Suzuki, T.
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THE SINGULARITY OF INFINITE PRODUCT MEASURES

TAKERU SUZUKI

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1. Introduction. In this paper we give a consideration on singularity and nonsingularity relations between two infinite direct product measures. In statistical asymptotic theory the concept of singularity is closely related to the existence of consistent sequence of test procedures. L. Shepp [4] considered such a problem in the case where P_t, the component distribution with a location parameter t of the product measure, has a constant carrier and the Fisher's information number exists and is finite. Later L. LeCam [3] extended the result of L. Shepp to the case where P_t satisfy the quadratic mean differentiability conditions. Our purpose is to seek conditions under which given two product measures are singular, when the components P_t do not necessarily satisfy such regular conditions.

In Section 2 we introduce new quantities $\bar{I}$ and $\underline{I}$, and using them conditions of singularity are described. Section 3 is devoted to the proof of the quantity $\bar{I}$ being an extension of the Fisher's information number. In Section 4 we consider the relation between the concept of nonsingularity and that of contiguity which was introduced in L. LeCam [2]. Two examples not satisfying the usual conditions are given in Section 5.

2. The condition of singularity. Let Θ be an open set containing zero. For each $t\in\Theta$ let P_t be a probability measure on a certain σ-field $\mathfrak{A}$ of subsets of a set X. Let $(X^N, \mathfrak{A}^N)$ be the Cartesian product of countably many copies of $(X, \mathfrak{A})$. Let $Q_0=\prod_{i=1}^{\infty} P_0^{(i)}$ (direct product), $P_0^{(i)}=P_0$ $(i=1, 2, \cdots)$ and $Q_\pi=\prod_{i=1}^{\infty} P_{h_i}$ for $\pi=(h_1, h_2, \cdots)$. We say that Q_0 and Q_π are singular if there exists a set B in $\mathfrak{A}^N$ such that $Q_0(B)=0$ and $Q_\pi(B)=1$. In this section conditions are given to the sequence π for which Q_0 and Q_π are singular. Let $H=\{h; h\geqq 0, h\in\Theta, P_0$ and P_h are not singular.$\}$ and $\bar{H}=\prod_{i=1}^{\infty} H^{(i)}$, $H^{(i)}=H$ $(i=1, 2, \cdots)$. In the following the lower-case letters h with or without suffixes always mean the elements taken from the set H. Throughout this paper the following assumptions (A–1) and (A–2) will be made.

(A–1) $\{P_t; t\in\Theta\}$ is dominated by a σ-finite measure μ on X.

We denote by $f(x, t)$ the density of P_t relative to μ and define a function $X(x, h)$ as follows.

$$(2.1) \qquad \begin{aligned} X(x, h) &= [f(x, h)/f(x, 0)]-1 && \text{if } f(x, 0)>0 \\ &= 0 && \text{if } f(x, 0) = 0 . \end{aligned}$$

(A–2) $X(x, h)\rightarrow 0$ in probability for P_0 as $h\rightarrow 0$.

We define

$$(2.2) \qquad \rho(h) = \int_X [f(x, h)f(x, 0)]^{1/2} d\mu .$$

S. Kakutani [1] gave useful criteria for determining singularity or equivalence (i.e. absolutely continuous with each other) of infinite product measures. As the proofs of Theorem 2, 3 and 4 lean heavily on his result, we now quote a version of it without proof.

Theorem 1. (Kakutani). *Let* $\pi=(h_1, h_2, \cdots)$ *be an element of* $\bar{H}$. (1) Q_0 *and* Q_π *are singular if and only if* $\prod_{i=1}^{\infty} \rho(h_i)=0$. (2) *If* P_0 *and* P_{h_i} *are equivalent for every* i, *then* Q_0 *and* Q_π *are equivalent if and only if* $\prod_{i=i}^{\infty} \rho(h_i)>0$.

Proposition 1. *Under our assumptions* (A–1) *and* (A–2) *we have*

$$(2.3) \qquad \lim_{h\rightarrow 0} \rho(h) = 1 .$$

Proof. Let $g(x, h)$ be a function such that $g(x, h)=1$ or 0 according as $|X(x, h)|\leqq 1$ or $|X(x, h)|>1$. Obviously we have $|g(x, h)\cdot X(x, h)|\leqq 1$ for all $x\in X$, and $g(x, h)X(x, h)\rightarrow 0$ in probability for P_0 as $h\rightarrow 0$. From the Lebesgue's convergence theorem we have

$$(2.4) \qquad \lim_{h\rightarrow 0}\int_X [f(x, h)/f(x, 0)]^{1/2} g(x, h)\, dP_0 = 1 .$$

This implies (2.3). This completes the proof of the proposition.

From this proposition it follows that for any sufficiently small neighborhood U of zero we have $U\cap[0, \infty)\subset H$ and $\inf\{\rho(h); h\in U\cap[0, \infty)\}>0$. We denote by Γ the class of functions $\gamma(x, h)$ satisfying the following conditions (Γ–1)—(Γ–3).

(Γ–1) $0\leqq\gamma(x, h)\leqq 1$ for all h and all $x\in X$.

(Γ–2) $\gamma(x, h)\rightarrow 1$ in probability for P_0 as $h\rightarrow 0$.

(Γ–3) Tnere exists $0<M<\infty$ and $0<\delta<\infty$ such that $|X(x, h)\gamma(x, h)|\leqq M$ for P_0-almost all $x\in X$ for every h satisfying $0\leqq h\leqq\delta$.

Let Φ be the class of strictly monotone increasing continuous functions ϕ from $[0, \infty)$ to $[0, \infty)$ satisfying $\phi(0)=0$. For each $\gamma\in\Gamma$ and each $\phi\in\Phi$ we define

$$
\begin{aligned}
\bar{I}(\gamma, \phi) &= \limsup_{h\to 0} (1/h^2)\int_X [X(x, \phi^{-1}(h))]^2\gamma(x, \phi^{-1}(h))\, dP_0 \\
\underline{I}(\gamma, \phi) &= \liminf_{h\to 0} (1/h^2)\int_X [X(x, \phi^{-1}(h))]^2\gamma(x, \phi^{-1}(h))\, dP_0 \\
\alpha(\gamma, h) &= 1-\int_{f(x,0)>0} \gamma(x, h)\, dP_h \\
\beta(\gamma, h) &= 1-\int_X \gamma(x, h)\, dP_0 .
\end{aligned}
\tag{2.5}
$$

Theorem 2. (1) *If there exist* $\gamma\in\Gamma$ *and* $\phi\in\Phi$ *such that* $\bar{I}(\gamma, \phi)<\infty$, $\limsup_{h\to 0} \alpha(\gamma, h)/[\phi(h)]^2<\infty$ *and* $\limsup_{h\to 0} \beta(\gamma, h)/[\phi(h)]^2<\infty$, *then* Q_0 *and* Q_π *are not singular for any* $\pi\in\bar{H}$ *satisfying* $\sum_{i=1}^{\infty} \phi(h_i)^2 <\infty$. (2) *If there exist* $\gamma\in\Gamma$ *and* $\phi\in\Phi$ *such that* $0<\underline{I}(\gamma, \phi)\leqq\infty$, *then* Q_0 *and* Q_π *are singular for any* $\pi\in\bar{H}$ *satisfying* $\sum_{i=1}^{\infty} \phi(h_i)^2=\infty$.

Proof. First we prove the part (1) of the theorem. Let

$$
\begin{aligned}
J(\gamma, h) &= \int_X [X(x, h)]^2\gamma(x, h)\, dP_0, \text{ and} \\
\rho^*(h) &= \int_X [f(x, h)f(x, 0)]^{1/2}\gamma(x, h)\, d\mu
\end{aligned}
\tag{2.6}
$$

then we have

$$
\begin{aligned}
J(\gamma, h) &\geqq \int_X [(f(x, h)/f(x, 0))^{1/2}-1]^2\gamma(x, h)\, dP_0 \\
&= 2-\alpha(\gamma, h)-\beta(\gamma, h)-2\rho^*(h) .
\end{aligned}
\tag{2.7}
$$

Since $\rho(h)\geqq\rho^*(h)$ we have from (2.7)

$$\rho(h)\geqq 1-(1/2)\{\alpha(\gamma, h)+\beta(\gamma, h)+J(\gamma, h)\} . \tag{2.8}$$

Using the inequality

$$1-x\geqq\exp(-2x) \qquad \text{(for sufficiently small } x\geqq 0) \tag{2.9}$$

we have for every integer $n\geqq 1$ and every sufficiently small $h_1, h_2, \cdots, h_n$,

$$\prod_{i=1}^{n} \rho(h_i)\geqq\exp[-\{ \sum_{i=1}^{n} \alpha(\gamma, h_i)+ \sum_{i=1}^{n} \beta(\gamma, h_i)+ \sum_{i=1}^{n} J(\gamma, h_i)\}] . \tag{2.10}$$

On the other hand $\bar{I}(\gamma, \phi)<\infty$ implies

$$0\leqq \sum_{i=1}^{n} J(\gamma, h_i)\leqq[\sum_{i=1}^{n} \phi(h_i)^2][\bar{I}(\gamma, \phi)+1] \tag{2.11}$$

for every $n\geqq 1$ and every sufficiently small $h_1, h_2, \cdots, h_n$.
According to Theorem 1 our result follows from (2.10) and (2.11).

Next we prove the part (2) of the theorem. It is easy to see that

$$(2.12)\qquad J(\gamma, h)\leqq L\cdot\int_X[(f(x, h)/f(x, 0))^{1/2}-1]^2dP_0 \\ \leqq 2L[1-\rho(h)] \qquad (L=2M+4)$$

where M is a constant appeared in $(\Gamma-3)$. Hence it follows that

$$(2.13)\qquad \rho(h)\leqq 1-(1/2)J(\gamma, h)L^{-1}.$$

By the inequality

$$(2.14)\qquad 1-x\leqq\exp(-x) \qquad (\text{for every } x)$$

we have for every $n\geqq 1$ and every $h_1, h_2, \cdots, h_n$,

$$(1.15)\qquad \prod_{i=1}^{n}\rho(h_i)\leqq\exp\left[-(1/2L)\left\{\sum_{i=1}^{n}J(\gamma, h_i)\right\}\right].$$

On the other hand $\underline{I}(\gamma, \phi)>0$ implies

$$(2.16)\qquad \sum_{i=1}^{n}J(\gamma, h_i)\geqq\underline{I}(\gamma, \phi)\left[\sum_{i=1}^{n}\phi(h_i)^2\right]/2$$

for every $n\geqq 1$ and every sufficiently small $h_1, h_2, \cdots, h_n$.
Again by Theorem 1 we have the desired result from (2.15) and (2.16). This completes the proof of the theorem.

The following theorem shows that for somewhat restricted class $\{P_t; t\in\Theta\}$ the converse of Theorem 2 is also true.

Theorem 3. *Let ϕ be an element of Φ. (1) If Q_0 and Q_π are not singular for any $\pi\in\bar{H}$ satisfying $\sum_{i=1}^{\infty}\phi(h_i)^2<\infty$, then $\bar{I}(\gamma, \phi)<\infty$ for any $\gamma\in\Gamma$. (2) If Q_0 and Q_π are singular for any $\pi\in\bar{H}$ satisfying $\sum_{i=1}^{\infty}\phi(h_i)^2=\infty$, then $0<\underline{I}(\gamma, \phi)\leqq\infty$ for any $\gamma\in\Gamma$ satisfying*

$$(2.17)\qquad \lim_{h\to 0}\alpha(\gamma, h)/[\phi(h)]^2=\lim_{h\to 0}\beta(\gamma, h)/[\phi(h)]^2=0.$$

Proof. To prove this theorem we shall use an analogous method to that employed in L. Shepp [4]. First we prove the part (1). Suppose that $\bar{I}(\gamma, \phi)=\infty$ for some $\gamma\in\Gamma$. Then we can choose a sequence $\{t_i\}\subset H$ such that

$$(2.18)\qquad 0<\phi(t_i)\leqq(1/2)^i \text{ and } J(\gamma, t_i)/[\phi(t_i)]^2\geqq 2^i$$

for every $i=1, 2, \cdots$. For each i let r_i be the integer such that

$$r_i 2^i[\phi(t_i)]^2 < 1 \leqq (r_i+1)2^i[\phi(t_i)]^2 . \tag{2.19}$$

Define $h_i = t_k$ if $j_{k-1} \leqq i < j_k$ where $j_1 = r_1$ and $j_k = r_1 + r_2 + \cdots + r_k$ $(k \geqq 2)$. It is easy to see that $\pi = (h_1, h_2, \cdots) \in \bar{H}$, $\sum_{i=1}^{\infty} \phi(h_i)^2 < \infty$ and $\sum_{i=1}^{\infty} J(\gamma, h_i) = \infty$. From (2.15) it follows that

$$\prod_{i=1}^{\infty} \rho(h_i) = 0 . \tag{2.20}$$

Hence Q_0 and Q_π are singular. This proves the part (1).

Next we show the part (2). Suppose that there exists $\gamma \in \Gamma$ such that

$$\lim_{h \to 0} \alpha(\gamma, h)/[\phi(h)]^2 = \lim_{h \to 0} \beta(\gamma, h)/[\phi(h)]^2 = 0 , \tag{2.21}$$

and that $\underline{I}(\gamma, \phi) = 0$. Then we can choose a sequence $\{t_i\} \subset H$ such that

$$\begin{aligned} &0 < \phi(t_i) \leqq (1/2)^i,\ J(\gamma, t_i)/[\phi(t_i)]^2 \leqq 2^{-i} , \\ &\alpha(\gamma, t_i)/[\phi(t_i)]^2 \leqq 2^{-i} \quad \text{and} \quad \beta(\gamma, t_i)/[\phi(t_i)]^2 \leqq 2^{-i} \end{aligned} \tag{2.22}$$

for every $i = 1, 2, \cdots$. Let $\pi = (h_1, h_2, \cdots)$ be a sequence constructed from $\{t_i\}$ by the same method as employed in previous section. Then we have

$$\begin{aligned} &\sum_{i=1}^{\infty} J(\gamma, h_i) < \infty,\ \sum_{i=1}^{\infty} \alpha(\gamma, h_i) < \infty,\ \sum_{i=1}^{\infty} \beta(\gamma, h_i) < \infty \quad \text{and that} \\ &\sum_{i=1}^{\infty} [\phi(h_i)]^2 = \infty . \end{aligned} \tag{2.23}$$

From the inequality (2.10) we have

$$\prod_{i=1}^{\infty} \rho(h_i) > 0 . \tag{2.24}$$

Hence Q_0 and Q_π are not singular. This completes the proof of the part (2).

3. The quantity $\bar{I}$. In the following proposition we shall show that the quantity $\bar{I}$ defined in (2.5) includes the concept of the usual Fisher's information number. Let $Y(x,h)$ be a function defined by

$$\begin{aligned} Y(x, h) &= [f(x, h)/f(x, 0)]^{1/2} - 1 \qquad &&\text{if } f(x, 0) > 0 \\ &= 0 &&\text{otherwise.} \end{aligned} \tag{3.1}$$

Define $\gamma_0 \in \Gamma$ and $\phi_0 \in \Phi$ as follows.

$$\begin{aligned} \gamma_0(x, h) &= 1 \qquad &&\text{if } |X(x, h)| \leqq 1 \\ &= 0 &&\text{otherwise.} \\ \phi_0(h) &= h . \end{aligned} \tag{3.2}$$

Proposition 2. *Suppose that $Y(x, h)$ is quadratic mean differentiable at $h=0$ with a derivative V i.e.,*

$$(3.3)\qquad \lim_{h\to 0}(1/h^2)\int_X [Y(x, h)-(1/2)hV]^2 dP_0 = 0 .$$

Then we have

$$(3.4)\qquad \bar{I}(\gamma_0, \phi_0) = \underline{I}(\gamma_0, \phi_0) = \int_X V^2 dP_0 < \infty .$$

Proof. From the quadratic mean differentiability of $Y(x, h)$ it follows that

$$(3.5)\qquad \lim_{h\to 0}(1/h^2)\int_X [Y(x, h)-(1/2)hV]^2[Y(x, h)+2]^2\gamma_0(x, h)dP_0 = 0 .$$

By the Lebesgue's dominated convergence theorem we have from (3.5)

$$(3.6)\qquad \lim_{h\to 0}(1/h^2)\int_X [X(x, h)-hV]^2\gamma_0(x, h)dP_0 = 0 .$$

Thus we have

$$(3.7)\qquad \lim_{h\to 0}(1/h^2)\int_X [X(x, h)]^2\gamma_0(x, h)dP_0 = \int_X V^2 dP_0 .$$

This concludes the proof of the proposition.

4. The concept of contiguity and that of nonsingularity. In this section we shall show that the two concepts of nonsingularity and contiguity coincide with each other. For each integer $n\geqq 1$ let $\mathfrak{A}^n = \prod_{i=1}^n \mathfrak{A}^{(i)}$, $\mathfrak{A}^{(i)}=\mathfrak{A}$, $Q_{0,n}= \prod_{i=1}^n P_0^{(i)}$, $P_0^{(i)}=P_0$ $(i=1, 2, \cdots)$ and $Q_{\pi,n}=\prod_{i=1}^n P_{h_i}$ for $\pi=(h_1, h_2, \cdots)$.

Theorem 4. *Let π be an element of $\bar{H}$. (1) If $\{Q_{0,n}\}$ and $\{Q_{\pi,n}\}$ are contiguous, then Q_0 and Q_π are nonsingular. (2) If each P_{h_i} is equivalent to P_0, then the converse of (1) is also true.*

Proof. First we prove the part (1). Assume that $\{Q_{0,n}\}$ and $\{Q_{\pi,n}\}$ are contiguous, and that

$$(4.1)\qquad \prod_{i=1}^{\infty}\rho(h_i) = 0 .$$

Let Λ_n be the logarithm of the likelihood ratio of $Q_{\pi,n}$ to $Q_{0,n}$. By Theorem 1 in L. LeCam [2] we can find a subsequence $\{n'\}\subset\{n\}$ such that the distribution of $\Lambda_{n'}$ under $P_{0,n'}$ converges weakly to a distribution $L[\Lambda]$, and furthermore

$$(4.2)\qquad \int_{R^1}\exp(\Lambda)\,dL[\Lambda] = 1 .$$

Let ε be any positive number. For the ε there exists $a>0$ which is a continuity point of $L[\Lambda]$ such that

$$\int_{|\Lambda|>a} \exp[\Lambda/2]\, dL[\Lambda] \leqq \varepsilon . \tag{4.3}$$

From (4.1) and (4.3) we have

$$\int_{R^1} \exp[\Lambda/2]\, dL[\Lambda] \leqq \lim_{n'\to\infty} \int_{|\Lambda_{n'}|\leqq a} \exp[\Lambda_{n'}/2]\, dP_{0,n'} + \varepsilon = \varepsilon . \tag{4.4}$$

Thus we have

$$\int_{R^1} \exp[\Lambda/2]\, dL[\Lambda] = 0 . \tag{4.5}$$

But this does not occur, since $L[\Lambda]$ is a probability distribution on R^1. Therefore we have

$$\prod_{i=1}^{\infty} \rho(h_i) > 0 . \tag{4.6}$$

Hence Q_0 and Q_π are nonsingular.

Next we prove the part (2). Let Q_0 and Q_π are nonsingular. Then by Theorem 1 Q_0 and Q_π are equivalent. Therefore for any sequence $\{B_n\}$ satisfying $B_n \in \mathfrak{A}^n$ and $Q_{0,n}(B_n) \to 0$ as $n\to\infty$, we have $Q_{\pi,n}(B_n)\to 0$ as $n\to\infty$, and vice versa. Thus $\{Q_{0,n}\}$ and $\{Q_{\pi,n}\}$ are contiguous. This concludes the proof of the proposition.

From Theorem 2, 3 and 4 we have the following result.

Corollary. *Let $\pi = (h_1, h_2, \cdots)$ be an element of $\bar{H}$. Suppose that P_t, $t \in \Theta$ are absolutely continuous with each other, and that there exist $\gamma \in \Gamma$ and $\phi \in \Phi$ such that*

$$0 < \underline{I}(\gamma, \phi) \leqq \bar{I}(\gamma, \phi) < \infty , \tag{4.7}$$
$$\lim_{h\to 0} \alpha(\phi, h)/[\phi(h)]^2 = \lim_{h\to 0} \beta(\gamma, h)/[\phi(h)]^2 = 0 .$$

Then $\{Q_{0,n}\}$ and $\{Q_{\pi,n}\}$ are contiguous if and only if π satisfies $\sum_{i=1}^{\infty} \phi(h_i)^2 < \infty$.

Finally we remark that all the same results as stated in this paper hold if we take an arbitrarily fixed $\theta \in \Theta$ instead of zero and if $H' = \{h; h \geqq 0, \theta + h \in \Theta, P_\theta$ and $P_{\theta+h}$ are not singular.$\}$ or $H'' = \{h; h \leqq 0, \theta + h \in \Theta, P_\theta$ and $P_{\theta+h}$ are not singular.$\}$ instead of H. In this paper the parameter space Θ is restricted to a subset of real line. It seems to the auther that an extension to a multidimensional case of Θ is easy.

5. Examples. In this section we shall give some examples which do not satisfy the usual conditions.

EXAMPLE 1. Let $X = \Theta = R^1$. Let P_t be the distributions having the following densities relative to the Lebesgue measure.

$$f(x, t) = [1-|x-t|]^+ . \tag{5.1}$$

Define $\gamma(x, h)$ as follows.

$$\begin{aligned} \gamma(x, h) &= 1 \quad \text{if} \quad |f'/f| \leqq 1/h \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{5.2}$$

where f' means the ordinary differential (which exists for almost all $x \in X$) of f with respect to t at $t=0$. Let $\phi(h)=(h^2 \cdot |\log h|)^{1/2}$ for $0 \leqq h \leqq \exp[-(1/2)], = [(1/2)\exp[-(1/2)]h]^{1/2}$ for $\exp[-(1/2)] < h < 2$. We then have $\gamma \in \Gamma$ and $\phi \in \Phi$, and that $\bar{I}(\gamma, \phi)=\underline{I}(\gamma, \phi)=2$. Since $\alpha(\gamma, h)=2h^2$ and $\beta(\gamma, h)=h^2/2$ for $0 \leqq h \leqq \exp[-(1/2)]$, it follows that

$$\lim_{h\to 0} \alpha(\gamma, h)/[\phi(h)]^2 = \lim_{h\to 0} \beta(\gamma, h)/[\phi(h)]^2 = 0 . \tag{5.3}$$

Thus, according to Theorem 2, for any sequence $\pi=(h_1, h_2, \cdots)$ satisfying $0 \leqq h_i \leqq \exp[-(1/2)]$ $(i=1, 2, \cdots)$ Q_0 and Q_π are singular if and only if $\sum_{i=1}^{\infty} (h_i^2)|\log h_i| = \infty$.

EXAMPLE 2. Let $X=R^1$, $\Theta=(0, \infty)$, and let $\theta \in \Theta$ be any fixed number. Let P_t be the uniform distribution on $(0, t)$. Define $\gamma(x, h)=1$ and $\phi(h)=(-h)^{1/2}$ for any h satisfying $-\theta < h < 0$. Then we have $\alpha(\gamma, h)=\beta(\gamma, h)=0$ and $J(\gamma, h)=(-h)/(\theta+h)$. Hence it follows that

$$\begin{gathered} \lim_{h\to 0, h<0} \alpha(\gamma, h)/[\phi(h)]^2 = \lim_{h\to 0, h<0} \beta(\gamma, h)/[\phi(h)]^2 = 0 , \\ \bar{I}(\gamma, \phi) = \underline{I}(\gamma, \phi) = 1/\theta . \end{gathered} \tag{5.4}$$

Thus for any sequence $\pi=(h_1, h_2, \cdots)$ of nonpositive numbers Q_0 and Q_π are singular if and only if $\sum_{i=1}^{\infty} (-h_i)=\infty$.

On the other hand, for any $h>0$ let $\gamma(x, h)=1$, $\phi_1(h)=h$ and $\phi_2(h)=h^{1/2}$. Then we have

$$\begin{gathered} \lim_{h\to 0, h>0} \beta(\gamma, h)/[\phi_i(h)]^2 = 0 \qquad (i=1, 2) \\ \bar{I}(\gamma, \phi_1) = \underline{I}(\gamma, \phi_1) = 1/\theta^2, \ \lim_{h\to 0, h>0} \alpha(\gamma, h)/[\phi_1(h)]^2 = \infty , \\ \bar{I}(\gamma, \phi_2) = \underline{I}(\gamma, \phi_2) = 0, \ \lim_{h\to 0, h>0} \alpha(\gamma, h)/[\phi_2(h)]^2 = 1/\theta . \end{gathered} \tag{5.5}$$

Hence it follows that for a sequence $\pi=(h_1, h_2, \cdots)$ of nonnegative numbers Q_0 and Q_π are singular if $\sum_{i=1}^{\infty} (h_i)^2 = \infty$, and Q_0 and Q_π are nonsingular if $\sum_{i=1}^{\infty} h_i < \infty$.

OSAKA CITY UNIVERSITY

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