfrak{t}/\Gamma) d\bar{\mu} ,$$

and then $d\bar{\mu} = \frac{1}{(2\pi)^l c^2} d\mu$. Therefore

$$(4.12) = \frac{1}{(2\pi)^{l/2} c^2} (\pi \cdot h_a)^{\sim}(\lambda+\delta) \qquad \left(a = \frac{1}{2t}\right)$$

$$= \frac{1}{(2\pi)^{l/2} c^2} c(-\sqrt{-1})^m \left(\frac{1}{2t}\right)^{-n/2} \pi(\lambda+\delta) e^{-|\lambda+\delta|^2 t}$$

$$= \frac{\pi(\lambda+\delta)}{c(\sqrt{-1})^m (2\pi)^{m+l}} (4\pi t)^{n/2} e^{-|\lambda+\delta|^2 t} .$$

Therefore (4.10) coincides with (4.9). Q.E.D.

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