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ON MULTIPLY TRANSITIVE GROUPS XII

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1. Introduction

The known 4-fold transitive groups are the symmetric groups S_n $(n\geq 4)$, the alternating groups A_n $(n\geq 6)$ and Mathieu groups M_n $(n=11, 12, 23, 24)$. The main purpose of this paper is to characterize these known 4-fold transitive groups. The result is as follows.

Theorem. *Let G be a* 4*-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$. Assume that*

(*) *t is the maximal number of fixed points of involutions of G.*

Furthermore assume that G contains a 2*-subgroup Q which satisfies the following conditions:*

(1) *$|I(Q)|=t$ and Q is a Sylow* 2*-subgroup of $G_{I(Q)}$,*

(2) *$N(Q)^{I(Q)}=S_t$ or A_t.*

Then G is one of the following groups; S_n $(n\geq 4)$, A_n $(n\geq 6)$ or M_n $(n=11, 12, 23, 24)$.

This theorem is a generalization of theorems of M. Hall ([2], Theorem 5.8.1), H. Nagao [10] and the author [11]: the case $t<4$ has been proved by M. Hall, the case $t=4$ or 5 by H. Nagao and the case $t=6$ or 7 and $N(Q)^{I(Q)}=A_t$ by the author.

The followings are corollaries.

Corollary 1. *Let G be a* 4*-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$, and P a Sylow* 2*-subgroup of a stabilizer of four points in G. Assume that n is even and $P\neq 1$.*

(1) *If $I(P)=I(Z(P))$, where $Z(P)$ is the center of P, then G is one of the following groups; S_n $(n\geq 6)$, A_n $(n\geq 8$ and $n\equiv 0$ $(mod\ 4))$ or M_{12}.*

(2) *For any point i of $\Omega-I(P)$ if P_i is semiregular $(\neq 1)$ on $\Omega-I(P_i)$ or* 1*, then G is one of the following groups; S_6, S_8, A_8, A_{10}, M_{12} or M_{24}.*

Corollary 2. *Let G be a* 4*-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$ and P a Sylow* 2*-subgroup of a stabilizer of four points in G. If P is a transitive group*

$(\neq 1)$ *on* $\Omega - I(P)$, *then* G *is one of the following groups*; S_{2^k+4} $(k \geq 1)$, S_{2^k+5} $(k \geq 1)$, A_{2^k+4} $(k \geq 2)$, A_{2^k+5} $(k \geq 2)$, M_{12} *or* M_{23}.

Corollary 2 is a generalization of Theorem 1 and Theorem 2 in [7] and Theorem in [8]. In the proof of Corollary 1 we make use of the following

Lemma. *Let* G *be a 4-fold transitive group on* $\Omega = \{1, 2, \cdots, n\}$. *Assume that the maximal number of fixed points of involutions of* G *is twelve. Then for any 2-subgroup* Q *fixing exactly twelve points* $N(Q)^{I(Q)} \neq M_{12}$.

We shall use the same notations in [12].

2. Proof of the theorem

We proceed by way of contradiction. From now on we assume that G is a counter-example to our theorem of the least possible degree. Since there is no 4-fold transitive group of degree less than thirty-five except known ones ([2], P. 80), the degree n of G is not less than thirty-five. Set $I(Q) = \{1, 2, \cdots, t\}$ and $\Delta = \Omega - I(Q)$. For any point $t+i$ of Δ set $i' = t+i$, $1 \leq i \leq n-t$.

2.1. $t \geq 6$. *In particular if* $N(Q)^{I(Q)} = A_t$, *then* $t \geq 8$.

Proof. If $t < 4$, then by a theorem of M. Hall ([2], Theorem 5.8.1) $G = S_4$, S_5, A_6, A_7 or M_{11}, which is a contradiction since $n \geq 35$. If $t = 4$ or 5, then by a theorem of H. Nagao [10] $G = S_6$, S_7, A_8, A_9 or M_{12}, which is also a contradiction. Thus $t \geq 6$.

Suppose that $N(Q)^{I(Q)} = A_t$, $t = 6$ or 7. Since Q is a Sylow 2-subgroup of $G_{I(Q)}$, Q is a Sylow 2-subgroup of a stabilizer of four points of $I(Q)$ in G. Hence by a theorem of [11] $G = M_{23}$, which is also a contradiction. Thus if $N(Q)^{I(Q)} = A_t$, then $t \geq 8$.

2.2. $|\Delta| \geq 17$.

Proof. G is a 4-fold transitive group and $n \geq 35$. Hence by a theorem of W. A. Manning [5]

$$|\Delta| \geq \frac{n-1}{2} \geq \frac{35-1}{2} = 17\,.$$

2.3. *Let* R *be a 2-subgroup of* $N(Q)$ *containing* Q, *and* X *a 2-subgroup of* $N(Q)$. *If* $\langle R, X\rangle^{I(Q)}$ *is a 2-group, then there is a 2-subgroup* X' *in* $N(Q)$ *such that* $X^{I(Q)} = X'^{I(Q)}$, $\langle R, X'\rangle$ *is a 2-group and* $\langle Q, X'\rangle$ *is conjugate to* $\langle Q, X\rangle$ *in* $N(Q)$.

Proof. Let P be a Sylow 2-subgroup of $\langle R, X\rangle$ containing R. Since $\langle R, X\rangle^{I(Q)}$ is a 2-group, $P^{I(Q)} = \langle R, X\rangle^{I(Q)}$. Then P contains a 2-group X' such that $X^{I(Q)} = X'^{I(Q)}$. Then $\langle R, X'\rangle$ is a 2-subgroup of P. Since Q is a Sylow 2-subgroup of $G_{I(Q)}$ and $\langle Q, X\rangle^{I(Q)} = \langle Q, X'\rangle^{I(Q)}$, both $\langle Q, X\rangle$ and $\langle Q, X'\rangle$ are

Sylow 2-subgroups of $\langle Q, X, X'\rangle$. Hence $\langle Q, X'\rangle$ is conjugate to $\langle Q, X\rangle$ in $\langle Q, X, X'\rangle$. Thus $\langle Q, X'\rangle$ is conjugate to $\langle Q, X\rangle$ in $N(Q)$.

2.4. *If $N(Q)^{I(Q)}=S_t$, then $N(Q)$ has a 2-group $\langle Q, x_1, x_2, \cdots, x_k\rangle$, where*

$$x_i = (1)\,(2)\cdots(2i-2)\,(2i-1\ 2i)\,(2i+1)\cdots(t)\cdots,$$

$1\leq i\leq k$, *$k=\dfrac{t}{2}$ if t is even and $k=\dfrac{t-1}{2}$ if t is odd.*

Furthermore since $N(Q)^{I(Q)}=S_t$ or A_t, $N(Q)$ has a 2-group $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$, where

$$y_i = (1\ 2)\,(3)\,(4)\cdots(2i)\,(2i+1\ 2i+2)\,(2i+3)\cdots(t)\cdots,$$
$$y_1' = (1\ 3)\,(2\ 4)\,(5)\,(6)\cdots(t)\cdots,$$

$1\leq i\leq k$, *$k=\dfrac{t-2}{2}$ if t is even and $k=\dfrac{t-3}{2}$ if t is odd.*

In either case $k\geq 3$.

Proof. Since $N(Q)^{I(Q)}=S_t$ or A_t, this follows immediately from (2.1) and (2.3).

From now on we denote that $\langle Q, x_1, x_2, \cdots, x_k\rangle$ and $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ are the groups in (2.4).

2.5. *Suppose that $N(Q)$ has the 2-group $\langle Q, x_1, x_2, \cdots, x_k\rangle$ in* (2.4), *which is abelian and fixes a subset Δ' of Δ. If $\langle Q, x_1, x_2\rangle$ is semiregular on Δ', then $\langle Q, x_1, x_2, \cdots, x_k\rangle$ is semiregular on Δ'.*

Proof. Suppose that $\langle Q, x_1, x_2, \cdots, x_i\rangle$, $i\geq 2$, is semiregular on Δ' and $\langle Q, x_1, x_2, \cdots, x_{i+1}\rangle$ is not semiregular on Δ'. Then $\langle Q, x_1, x_2, \cdots, x_i\rangle x_{i+1}$ has an element x fixing a $\langle Q, x_1, x_2, \cdots, x_i\rangle$-orbit of length $2^i\cdot|Q|\,(\geq 2^{i+1})$ in Δ' pointwise since $\langle Q, x_1, x_2, \cdots, x_{i+1}\rangle$ is abelian and $\langle Q, x_1, x_2, \cdots, x_i\rangle$ is semiregular on Δ'. Then since x has at most $i+1$ 2-cycles in $I(Q)$ and $i\geq 2$, $|I(x)|\geq t-2(i+1)+2^{i+1}>t$, contrary to the assumption $(*)$. Thus if $\langle Q, x_1, x_2, \cdots, x_i\rangle$, $i\geq 2$, is semiregular on Δ', then $\langle Q, x_1, x_2, \cdots, x_{i+1}\rangle$ is semiregular on Δ'. Then since $\langle Q, x_1, x_2\rangle$ is semiregular on Δ', this implies by induction that $\langle Q, x_1, x_2, \cdots, x_k\rangle$ is semiregular on Δ'.

2.6. *$N(Q)$ has the 2-group $\langle Q, y_1, y_2, \cdots, y_k\rangle$ in* (2.4). *Suppose that $\langle Q, y_1, y_2, \cdots, y_k\rangle$ is abelian and fixes a subset Δ' of Δ. If $\langle Q, y_1, y_2, y_3\rangle$ is semiregular on Δ', then $\langle Q, y_1, y_2, \cdots, y_k\rangle$ is semiregular on Δ'.*

Proof. Suppose that $\langle Q, y_1, y_2, \cdots, y_i\rangle$, $i\geq 3$, is semiregular on Δ' and $\langle Q, y_1, y_2, \cdots, y_{i+1}\rangle$ is not semiregular on Δ'. Then $\langle Q, y_1, y_2, \cdots, y_i\rangle y_{i+1}$ has an element y fixing a $\langle Q, y_1, y_2, \cdots, y_i\rangle$-orbit of length $2^i\cdot|Q|\,(\geq 2^{i+1})$ in Δ' pointwise since $\langle Q, y_1, y_2, \cdots, y_{i+1}\rangle$ is abelian and $\langle Q, y_1, y_2, \cdots, y_i\rangle$ is semiregular on Δ'. Then since y has at most $i+2$ 2-cycles in $I(Q)$ and $i\geq 3$, $|I(y)|\geq t-2(i+2)+2^{i+1}>t$, contrary to the assumption $(*)$. Thus if $\langle Q, y_1, y_2, \cdots, y_i\rangle$, $i\geq 3$, is semiregular on Δ', then $\langle Q, y_1, y_2, \cdots, y_{i+1}\rangle$ is semiregular on Δ'. Then since $\langle Q, y_1, y_2, y_3\rangle$ is semiregular on Δ', this implies by induction that $\langle Q, y_1, y_2, \cdots, y_k\rangle$ is semiregular on Δ'.

2.7. $|\Delta|\equiv 0$ (*mod* 4).

Proof. Since Q is semiregular $(\neq 1)$ on Δ, $|\Delta|$ is even, i.e., $|\Delta|\equiv 0$ or 2 (mod 4). Suppose by way of contradiction that $|\Delta|\equiv 2$ (mod 4). Then $|Q|=2$. Hence we may assume that $Q=\langle a\rangle$ and

$$a=(1)\,(2)\cdots(t)\,(1'\ 2')\,(3'\ 4')\cdots(n-1\ n)\,.$$

Then $N(Q)=C(Q)=C(a)$ and $C(a)^{I(a)}=S_t$ or A_t. We treat these cases separately.

(i) Suppose that $C(a)^{I(a)}=S_t$. Then $C(a)$ has the 2-group $\langle a, x_1, x_2, \cdots, x_k\rangle$ in (2.4). First we show that $\langle a, x_1, x_2, \cdots, x_k\rangle$ has exactly one orbit Γ of length two in Δ and is semiregular on $\Delta-\Gamma$.

Since $|\Delta|\equiv 2$ (mod 4) and Δ is a union of $\langle a, x_1, x_2, \cdots, x_k\rangle$-orbits, $\langle a, x_1, x_2, \cdots, x_k\rangle$ has at least one orbit of length two in Δ. Hence we may assume that $\{1', 2'\}$ is the $\langle a, x_1, x_2, \cdots, x_k\rangle$-orbit of length two. Then x_i or ax_i, $1\leq i\leq k$, fixes $\{1', 2'\}$ pointwise. Hence we may assume that x_i fixes $\{1', 2'\}$ pointwise. Then $I(x_i)$ contains $(I(a)-\{2i-1, 2i\})\cup\{1', 2'\}$ of length t. Hence by the assumption $(*)$ $|I(x_i)|=t$ and $I(x_i)\cap\Delta=\{1', 2'\}$. Since $I(x_i^{x_j}\cdot x_i)$ contains $I(a)\cup\{1', 2'\}$ of length $t+2$, $1\leq i, j\leq k$, $x_i^{x_j}\cdot x_i=1$ by the assumption $(*)$. Thus $x_i^2=1$ and $x_ix_j=x_jx_i$. Hence $\langle a, x_1, x_2, \cdots, x_k\rangle$ is elementary abelian.

Since a and x_i, $1\leq i\leq k$, has no fixed point in $\Delta-\{1', 2'\}$ and $|\Delta-\{1', 2'\}|\equiv 0$ (mod 4), $|I(ax_i)\cap(\Delta-\{1', 2'\})|\equiv 0$ (mod 4). On the other hand since $|I(ax_i)\cap I(a)|=t-2$, $|I(ax_i)\cap\Delta|=2$ or 0 by the assumption $(*)$. Hence $|I(ax_i)\cap(\Delta-\{1', 2'\})|=0$. Thus $\langle a, x_i\rangle$ is semiregular on $\Delta-\{1', 2'\}$.

Suppose that $\langle a, x_1, x_2\rangle$ is not semiregular on $\Delta-\{1', 2'\}$. Then $\langle a, x_1, x_2\rangle$ has an orbit Δ' of length four in $\Delta-\{1', 2'\}$. Since $\langle a, x_1, x_2\rangle$ is an elementary abelian group of order eight, there is exactly one element $(\neq 1)$ in $\langle a, x_1, x_2\rangle$ fixing Δ' pointwise. Since $\langle a, x_1\rangle$ and $\langle a, x_2\rangle$ are semiregular on $\Delta-\{1', 2'\}$, x_1x_2 or ax_1x_2 fixes Δ' pointwise. Since $I(x_1x_2)$ contains $(I(a)-\{1, 2, 3, 4\})\cup\{1', 2'\}$ of length $t-2$, x_1x_2 does not fix Δ' pointwise by the assumption $(*)$. Hence ax_1x_2 fixes Δ' pointwise. Then $|I(ax_1x_2)|=t$ and so ax_1x_2 has no fixed point in $\Delta-(\{1', 2'\}\cup\Delta')$. This shows that $\langle a, x_1, x_2\rangle$ is semiregular on $\Delta-(\{1', 2'\}\cup\Delta')$. By (2.4) $k\geq 3$ and so $C(a)$ has x_3 in (2.4). Since x_3 normalizes $\langle a, x_1, x_2\rangle$, x_3 fixes Δ'. Then by the same argument as above ax_1x_3 fixes Δ' pointwise. Thus $I(ax_1x_2\cdot ax_1x_3)=I(x_2x_3)$ contains $(I(a)-\{3, 4, 5, 6\})\cup\{1', 2'\}\cup\Delta'$ of length $t+2$, contrary to the assumption $(*)$. Thus $\langle a, x_1, x_2\rangle$ is semiregular on $\Delta-\{1', 2'\}$. Hence by (2.5) $\langle a, x_1, x_2, \cdots, x_k\rangle$ is semiregular on $\Delta-\{1', 2'\}$.

On the other hand a normalizes $G_{1'2'3'4'}$, which is even order. Hence a commutes with an involution u of $G_{1'2'3'4'}$. Since $C(a)^{I(a)}=S_t$, $\langle a, x_1, x_2, \cdots, x_k\rangle$ has a subgroup which is conjugate to $\langle a, u\rangle$ in $C(a)$. Since u fixes at least four points of Δ, $\langle a, x_1, x_2, \cdots, x_k\rangle$ has an element $(\neq 1)$ fixing at least four points in Δ, which is a contradiction. Thus $C(a)^{I(a)}\neq S_t$.

(ii) Suppose that $C(a)^{I(a)}=A_t$. Let y be a 2-element such that $y^{I(a)}$ is an involution consisting two 2-cycles. Since $|I(y)|\leq t$, $|I(y)\cap\Delta|=0$, 2 or 4.

(ii.i) First assume that $|I(y)\cap\Delta|=4$. By (2.4) $C(a)$ has the 2-group $\langle a, y_1, y_2, y_3\rangle$. Since $\langle a, y_1\rangle$ is conjugate to $\langle a, y\rangle$ in $C(a)$, y_1 or ay_1 is conjugate to y. Hence we may assume that y_1 is conjugate to y and

$$y_1=(1\ 2)\ (3\ 4)\ (5)\ (6)\cdots(t)\ (1')\ (2')\ (3')\ (4')\cdots .$$

Since $|\Delta-\{1', 2', 3', 4'\}|\equiv 2 \pmod 4$ and $\Delta-\{1', 2', 3', 4'\}$ is a union of $\langle a, y_1\rangle$-orbits, the number of $\langle a, y_1\rangle$-orbits of length two in $\Delta-\{1', 2', 3', 4'\}$ is odd. Hence we may assume that $\{5', 6'\}$ is the orbit of length two. Then $y_1=(5'\ 6')$ on $\{5', 6'\}$, and $\langle a, y_1\rangle$ is semiregular on $\Delta-\{1', 2', \cdots, 6'\}$ since $|I(ay_1)|\leqq t$. Furthermore $C(a)$ has a 2-element

$$y_2'=(1)\ (2)\ (3\ 4)\ (5\ 7)\ (6)\ (8)\ (9)\cdots(t)\cdots .$$

By (2.3) we may assume that $\langle a, y_1, y_2'\rangle$ is a 2-group. Then y_2, y_3 and y_2' normalize $\langle a, y_1\rangle$. Since $|I(y_1)|\neq|I(ay_1)|$, $y_1^{y_2}=y_1^{y_3}=y_1^{y_2'}=y_1$. Thus y_2, y_3 and y_2' centralize $\langle a, y_1\rangle$, and so fix $\{1', 2', 3', 4'\}$ and $\{5', 6'\}$. Since y_i or ay_i, $i=2, 3$, and y_2' or ay_2' fix $\{5', 6'\}$ pointwise, we may assume that y_2, y_3 and y_2' fix $\{5', 6'\}$ pointwise. Since $I(y_i^{y_j}\cdot y_i)$ contains $I(a)\cup\{5', 6'\}$ of length $t+2$, $2\leq i, j\leq 3$, $y_2^2=y_3^2=1$ and $y_2y_3=y_3y_2$ by the assumption $(*)$. Similarly y_2' is of order two. Thus $\langle a, y_1, y_2, y_3\rangle$ and $\langle a, y_1, y_2'\rangle$ are elementary abelian. Since y_2, y_3 and y_2' fix $\{1', 2', 3', 4'\}$, y_2, y_3 and y_2' are $(1')\ (2')\ (3')\ (4')$, $(1'\ 2')\ (3')\ (4')$, $(1')\ (2')\ (3'\ 4')$, $(1'\ 2')\ (3'\ 4')$, $(1'\ 3')\ (2'\ 4')$ or $(1'\ 4')\ (2'\ 3')$ on $\{1', 2', 3', 4'\}$. Since $I(y_2)$ contains $(I(a)-\{1, 2, 5, 6\})\cup\{5', 6'\}$ of length $t-2$, y_2 does not fix $\{1', 2', 3', 4'\}$ pointwise. Similarly y_3 and y_2' do not fix $\{1', 2', 3', 4'\}$ pointwise. If $y_2=(1'\ 2')\ (3'\ 4')\cdots$, then $I(ay_1y_2)$ contains $(I(a)-\{3, 4, 5, 6\})\cup\{1', 2', \cdots, 6'\}$ of length $t+2$, contrary to the assumption $(*)$. Thus $y_2\neq(1'\ 2')\ (3'\ 4')\cdots$. Similarly y_3 and $y_2'\neq(1'\ 2')\ (3'\ 4')\cdots$. Next suppose that $y_2=(1'\ 2')\ (3')\ (4')\cdots$. The proof in the case $y_2=(1')\ (2')\ (3'\ 4')\cdots$ is similar. Since y_3 commutes with y_2, $y_3=(1'\ 2')\ (3')\ (4')\cdots$ or $(1')\ (2')\ (3'\ 4')\cdots$. If $y_3=(1'\ 2')\ (3')\ (4')\cdots$, then $I(y_2y_3)$ contains $(I(a)-\{5, 6, 7, 8\})\cup\{1', 2', \cdots, 6'\}$ of length $t+2$, contrary to the assumption $(*)$. Thus $y_3=(1')\ (2')\ (3'\ 4')\cdots$. On the other hand as we have seen above $y_2'=(1'\ 2')\ (3')\ (4')\ (5')\ (6')$, $(1')\ (2')\ (3'\ 4')\ (5')\ (6')$, $(1'\ 3')\ (2'\ 4')\ (5')\ (6')$ or $(1'\ 4')\ (2'\ 3')\ (5')\ (6')$ on $\{1', 2', \cdots, 6'\}$. If y_2' is of the first form, then

$(y_2 y_2')^3$ is of even order and $|I((y_2 y_2')^3)| \geq t+2$, contrary to the assumption (*). If y_2' is of the second form, then $(y_3 y_2')^3$ is of even order and $|I((y_3 y_2')^3)| \geq t+2$, contrary to the assumption (*). If y_2' is of the third or fourth form, then $(y_2 y_2')^6$ is of even order and $|I((y_2 y_2')^6)| \geq t+2$, contrary to the assumption (*). Thus $y_2 \neq (1'\ 2')\ (3')\ (4')\cdots$ and so $y_2 \neq (1')\ (2')\ (3'\ 4')\cdots$. Finally suppose that $y_2 = (1'\ 3')\ (2'\ 4')\cdots$. The proof in the case $y_2 = (1'\ 4')\ (2'\ 3')\cdots$ is similar. Then by the same argument as is used for y_2, y_3 and y_2' are $(1'\ 3')\ (2'\ 4')$ or $(1'\ 4')\ (2'\ 3')$ on $\{1', 2', 3', 4'\}$. If y_3 or $y_2' = (1'\ 3')\ (2'\ 4')\cdots$, then $|I(y_2 y_3)|$ or $|I((y_2 y_2')^3)| \geq t+2$ respectively, contrary to the assumption (*). Thus y_3 and $y_2' = (1'\ 4')\ (2'\ 3')\cdots$. Then $(y_3 y_2')^3$ is of even order and $|I((y_3 y_2')^3)| \geq t+2$, contrary to the assumption (*). Thus if y is a 2-element of $C(a)$ such that $y^{I(a)}$ is an involution consisting of two 2-cycles, then $|I(y) \cap \Delta| \neq 4$.

(ii.ii) By (ii.i) for any 2-element y of $C(a)$ such that $y^{I(a)}$ is an involution consisting of two 2-cycles, $|I(y) \cap \Delta| = 0$ or 2. By (2.4) $C(a)$ has the 2-group $\langle a, y_1, y_2, \cdots, y_k \rangle$. First we show that $\langle a, y_1, y_2, \cdots, y_k \rangle$ has exactly one orbit Γ of length two in Δ and is semiregular on $\Delta - \Gamma$.

Since $|\Delta| \equiv 2 \pmod 4$ and Δ is a union of $\langle a, y_1, y_2, \cdots, y_k \rangle$-orbits, $\langle a, y_1, y_2, \cdots, y_k \rangle$ has at least one orbit of length two in Δ. We may assume that $\{1', 2'\}$ is the $\langle a, y_1, y_2, \cdots, y_k \rangle$-orbit of length two. Then y_i or ay_i, $1 \leq i \leq k$, fixes $\{1', 2'\}$ pointwise. Hence we may assume that y_i fixes $\{1', 2'\}$ pointwise. Since $|I(y_i) \cap \Delta| = 0$ or 2, $I(y_i) \cap \Delta = \{1', 2'\}$. Since $I(y_i^{y_j} \cdot y_i)$ contains $I(a) \cup \{1', 2'\}$ of length $t+2$, $1 \leq i, j \leq k$, $y_i^{y_j} \cdot y_i = 1$ by the assumption (*). Hence $y_i^2 = 1$ and $y_i y_j = y_j y_i$. Thus $\langle a, y_1, y_2, \cdots, y_k \rangle$ is an elementary abelian group.

Since a and y_1 has no fixed point in $\Delta - \{1', 2'\}$ and $|\Delta - \{1', 2'\}| \equiv 0 \pmod 4$, $|I(ay_1) \cap (\Delta - \{1', 2'\})| \equiv 0 \pmod 4$. Hence by (ii.i) $|I(ay_1) \cap (\Delta - \{1', 2'\})| = 0$. Thus $\langle a, y_1 \rangle$ is semiregular on $\Delta - \{1', 2'\}$.

Suppose that $\langle a, y_1, y_2 \rangle$ is not semiregular on $\Delta - \{1', 2'\}$. Then $\langle a, y_1, y_2 \rangle$ has an orbit Δ' of length four in $\Delta - \{1', 2'\}$. Since $\langle a, y_1, y_2 \rangle$ is an abelian group, there is an involution y' in $\langle a, y_1 \rangle y_2$ fixing Δ' pointwise. Then $y'^{I(a)}$ is an involution consisting of two 2-cycles and $I(y') \cap \Delta \supseteq \Delta'$, contrary to (ii.i). Thus $\langle a, y_1, y_2 \rangle$ is semiregular on $\Delta - \{1', 2'\}$.

Suppose that $\langle a, y_1, y_2, y_3 \rangle$ is not semiregular on $\Delta - \{1', 2'\}$. Then $\langle a, y_1, y_2, y_3 \rangle$ has an orbit Δ' of length eight in $\Delta - \{1', 2'\}$. Since $\langle a, y_1, y_2, y_3 \rangle$ is an abelian group of order sixteen, there is exactly one involution y' in $\langle a, y_1, y_2, y_3 \rangle$ fixing Δ' pointwise. Since $|\Delta'| = 8$, y' has at least four 2-cycles on $I(a)$. Thus $y' = y_1 y_2 y_3$ or $ay_1 y_2 y_3$. If $y' = y_1 y_2 y_3$, then $I(y')$ contains $(I(a) - \{1, 2, \cdots, 8\}) \cup \{1', 2'\} \cup \Delta'$ of length $t+2$, contrary to the assumption (*). Thus $y' = ay_1 y_2 y_3$. Then $I(ay_1 y_2 y_3) = (I(a) - \{1, 2, \cdots, 8\}) \cup \Delta'$ since $|(I(a) - \{1, 2, \cdots, 8\}) \cup \Delta'| = t$. Furthermore this shows that $\langle a, y_1, y_2, y_3 \rangle$ has no orbit of length eight in $\Delta - (\{1', 2'\} \cup \Delta')$. On the other hand $C(a)$ has a 2-element

$$y_1' = (1\ 3)\ (2\ 4)\ (5)\ (6)\cdots(t)\cdots .$$

By (2.3) we may assume that $\langle a, y_1, y_2, y_3, y_1'\rangle$ is a 2-group. Then y_1' normalizes $\langle a, y_1, y_2, y_3\rangle$ and so y_1' fixes $\{1', 2'\}$ and Δ'. Set $R=\langle a, y_1, y_2, y_3, y_1'\rangle_i$, where $i\in\Delta'$. Then the order of R is four and so R is cyclic or elementary abelian. Since $\langle a, y_1\rangle$ is contained in the center of $\langle a, y_1, y_2, y_3, y_1'\rangle$ and semiregular on Δ', any element of R fixes at least four points of Δ. Suppose that R is a cyclic group generated by an element z. Then since $ay_1y_2y_3$ is the involution of R, $z^2=ay_1y_2y_3$. Thus $z^{I(a)}$ has two 4-cycles since $(ay_1y_2y_3)^{I(a)}=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)$. However this is impossible since $\langle a, y_1, y_2, y_3, y_1'\rangle^{I(a)}$ has no such element. Next suppose that R is elementary abelian. Since $R_{I(a)}=1$, $R^{I(a)}$ is also an elementary abelian group of order four. Furthermore since any element of R fixes at least four points of Δ, every element $(\neq 1)$ of $R^{I(a)}$ has at least three 2-cycles by the assumption (*) and (ii.i). This is a contradiction since $\langle a, y_1, y_2, y_3, y_1'\rangle^{I(a)}$ has no such group. Thus $\langle a, y_1, y_2, y_3\rangle$ is semiregular on $\Delta-\{1', 2'\}$. Hence by (2.6) $\langle a, y_1, y_2, \cdots, y_k\rangle$ is semiregular on $\Delta-\{1', 2'\}$.

On the other hand a normalizes $G_{1'2'3'4'}$, which is of even order. Hence a commutes with an involution u of $G_{1'2'3'4'}$. Since $C(a)^{I(a)}=A_t$, $\langle a, y_1, y_2, \cdots, y_k\rangle$ has a subgroup which is conjugate to $\langle a, u\rangle$ in $C(a)$. Since u fixes at least four points of Δ, $\langle a, y_1, y_2, \cdots, y_k\rangle$ has an element $(\neq 1)$ fixing at least four points of Δ, which is a contradiction. Thus $C(a)^{I(a)}\neq A_t$. Hence $|\Delta|\equiv 0 \pmod 4$.

2.8. *Let x be a 2-element of $N(Q)$ such that $x^{I(Q)}$ is an involution consisting of m 2-cycles. If x fixes r Q-orbits in Δ, then $r\leqq 2m$ and Qx has at least $\frac{r}{2m}|Q|$ involutions which have fixed points in Δ.*

Proof. Assume that x fixes r Q-orbits $\Delta_1, \Delta_2, \cdots, \Delta_r$ in Δ. Set $\Gamma=\Delta_1\cup\Delta_2\cup\cdots\cup\Delta_r$. Then

$$r\cdot|\langle Q, x\rangle| = \sum_{u\in\langle Q,x\rangle}|I(u^\Gamma)|\ .$$

Since $\langle Q, x\rangle=Q+Qx$ and $|Q|=|\Delta_1|=\cdots=|\Delta_r|$,

$$\begin{aligned} r\cdot 2\cdot|Q| &= \sum_{u\in Q}|I(u^\Gamma)|+\sum_{u\in Q}|I((ux)^\Gamma)| \\ &= r\cdot|Q|+\sum_{u\in Q}|I((ux)^\Gamma)|\ . \end{aligned}$$

Hence

$$\sum_{u\in Q}|I((ux)^\Gamma)| = r\cdot|Q|\ .$$

On the other hand $|I(x)\cap I(Q)|=t-2m$. Hence for any element u of Q $|I(ux)\cap\Delta|\leq 2m$ by the assumption (*). Hence $|I((ux)^\Gamma)|\leq 2m$. Suppose that Qx has s elements which have fixed points in Γ. Then

$$\sum_{u\in Q}|I((ux)^\Gamma)|\leqq 2ms\ .$$

Hence $r\cdot|Q|\leqq 2ms$. Thus $\frac{r}{2m}\cdot|Q|\leqq s$. Furthermore since $s\leqq|Q|$, $\frac{r}{2m}\cdot|Q|\leqq|Q|$. Hence $r\leqq 2m$.

Let x' be any element of Qx such that $|I(x')\cap\Delta|\neq 0$. Then $|I(x'^2)|>t$. Hence $x'^2=1$ by theassumption $(*)$.

We use the following notations: Assume that the Q-orbits on Δ consist of $\Delta_1, \Delta_2, \cdots, \Delta_r$. For any element $x\in N(Q)$ let $\bar{x}$ be the permutation on $\{\Delta_1, \Delta_2, \cdots, \Delta_r\}$ induced by x,

$$\bar{x}=\begin{pmatrix}\Delta_1 & \Delta_2 & \cdots\Delta_r \\ \Delta_1^x & \Delta_2^x & \Delta_r^x\end{pmatrix}.$$

Then $\bar{x}$ form a permutation group $\overline{N(Q)}$ on $\bar{\Delta}=\{\Delta_1, \Delta_2, \cdots, \Delta_r\}$.

2.9. *Suppose that $N(Q)$ has the 2-group $\langle Q, x_1, x_2, \cdots, x_k\rangle$ as in* (2.4), *and $\langle Q, x_1, x_2, \cdots, x_k\rangle$ fixes a subset Δ' of Δ. If $\langle Q, x_1, x_2, x_3, x_4\rangle$ is semiregular on Δ', then $\langle Q, x_1, x_2, \cdots, x_k\rangle$ is semiregular on Δ'.*

Proof. Suppose that $\langle Q, x_1, x_2, \cdots, x_i\rangle$, $i\geq 4$, is semiregular on Δ' and $\langle Q, x_1, x_2, \cdots, x_{i+1}\rangle$ is not semiregular on Δ'. Then $\langle Q, x_1, x_2, \cdots, x_i\rangle x_{i+1}$ has an element x having fixed points in Δ'. Since $\langle\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{i+1}\rangle$ is abelian and $\langle\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_i\rangle$ is semiregular on the set of the Q-orbits contained in Δ', $\bar{x}$ fixes at least 2^i Q-orbits in Δ'. On the other hand since $x\in\langle Q, x_1, x_2, \cdots, x_{i+1}\rangle$, x has at most $i+1$ 2-cycles on $I(Q)$. Hence by (2.8) $2^i\leq 2(i+1)$, so $i\leq 3$, which is a contradiction. Thus if $\langle Q, x_1, x_2, \cdots, x_i\rangle$, $i\geq 4$, is semiregular on Δ', then $\langle Q, x_1, x_2, \cdots, x_{i+1}\rangle$ is semiregular on Δ'. Since $\langle Q, x_1, x_2, x_3, x_4\rangle$ is semiregular on Δ', this implies by induction that $\langle Q, x_1, x_2, \cdots, x_k\rangle$ is semiregular on Δ'.

2.10. *Suppose that $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ as in* (2.4) *fixes a subset Δ' of Δ. If $\langle Q, y_1, y_2, y_3, y_4, y_1'\rangle$ is semiregular on Δ', then $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ is semiregular on Δ'.*

Proof. Suppose that $\langle Q, y_1, y_2, \cdots, y_i, y_1'\rangle$, $i\geq 4$, is semiregular on Δ' and $\langle Q, y_1, y_2, \cdots, y_{i+1}, y_1'\rangle$ is not semiregular on Δ'. Then there is an element y $(\neq 1)$ in $\langle Q, y_1, y_2, \cdots, y_{i+1}, y_1'\rangle$ such that $\bar{y}$ fixes Q-orbits in Δ'. Then $y^{I(Q)}$ is of order four or two. If $y^{I(Q)}$ is of order four, then $y^{I(Q)}$ consists of exactly one 4-cycle (1 3 2 4) or (1 4 2 3) and some 2-cycles. Hence $(y^2)^{I(Q)}=y_1^{I(Q)}$ and so $\bar{y}^2=\bar{y}_1$. This is a contradiction since $\bar{y}_1$ has no fixed point in the set of the Q-orbits in Δ'. Thus $y^{I(Q)}$ is of order two and consists of at most $i+2$ 2-cycles. Then $\bar{y}$ centralizes $\langle\bar{y}_1, \bar{y}_2\bar{y}_3, \bar{y}_2\bar{y}_4, \cdots, \bar{y}_2\bar{y}_i, \bar{y}_1'\rangle$ or $\langle\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_i\rangle$, which is semiregular on the set of Q-orbits in Δ' and of order 2^i. Hence $\bar{y}$ fixes at least 2^i Q-orbits in Δ' and so by (2.8) $2^i\leq 2(i+2)$. Hence $i\leq 3$, which is a contradiction. Thus if $\langle Q, y_1, y_2, \cdots, y_i, y_1'\rangle$, $i\geq 4$, is semiregular on Δ', then $\langle Q, y_1, y_2,$

$\cdots, y_{i+1}, y_1'\rangle$ is semiregular on Δ'. Since $\langle Q, y_1, y_2, y_3, y_4, y_1'\rangle$ is semiregular on Δ', this implies by induction that $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ is semiregular on Δ'.

2.11. *G is not 5-fold transitive on Ω.*

Proof. If G is 5-fold transitive on Ω, then G_1 is 4-fold transitive on $\Omega-\{1\}$ and satisfies the assumptions of the theorem. Hence by the minimal nature of the degree of G, G_1 contains A_{n-1}, so G contains A_n. This is a contradiction. Thus G is not 5-fold transitive.

2.12. *Let x be an involution of $N(Q)$. If there is a Q-orbit Δ' in Δ such that $|I(x)\cap\Delta'|=2$, then $C(Q)^{I(Q)}=A_t$ or S_t.*

Proof. Since x is an involution and $|I(x)\cap\Delta'|=2$, x induces an involutory automorphism of Q which fixes exactly two elements. By a theorem of H. Zassenhaus ([16], Satz 5) Q contains a cyclic group of index two. Then the automorphism group of Q is S_3, S_4 or a 2-group (cf. H. Zassenhaus [17], IV, §3, Exercise 4). Since $N(Q)^{I(Q)}=A_t$ or S_t, $t\geq 6$ and $N(Q)^{I(Q)}/C(Q)^{I(Q)}$ is involved in the automorphism group of Q, $C(Q)^{I(Q)}$ contains A_t.

2.13. *Let x be a 2-element of $N(Q)$. If $x^{I(Q)}$ is an involution consisting of exactly one 2-cycle, then $|I(x)\cap\Delta|=0$.*

Proof. Since $|I(x)|\leq t$, $|I(x)\cap\Delta|=0$ or 2. Suppose by way of contradiction that $x^{I(Q)}$ is an involution consisting of exactly one 2-cycle and $|I(x)\cap\Delta|=2$. Then $|I(x^2)|\geq t+2$. Hence $x^2=1$. Since $x^{I(Q)}$ is an odd permutation, $N(Q)^{I(Q)}=S_t$. Furthermore by (2.12) $C(Q)^{I(Q)}=S_t$ or A_t. We treat these cases separately.

(i) Suppose that $C(Q)^{I(Q)}=S_t$. Then $C(Q)$ has a 2-element x' such that $x'^{I(Q)}=x^{I(Q)}$. Since Q is a Sylow 2-subgroup of $G_{I(Q)}$, $\langle Q, x\rangle$ and $\langle Q, x'\rangle$ are Sylow 2-subgroups of $\langle Q, x, x'\rangle$. Hence $\langle Q, x\rangle$ is conjugate to $\langle Q, x'\rangle$. Thus x is conjugate to $x'c$, where $c\in Q$, and so $|I(x'c)\cap\Delta|=2$. Hence $x'c$ commutes with exactly one element of Q other than 1, which is a central involution of Q. On the other hand since $x'\in C(Q)$, x' commutes with c. Hence $x'c$ commutes with c. Thus c is 1 or a central involution of Q. Hence $x'c\in C(Q)$ and so Q is of order two. Set $Q=\langle a\rangle$. Then we may assume that

$$a=(1)\,(2)\cdots(t)\,(1'\ 2')\,(3'\ 4')\cdots(n-1\ n)\,.$$

Since $|\Delta|\equiv 0 \pmod 4$ and $|I(x)\cap\Delta|=2$, $|I(ax)\cap\Delta|\equiv 2 \pmod 4$. Hence $|I(ax)\cap\Delta|=2$ because $|I(ax)|\leq t$. Since $C(a)^{I(a)}=S_t$, $C(a)$ has the 2-group $\langle a, x_1, x_2, \cdots, x_k\rangle$ as in (2.4). Since $\langle a, x_i\rangle$, $1\leq i\leq k$, is conjugate to $\langle a, x\rangle$ in $C(a)$, $\langle a, x_i\rangle$ is elementary abelian and $|I(x_i)\cap\Delta|=|I(ax_i)\cap\Delta|=2$. Hence we may assume that

$$x_1=(1\ 2)\ (3)\ (4)\cdots(t)\ (1')\ (2')\ (3'\ 4')\ (5'\ 7')\ (6'\ 8')\cdots.$$

Then $\langle a, x_1\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$.

Now we show that $\langle a, x_1, x_2, \cdots, x_k\rangle$ is elementary abelian and semiregular on $\Delta-\{1', 2', 3', 4'\}$, where $\{1', 2'\}$ and $\{3', 4'\}$ are $\langle a, x_1, x_2, \cdots, x_k\rangle$-orbits of length two. Since x_2 normalizes $\langle a, x_1\rangle$, $x_1^{x_2}=x_1$ or ax_1. Suppose that $x_1^{x_2}=ax_1$. Then $(x_1 x_2)^2=a$. Hence $\langle x_1 x_2\rangle$ is a cyclic group of order four and contains a. On the other hand since $C(a)^{I(a)}=S_t$, $\langle a, x_1, x_3\rangle$ is conjugate to $\langle a, x_1, x_2\rangle$ in $C(a)$. Hence $x_1^{x_3}=ax_1$. Thus $x_1^{x_2x_3}=x_1$ and so $x_2 x_3$ centralizes $\langle a, x_1\rangle$. Furthermore since $I(x_1)\cap\Delta=\{1', 2'\}$ and $I(ax_1)\cap\Delta=\{3', 4'\}$, $x_2 x_3$ fixes $\{1', 2'\}$ and $\{3', 4'\}$. Thus $I((x_2 x_3)^2)$ contains $I(a)\cap\{1', 2', 3', 4'\}$ of length $t+4$. Hence $(x_2 x_3)^2=1$. This is a contradiction since $\langle a, x_2x_3\rangle$ is conjugate to the cyclic group $\langle x_1 x_2\rangle$. Thus x_2 commutes with x_1 and so $\langle a, x_1, x_2\rangle$ is elementary abelian. Furthermore $\langle a, x_1, x_2\rangle$ is conjugate to $\langle a, x_i, x_j\rangle$, $i\neq j$ and $1\leq i, j\leq k$. Hence $\langle a, x_i, x_j\rangle$ is also elementary abelian. Thus $\langle a, x_1, x_2 \cdots, x_k\rangle$ is elementary abelian. Since $I(x_1)\cap\Delta=\{1', 2'\}$ and $I(ax_1)\cap\Delta=\{3', 4'\}$, $\{1', 2'\}$ and $\{3', 4'\}$ are $\langle a, x_1, x_2, \cdots, x_k\rangle$-orbits of length two. Since x_i or ax_i, $2\leq i\leq k$, fixes $\{1', 2'\}$ pointwise, we may assume that x_i fixes $\{1', 2'\}$ pointwise.

Suppose that $\langle a, x_1, x_2\rangle$ is not semiregular on $\Delta-\{1', 2', 3', 4'\}$. Then $\langle a, x_1, x_2\rangle$ has an orbit Δ' of length four in $\Delta-\{1', 2', 3', 4'\}$. Since $\langle a, x_1, x_2\rangle$ is an elementary abelian group of order eight, there is exactly one involution x' in $\langle a, x_1, x_2\rangle$ fixing Δ' pointwise. Since $|\Delta'|=4$, x' has at least two 2-cycles in $I(a)$. Hence $x'=x_1 x_2$ or $ax_1 x_2$. If $x'=x_1 x_2$, then $I(x')$ contains $(I(a)-\{1,2,3,4\})\cup\{1', 2'\}\cup\Delta'$ of length $t+2$, contrary to the assumption $(*)$. Thus $x'=ax_1 x_2$. Then $I(ax_1 x_2)=(I(a)-\{1,2,3,4\})\cup\Delta'$ since $|(I(a)-\{1,2,3,4\})\cup\Delta'|=t$. This shows that $\langle a, x_1, x_2\rangle$ is semiregular on $\Delta-(\{1', 2', 3', 4'\}\cup\Delta')$. By (2.4) $C(a)$ has x_3. Then x_3 normalizes $\langle a, x_1, x_2\rangle$ and so fixes Δ'. Hence by the same argument as above $ax_1 x_3$ fixes Δ' pointwise. Thus $I(ax_1 x_2\cdot ax_1 x_3)=I(x_2 x_3)$ contains $(I(a)-\{3, 4, 5, 6\})\cup\{1', 2', 3', 4'\}\cup\Delta'$ of length $t+4$, contrary to the assumption $(*)$. Thus $\langle a, x_1, x_2\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$. Hence by (2.5) $\langle a, x_1, x_2, \cdots, x_k\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$.

On the other hand $\langle a, x_1\rangle$ normalizes $G_{5'6'7'8'}$, which is even order. Hence a and x_1 commute with an involution u of $G_{5'6'7'8'}$. Since $I(x_1)\cap\Delta=\{1', 2'\}$ and $I(ax_1)\cap\Delta=\{3', 4'\}$, $\langle a, u\rangle$ has at least four orbits $\{1', 2'\}$, $\{3', 4'\}$, $\{5', 6'\}$ and $\{7', 8'\}$ of length two in Δ. Since $C(a)^{I(a)}=S_t$, $\langle a, x_1, x_2, \cdots, x_k\rangle$ has a subgroup $\langle a, u'\rangle$ which is conjugate to $\langle a, u\rangle$ in $C(a)$. This is a contradiction since $\langle a, u'\rangle$ has exactly two orbits $\{1', 2'\}$ and $\{3', 4'\}$ of length two in Δ. Thus $C(Q)^{I(Q)}\neq S_t$.

(ii) Suppose that $C(Q)^{I(Q)}=A_t$.

(ii.i) We show that x fixes exactly one Q-orbit in Δ. Since $|I(x)\cap\Delta|=2$, x fixes at least one Q-orbit in Δ. On the other hand by (2.8) x fixes at most two Q-orbits. Suppose that x fixes exactly two Q-orbits Δ_1 and Δ_2 in Δ. Let u be any element of Q. Then by (2.8) ux is an involution having fixed points in Δ_1 or Δ_2. Since ux consists of one 2-cycle on $I(Q)$, ux fixes two points and these two points are contained in either Δ_1 or Δ_2. Hence $\langle Q, x\rangle$ is semiregular on $\Delta-(\Delta_1\cup\Delta_2)$. Since $(ux)^2=1$, $u^x=u^{-1}$. In particular if u is an involution, then x commutes with u. On the other hand since $|I(x)\cap\Delta|=2$, x commutes with exactly one involution of Q. Hence Q has exactly one involution and so Q is a cyclic or generalized quaternion group. Let u and u' be any two elements of Q. Then $(uu')^x=(uu')^{-1}$, and $(uu')^x=u^xu'^x=u^{-1}u'^{-1}=(u'u)^{-1}$. Hence $uu'=u'u$ and so Q is a cyclic group. Furthermore since $C(Q)^{I(Q)}=A_t$, any 2-element of $N(Q)$ whose restriction on $I(Q)$ is an even permutation belongs to $C(Q)$.

$N(Q)$ has the 2-group $\langle Q, x_1, x_2, x_3\rangle$ as in (2.4). Since $\langle Q, x_1\rangle$ is conjugate to $\langle Q, x\rangle$, we may assume that $x_1=x$,

$$x_1=(1\ 2)\ (3)\ (4)\cdots(t)\ (1')\ (2')\ (3'\ 4')\cdots$$

and $\{1', 2'\}\subset\Delta_1$. Since x_2 normalizes $\langle Q, x_1\rangle$ and $\langle Q, x_1\rangle$ has exactly two orbits Δ_1 and Δ_2 of length $|Q|$, $\Delta_1^{x_2}=\Delta_1$ or Δ_2. First assume that $\Delta_1^{x_2}=\Delta_1$. Since $\langle Q, x_1, x_3\rangle$ is conjugate to $\langle Q, x_1, x_2\rangle$ in $N(Q)$, $\Delta_1^{x_3}=\Delta_1$. Hence $\Delta_1^{x_2x_3}=\Delta_1$. Next assume that $\Delta_1^{x_2}=\Delta_2$. Then similarly $\Delta_1^{x_3}=\Delta_2$. Hence $\Delta_1^{x_2x_3}=\Delta_1$. Thus in either case $\Delta_1^{x_2x_3}=\Delta_1$. Hence there is an element y in Qx_2x_3 such that $|I(y)\cap\Delta_1|\neq 0$. Since $y^{I(Q)}=(3\ 4)\ (5\ 6)$, $|I(y)\cap\Delta_1|=2$ or 4. Furthermore as we have seen above $y\in C(Q)$. Hence $|Q|=2$ or 4. However we assumed that $N(Q)\neq C(Q)$. Hence $|Q|=4$. Let $Q=\langle b\rangle$. Since $b^{x_1}=b^{-1}$, we may assume that

$$b=(1)\ (2)\cdots(t)\ (1'\ 3'\ 2'\ 4')\ (5'\ 7'\ 6'\ 8')\cdots,$$

$\Delta_1=\{1', 2', 3', 4'\}$ and $\Delta_2=\{5', 6', 7', 8'\}$. Then

$$y=(1)\ (2)\ (3\ 4)\ (5\ 6)\ (7)\ (8)\cdots(t)\ (1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\cdots.$$

On the other hand $C(Q)$ has a 2-element

$$y'=(1)\ (2)\ (3\ 5)\ (4\ 6)\ (7)\ (8)\cdots(t)\cdots.$$

By (2.3) we may assume that $\langle Q, x_1, y, y'\rangle$ is a 2-group. Since $\langle Q, x_1, y'\rangle$ is conjugate to $\langle Q, x_1, y\rangle$ in $N(Q)$, $\Delta_1^{y'}=\Delta_1$ and $\Delta_2^{y'}=\Delta_2$. Then Qy' has an element

$$y''=(1)\ (2)\ (3\ 5)\ (4\ 6)\ (7)\ (8)\cdots(t)\ (1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\cdots.$$

Then yy'' is of even order and $I(yy'')$ contains $(I(Q)-\{3, 4, 5, 6\})\cup\Delta_1\cup\Delta_2$ of length $t+4$, contrary to the assumption $(*)$. Thus x_1 fixes exactly one Q-orbit in Δ.

(ii.ii) We show that $|Q|=4$. Since $N(Q)^{I(Q)}\neq C(Q)^{I(Q)}$, $|Q|\neq 2$. Suppose by way of contradiction that $|Q|\geq 8$. By (2.4) $N(Q)$ has the 2-group $\langle Q, x_1, x_2,$

$x_3\rangle$. Since $\langle Q, x_1\rangle$ is conjugate to $\langle Q, x\rangle$, we may assume that $x_1=x$ and

$$x_1 = (1\ 2)\ (3)\ (4)\cdots(t)\ (1')\ (2')\ (3'\ 4')\ (5'\ 7')\ (6'\ 8')\cdots.$$

Then there is exactly one involution a in Q commuting with x_1. Then we may assume that

$$a = (1)\ (2)\cdots(t)\ (1'\ 2')\ (3'\ 4')\ (5'\ 6')\ (7'\ 8')\cdots(n-1\ n).$$

By (ii.i) there is exactly one Q-orbit Δ_1 in Δ fixed by x_1. Since $|\Delta_1|=|Q|\geq 8$, we may assume that $\Delta_1\supseteq\{1', 2', \cdots, 8'\}$. Since x_2 and x_3 normalizes $\langle Q, x_1\rangle$, x_2 and x_3 fix Δ_1. Thus Qx_2 and Qx_3 have elements fixing $1'$ of Δ_1. We may assume that x_2 and x_3 fix $1'$. Then $I(x_i^{x_j}\cdot x_i)\supseteq I(a)\cup\{1'\}$, $1\leq i, j\leq 3$. Hence $x_2^2=x_3^2=1$ and x_i commutes with x_j. Since $I(x_1)\cap\Delta=\{1', 2'\}$ and $|I(x_i)|\leq t$, $i=2, 3$, $I(x_i)\cap\Delta=\{1', 2'\}$. This implies that x_2 and x_3 commute with a. Thus $\langle a, x_1, x_2, x_3\rangle$ is elementary abelian. Furthermore $I(ax_1)\cap\Delta=\{3', 4'\}$. Hence x_2 and $x_3=(1')\ (2')\ (3'\ 4')$ on $\{1', 2', 3', 4'\}$. On the other hand $|\Delta_1-\{1', 2', 3', 4'\}|\equiv 4$ (mod 8). Hence $\langle a, x_1, x_2, x_3\rangle$ has an orbit of length four in $\Delta_1-\{1', 2', 3', 4'\}$. Hence we may assume that $\{5', 6', 7', 8'\}$ is the $\langle a, x_1, x_2, x_3\rangle$-orbit of length four. Since $|\langle a, x_1, x_i\rangle|=8$, $i=2, 3$, there is an involution x_i' in $\langle a, x_1, x_i\rangle$ fixing $\{5', 6', 7', 8'\}$ pointwise. Since $|I(x_i')|\leq t$, $x_i'=x_1x_i$ or ax_1x_i. If $x_i'=x_1x_i$, then $I(x_1x_i)\cap\Delta\supseteq\{1', 2', \cdots, 8'\}$ and so $|I(x_1x_i)|\geq t+4$, contrary to the assumption (*). Thus $x_i'=ax_1x_i$. Hence $I(ax_1x_2\cdot ax_1x_3)=I(x_2x_3)$ contains $(I(a)-\{3, 4, 5, 6\})\cup\{1', 2', \cdots, 8'\}$ of length $t+4$, contrary to the assumption (*). Thus $|Q|=4$.

(ii.iii) We show that $|Q|=4$ implies a contradiction. $N(Q)$ has the 2-group $\langle Q, x_1, x_2, \cdots, x_k\rangle$ as in (2.4). Since $\langle Q, x_1\rangle$ is conjugate to $\langle Q, x\rangle$, we may assume that $x_1=x$ and

$$x_1 = (1\ 2)\ (3)\ (4)\cdots(t)\ (1')\ (2')\ (3'\ 4')\ (5'\ 7')\ (6'\ 8')\cdots.$$

Let a be an involution of Q commuting with x_1. Then we may assume that

$$a = (1)\ (2)\cdots(t)\ (1'\ 2')\ (3'\ 4')\cdots(n-1\ n).$$

Then by (ii.i) and (ii.ii) $\{1', 2', 3', 4'\}$ is a $\langle Q, x_1\rangle$-orbit and $\langle Q, x_1\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$. Since x_i normalizes $\langle Q, x_1\rangle$, $2\leq i\leq k$, x_i fixes $\{1', 2', 3', 4'\}$. Hence Qx_i has an element fixing $1'$. We may assume that x_i fixes $1'$. Then $I(x_i^{x_j}\cdot x_i)$, $1\leq i, j\leq k$, contains $I(Q)\cup\{1'\}$ of length $t+1$. Hence $x_i^{x_j}\cdot x_i=1$. Thus $x_i^2=1$ and $x_ix_j=x_jx_i$. Furthermore $I(x_1)\cap\Delta=\{1', 2'\}$. Hence $I(x_i)\cap\Delta=\{1', 2'\}$, $i\geq 2$. This implies that x_i commutes with a. Thus $\langle a, x_1, x_2, \cdots, x_k\rangle$ is elementary abelian and $x_i=(1')\ (2')\ (3'\ 4')$ on $\{1', 2', 3', 4'\}$, $1\leq i\leq k$. Furthermore since x_ix_j, $1\leq i, j\leq k$, fixes $\{1', 2', 3', 4'\}$ pointwise, $\langle a, x_ix_j\rangle<Z(\langle Q, x_1, x_2, \cdots, x_k\rangle)$.

Now we show that $\langle Q, x_1, x_2, \cdots, x_k\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$.

Suppose that $\langle Q, x_1, x_2\rangle$ is not semiregular on $\Delta-\{1', 2', 3', 4'\}$. Then there is a $\langle Q, x_1, x_2\rangle$-orbit Δ' of length eight. Since $\langle Q, x_1\rangle$ and $\langle Q, x_2\rangle$ are semiregular on $\Delta-\{1', 2', 3', 4'\}$, there is an element u in Q such that ux_1x_2 has fixed points in Δ'. If $u=1$ or a, then $ux_1x_2\in Z(\langle Q, x_1, x_2\rangle)$. Thus ux_1x_2 fixes Δ' pointwise and so $|I(ux_1x_2)|\geq t+4$, contrary to the assumption (*). Thus $u\neq 1, a$. Since $0<|I(ux_1x_2)\cap\Delta'|\leq 4$ and $ux_1x_2\in C(Q)$, ux_1x_2 fixes exactly four points of Δ'. Since $|\Delta'|=8$, there is an element u' in Q such that $u'x_1x_2$ fixes exactly four points of Δ' which are not fixed by ux_1x_2. By the same reason as above $u'\neq 1, a$. Hence $u'=ua$. Furthermore this shows that $\langle Q, x_1, x_2\rangle$ is semiregular on $\Delta-(\{1', 2', 3', 4'\}\cup\Delta')$. By (2.4) $N(Q)$ has x_3. Then x_3 normalizes $\langle Q, x_1, x_2\rangle$ and so fixes Δ'. Hence by the same argument as above $u''x_1x_3$, where $u''=u$ or ua, fixes the same points of Δ' that ux_1x_2 fixes. Then $ux_1x_2\cdot u''x_1x_3=uu''x_2x_3$ has fixed points in Δ'. Since $uu''=u^2$ or u^2a and $u^2=1$ or a, $uu''=1$ or a. Hence $uu''x_2x_3\in C(\langle Q, x_1, x_2\rangle)$ and so $uu''x_2x_3$ fixes Δ' pointwise. Thus $|I(uu''x_2x_3)|\geq t+4$, contrary to the assumption (*). Thus $\langle Q, x_1, x_2\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$.

Suppose that $\langle Q, x_1, x_2, x_3\rangle$ is not semiregular on $\Delta-\{1', 2', 3', 4'\}$. Then there is a $\langle Q, x_1, x_2, x_3\rangle$-orbit Δ' of length sixteen. Since $\langle Q, x_1, x_3\rangle$ and $\langle Q, x_2, x_3\rangle$ are conjugate to $\langle Q, x_1, x_2\rangle$ in $N(Q)$, $\langle Q, x_1, x_3\rangle$ and $\langle Q, x_2, x_3\rangle$ are semiregular on $\Delta-\{1', 2', 3', 4'\}$. Hence there is an elemenet x' in $Qx_1x_2x_3$ such that x' has fixed points in Δ'. Since $\langle a, x_1x_2, x_1x_3\rangle<Z(\langle Q, x_1, x_2, x_3\rangle)$, $x'\in C(\langle a, x_1x_2, x_1x_3\rangle)$. On the other hand $\langle Q, x_1, x_2\rangle$, $\langle Q, x_1, x_3\rangle$ and $\langle Q, x_2, x_3\rangle$ are semiregular on Δ'. Hence $\langle a, x_1x_2, x_1x_3\rangle$ is semiregular on Δ'. Since x' has fixed points in Δ' and $|\langle a, x_1x_2, x_1x_3\rangle|=8$, x' fixes at least eight points of Δ'. Thus $|I(x')|\geq t-6+8=t+2$, contrary to the assumption (*). Thus $\langle Q, x_1, x_2, x_3\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$.

Suppose that $\langle Q, x_1, x_2, x_3, x_4\rangle$ is not semiregular on $\Delta-\{1', 2', 3', 4'\}$. Then $\langle Q, x_1, x_2, x_3, x_4\rangle$ has an orbit Δ' of length 2^5. Since $\langle Q, x_2, x_3, x_4\rangle$, $\langle Q, x_1, x_2, x_4\rangle$ and $\langle Q, x_1, x_3, x_4\rangle$ ar conjugate to $\langle Q, x_1, x_2, x_3\rangle$ in $N(Q)$, these groups are semiregular on $\Delta-\{1', 2', 3', 4'\}$. Hence there is an element x' in $Qx_1x_2x_3x_4$ such that x' has fixed points in Δ'. Since $\langle Q, x_1x_2, x_3x_4\rangle<C(Q)$, $x'\in C(Q)$. Furthermore since x_1x_2 and $x_3x_4\in Z(\langle Q, x_1, x_2, x_3, x_4\rangle)$, x_1x_2 and x_1x_3 commute with x'. Thus $x'\in C(\langle Q, x_1x_2, x_1x_3\rangle)$. Since $\langle Q, x_1x_2, x_1x_3\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$ and of order 2^4, x' fixes at least 2^4 points in Δ'. Then $|I(x')|\geq t-2\cdot 4+2^4=t+8$, contrary to the assumption (*). Thus $\langle Q, x_1, x_2, x_3, x_4\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$. Hence by (2.9) $\langle Q, x_1, x_2, \cdots, x_k\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$.

On the other hand $\langle a, x_1\rangle$ normalizes $G_{5'6'7'8'}$, which is even order. Hence a and x_1 commute with an involution u of $G_{5'6'7'8'}$. Then $\langle a, x_1, u\rangle$ normalizes $G_{I(Q)}$. Hence there is a Sylow 2-subgroup Q' of $G_{I(Q)}$ such that $\langle a, x_1, u\rangle$ normalizes Q'. Since Q' is conjugate to Q in $G_{I(Q)}$ and $N(Q)^{I(Q)}=S_t$, $\langle Q', a, x_1, u\rangle$

is conjugate to a subgroup of $\langle Q, x_1, x_2, \cdots, x_k\rangle$ in $N(G_{I(Q)})$. Then $\langle Q', a, x_1, u\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$ since $I(x_1)\cap\Delta=\{1', 2'\}$ and $I(ax_1)\cap\Delta=\{3', 4'\}$. This is a contradiction since $I(u)\cap\Delta\supseteq\{5', 6', 7', 8'\}$. Thus $C(Q)^{I(Q)}\neq A_t$ and so we complete the proof of (2.13)

2.14. *Let y be a 2-element of $N(Q)$. If $y^{I(Q)}$ is an involution consisting of exactly two 2-cycles, then $|I(y)\cap\Delta|\neq 2$.*

Proof. Suppose by way of contradiction that $y^{I(Q)}$ is an involution consisting of exactly two 2-cycles and $|I(y)\cap\Delta|=2$. Then $|I(y^2)|\geq t+2$. Hence $y^2=1$. We may assume that

$$y=(1\ 2)\ (3\ 4)\ (5)\ (6)\cdots(t)\ (1')\ (2')\ (3'\ 4')\cdots.$$

Then by (2.12) $C(Q)^{I(Q)}=S_t$ or A_t. Then since $y^{I(Q)}$ is an even permutation, $y^{I(Q)}\in C(Q)^{I(Q)}$. Thus there is an element a of Q such that $ay\in C(Q)$. Hence ay commutes with a and so y commutes with a. On the other hand y commutes with exactly one involution of Q, which is a central involution of Q. Hence $a\in Z(Q)$ and so $y\in C(Q)$. Thus $|Q|=2$ and so $Q=\langle a\rangle$. Since $I(y)\cap\Delta=\{1', 2'\}$ and $|\Delta-\{1', 2'\}|\equiv 2 \pmod 4$, $|I(ay)\cap\Delta|\equiv 2 \pmod 4$. Hence $|I(ay)\cap\Delta|=2$. Thus we may assume that

$$a=(1)\ (2)\cdots(t)\ (1'\ 2')\ (3'\ 4')\cdots(n-1\ n).$$

Then $\langle a, y\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$. Since $C(a)^{I(a)}\geq A_t$, there is an element z in $C(Q)$ of the form

$$z=(1\ 3\ 2\ 4)\ (5\ 6)\cdots(t)\cdots.$$

By (2.3) we may assume that $\langle a, y, z\rangle$ is a 2-group. Then $z^2=y$ or ay, and so $I(z^2)\cap\Delta=\{1', 2'\}$ or $\{3', 4'\}$. Thus z consists of 4-cycles on $\Delta-\{1', 2'\}$ or $\Delta-\{3', 4'\}$. Hence $|\Delta|\equiv 2 \pmod 4$, contrary to (2.7). Thus we complete the proof.

2.15. *Let y be a 2-element of $N(Q)$. If $y^{I(Q)}$ is an involution consisting of exactly two 2-cycles, then $|I(y)\cap\Delta|=0$.*

Proof. Since $|I(y)\cap I(Q)|=t-4$, $|I(y)\cap\Delta|=0$, 2 or 4. By (2.14) $|I(y)\cap\Delta|\neq 2$. Hence suppose by way of contradiction that $|I(y)\cap\Delta|=4$. By (2.4) $N(Q)$ has the 2-group $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$. Since $\langle Q, y_1\rangle$ is conjugate to $\langle Q, y\rangle$, we may assume that $y_1=y$.

First we show that y_1 fixes at least two Q-orbits in Δ. Suppose by way of contradiction that y_1 fixes exactly one Q-orbit Δ_1 in Δ. Then $|I(y_1)\cap\Delta_1|=4$, so $|Q|=|\Delta_1|\geq 4$.

Since $N(Q)^{I(Q)}=S_t$ or A_t, first assume that $N(Q)^{I(Q)}=S_t$. Then $N(Q)$ has a 2-element

$$x = (1\ 2)\ (3)\ (4) \cdots (t) \cdots .$$

By (2.3) we may assume that $\langle Q, y_1, x\rangle$ is a 2-group. Then x normalizes $\langle Q, y_1\rangle$. Hence x fixes Δ_1, contrary to (2.13). Thus $N(Q)^{I(Q)} \neq S_t$.

Hence $N(Q)^{I(Q)} = A_t$. First we show that $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ fixes Δ_1 and is semiregular on $\Delta - \Delta_1$. Since y_1' normalizes $\langle Q, y_1\rangle$, y_1' fixes Δ_1. Since $\langle Q, y_1'\rangle$ and $\langle Q, y_1 y_1'\rangle$ are conjugate to $\langle Q, y_1\rangle$ in $N(Q)$, $\langle Q, y_1'\rangle$ and $\langle Q, y_1 y_1'\rangle$ are semiregular on $\Delta - \Delta_1$. Thus $\langle Q, y_1, y_1'\rangle$ are semiregular on $\Delta - \Delta_1$.

Since $(y_i y_j)^{I(Q)} = (y_j y_i)^{I(Q)}$, $1 \leq i, j \leq k$, $\bar{y}_i \bar{y}_j = \bar{y}_j \bar{y}_i$. Thus $\langle \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_k\rangle$ is elementary abelian. Similarly since $(y_1 y_1')^{I(Q)} = (y_1' y_1)^{I(Q)}$ and $(y_i y_j \cdot y_1')^{I(Q)} = (y_1' \cdot y_i y_j)^{I(Q)}$, $2 \leq i, j \leq k$, $\langle \bar{y}_1, \bar{y}_1', \bar{y}_i \bar{y}_j\rangle$ is elementary abelian. Since $\bar{y}_1$ fixes exactly one Q-orbit Δ_1 in $\bar{\Delta}$, $\langle \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_k, \bar{y}_1'\rangle$ fixes Δ_1. Thus Δ_1 is the $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$-orbit.

Suppose that $\langle Q, y_1, y_2, y_1'\rangle$ is not semiregular on $\Delta - \Delta_1$. Then there is an element y' in $\langle Q, y_1, y_1'\rangle y_2$ such that $\bar{y}'$ has fixed points in $\bar{\Delta} - \{\Delta_1\}$. Then $y'^{I(Q)}$ is of order two or four. If $y'^{I(Q)}$ is of order two, then $y'^{I(Q)}$ consists of two 2-cycles. Thus $\langle Q, y'\rangle$ is conjugate to $\langle Q, y_1\rangle$ which fixes exactly one Q-orbit Δ_1. This is a contradiction. Thus $y'^{I(Q)}$ is of order four and consists of one 4-cycle and one 2-cycle. Then y'^2 consists of two 2-cycles on $I(Q)$ and fixes at least two Q-orbits in Δ, which is also a contradiction. Thus $\langle Q, y_1, y_2, y_1'\rangle$ is semiregular on $\Delta - \Delta_1$.

Suppose that $\langle Q, y_1, y_2, y_3, y_1'\rangle$ is not semiregular on $\Delta - \Delta_1$. Then there is an element y' in $\langle Q, y_1, y_2, y_1'\rangle y_3$ such that $\bar{y}'$ has fixed points in $\bar{\Delta} - \{\Delta_1\}$. Then $\langle Q, y'\rangle$ is not conjugate to any subgroup of $\langle Q, y_1, y_2, y_1'\rangle$. Hence $y'^{I(Q)} = (y_1 y_2 y_3)^{I(Q)}$, $(y_1' y_2 y_3)^{I(Q)}$ or $(y_1 y_1' y_2 y_3)^{I(Q)}$. Suppose that $y'^{I(Q)} = (y_1 y_2 y_3)^{I(Q)}$. Then $\bar{y}' = \bar{y}_1 \bar{y}_2 \bar{y}_3$ commutes with $\bar{y}_1, \bar{y}_2$ and $\bar{y}_1'$. Since $\langle \bar{y}_1, \bar{y}_2, \bar{y}_1'\rangle$ is semiregular on $\bar{\Delta} - \{\Delta_1\}$, $\bar{y}'$ fixes at least eight Q-orbits in $\bar{\Delta} - \{\Delta_1\}$. Thus y' fixes at least eight Q-orbits other than Δ_1. However since $y'^{I(Q)}$ consists of four 2-cycles, y' fixes at most eight Q-orbits in Δ by (2.8). Thus we have a contradiction. Hence $y'^{I(Q)} \neq (y_1 y_2 y_3)^{I(Q)}$. Suppose that $y'^{I(Q)} = (y_1' y_2 y_3)^{I(Q)}$ or $(y_1 y_1' y_2 y_3)^{I(Q)}$. Then $\langle Q, y'\rangle$ is conjugate to $\langle Q, y_1 y_2 y_3\rangle$ in $N(Q)$ and so semiregular on $\Delta - \Delta_1$, which is a contradiction. Thus $\langle Q, y_1, y_2, y_3, y_1'\rangle$ is semiregular on $\Delta - \Delta_1$.

Suppose that $\langle Q, y_1, y_2, y_3, y_4, y_1'\rangle$ is not semiregular on $\Delta - \Delta_1$. Then there is an element y' in $\langle Q, y_1, y_2, y_3, y_1'\rangle y_4$ such that $\bar{y}'$ has fixed points in $\bar{\Delta} - \{\Delta_1\}$. Then $\langle Q, y'\rangle$ is not conjugate to any subgroup of $\langle Q, y_1, y_2, y_3, y_1'\rangle$. Hence y' consists of one 4-cycle and three 2-cycles on $I(Q)$. Then $\langle Q, y'^2\rangle = \langle Q, y_1\rangle$, which is semiregular on $\Delta - \Delta_1$. Thus we have a contradiction. Hence $\langle Q, y_1, y_2, y_3, y_4, y_1'\rangle$ is semiregular on $\Delta - \Delta_1$. Hence by (2.10) $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ is semiregular on $\Delta - \Delta_1$.

Let a be an involution of Q commuting with y_1 and $\{i_1, i_2, i_3, i_4\}$ be any

$\langle a, y_1\rangle$-orbit in $\Delta-\Delta_1$. Then $\langle a, y_1\rangle$ normalizes $G_{i_1 i_2 i_3 i_4}$, which is of even order. Hence a and y_1 commute with an involution u of $G_{i_1 i_2 i_3 i_4}$. Then the 2-group $\langle y_1, u\rangle$ normalizes $G_{I(Q)}$. Hence $\langle y_1, u\rangle$ normalizes a Sylow 2-subgroup Q' of $G_{I(Q)}$. Since Q' is conjugate to Q in $G_{I(Q)}$ and $N(Q)^{I(Q)}=A_t$, $\langle Q', y_1, u\rangle$ is conjugate to a subgroup of $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ in $N(G_{I(Q)})$. Hence $I(y_1)\cap\Delta$ and $\{i_1, i_2, i_3, i_4\}$ are contained in the same Q'-orbit. Since $\{i_1, i_2, i_3, i_4\}$ is any $\langle a, y_1\rangle$-orbit in $\Delta-\Delta_1$, $G_{I(Q)}$ is transitive on Δ. Hence G_{1234} is transitive or has two orbits $\{5, 6, \cdots, t\}$ and Δ on $\Omega-\{1, 2, 3, 4\}$. If G_{1234} is transitive on $\Omega-\{1, 2, 3, 4\}$, then G is 5-fold transitive on Ω, contrary to (2.11). Hence G_{1234} has two orbits $\{5, 6, \cdots, t\}$ and Δ on $\Omega-\{1, 2, 3, 4\}$. Since $N(Q)^{I(Q)}=A_t$, for any four points j_1, j_2, j_3, j_4 of $I(Q)$ the $G_{j_1 j_2 j_3 j_4}$-orbits on $\Omega-\{j_1, j_2, j_3, j_4\}$ consist of two orbits $I(Q)-\{j_1, j_2, j_3, j_4\}$ and Δ. Furthermore since G is 4-fold transitive, for any four points k_1, k_2, k_3, k_4 of Ω $G_{k_1 k_2 k_3 k_4}$ has two orbits Γ_1 and Γ_2, where $|\Gamma_1|=t-4$, $|\Gamma_2|=|\Delta|$. By a theorem of W. A. Manning [5] $|\Gamma_2|>|\Gamma_1|$. Set $\Gamma(k_1, k_2, k_3, k_4)=\Gamma_1\cup\{k_1, k_2, k_3, k_4\}$. Since $|I(y_1)\cap\Delta|=4$ and y_1 commutes with a, we may assume that

$$a=(1)\,(2)\cdots(t)\,(1'\,2')\,(3'\,4')\cdots,$$
$$y_1=(1\,2)\,(3\,4)\,(5)\,(6)\cdots(t)\,(1')\,(2')\,(3')\,(4')\cdots.$$

Let i, j be any two points of $I(Q)-\{1, 2, 3, 4\}$. Then $y_1\in G_{1'2'ij}$ and a normalizes $G_{1'2'ij}$. Since $|\Gamma(1', 2', i, j)-\{1', 2', i, j\}|\neq|\Omega-\Gamma(1', 2', i, j)|$, a fixes $\Gamma(1', 2', i, j)$. Suppose that $\Gamma(1', 2', i, j)$ contains $\{1, 2\}$. Then as we have seen above $\Gamma(1, 2, i, j)$ contains $\{1', 2'\}$. This is a contradiction since $\Gamma(1, 2, i, j)=I(Q)$. Similarly $\Gamma(1', 2', i, j)$ does not contain $\{3, 4\}$. On the other hand since $N(G_{\Gamma(1', 2', i, j)})^{\Gamma(1', 2', i, j)}=A_t$, a and y_1 are even permutations on $\Gamma(1', 2', i, j)$. Hence $\Gamma(1', 2', i, j)$ contains $\{3', 4'\}$. Hence $\Gamma(1', 2', 3', 4')$ contains $\{i, j\}$. Since i, j are any two points of $I(Q)-\{1, 2, 3, 4\}$, $\Gamma(1', 2', 3', 4')$ contains $I(Q)-\{1, 2, 3, 4\}$. By (2.1) $|I(Q)|\geq 8$. Hence $I(Q)-\{1, 2, 3, 4\}$ contains $\{5, 6, 7, 8\}$, which is contained in $\Gamma(1', 2', 3', 4')$. Hence $\Gamma(5, 6, 7, 8)$ contains $\{1', 2', 3', 4'\}$. This is a contradiction since $\Gamma(5, 6, 7, 8)=I(Q)$. Thus y_1 fixes at least two Q-orbits in Δ.

Since $C(Q)^{I(Q)}=S_t$, A_t or 1, we treat the following two cases separately:

Case 1. $C(Q)^{I(Q)}=S_t$ or A_t.

Case 2. $C(Q)^{I(Q)}=1$.

Case 1. $C(Q)^{I(Q)}=S_t$ or A_t. Then we may assume that

$$y_1=(1\,2)\,(3\,4)\,(5)\,(6)\cdots(t)\,(1')\,(2')\,(3')\,(4')\cdots,$$
$$a=(1)\,(2)\cdots(t)\,(1'\,2')\,(3'\,4')\cdots(n-1\;n),$$

where a is a central involution of Q commuting with y_1.

(i) Assume that $y_1\notin C(Q)$. Since $C(Q)^{I(Q)}\geq A_t$, there is an element b in Q such that $by_1\in C(Q)$. Then by_1 commutes with b, so y_1 commutes with b.

Since $y_1 \notin C(Q)$, $b \notin Z(Q)$. Thus Q is non-abelian and so $|Q|>4$. Since b fixes $\{1', 2', 3', 4'\}$ and commutes with a, b is an involution or $b^2=a$. Furthermore $Z(\langle Q, y_1\rangle) \geq \langle a, by_1\rangle$. Let y' be any element of $Z(\langle Q, y_1\rangle)$. Since $I(y_1) \cap \Delta = \{1', 2', 3', 4'\}$, y' fixes $\{1', 2', 3', 4'\}$. Furthermore since $\langle a, b\rangle$ is regular on $\{1', 2', 3', 4'\}$, $y'^{\{1',2',3',4'\}} \in \langle a, b\rangle^{\{1',2',3',4'\}}$. Hence there is an element u in $\langle a, b\rangle$ such that uy' fixes $\{1', 2', 3', 4'\}$ pointwise. Thus $uy' \in \langle y_1\rangle$ because $\langle Q, y_1\rangle_{1'} = \langle y_1\rangle$. Hence $uy'=1$ or y_1. If $uy'=1$, then $y' \in \langle a, b\rangle \cap Z(\langle Q, y_1\rangle)$ since $y' \in Z(\langle Q, y_1\rangle)$ and $u \in \langle a, b\rangle$. Hence $y'=a$ or 1. Next suppose that $uy'=y_1$. If $u=a$ or 1, then $y_1=uy' \in C(Q)$ since $y' \in C(Q)$. This is a contrdiction since $y_1 \notin C(Q)$. Thus $u=b$ or ab. Hence $y'=by_1$ or aby_1. Thus in either case $y' \in \langle a, by_1\rangle$. Hence $Z(\langle Q, y_1\rangle)=\langle a, by_1\rangle$.

Since $C(Q)^{I(Q)} \geq A_t$, Qy_2 has an element which belongs to $C(Q)$. Hence we may assume that $y_2 \in C(Q)$. Since y_2 normalizes $\langle Q, y_1\rangle$, y_2 normalizes the center $\langle a, by_1\rangle$ of $\langle Q, y_1\rangle$. Hence $(by_1)^{y_2}=by_1$ or a aby_1. First assume that $(by_1)^{y_2}=by_1$. Since y_2 commutes with b, y_2 commutes with y_1. Hence y_2 fixes $\{1', 2', 3', 4'\}$. Since $\langle a, by_1, y_2\rangle$ is an abelian group of order eight and $\langle a, by_1\rangle$ is regular on $\{1', 2', 3', 4'\}$, there is an element u in $\langle a, by_1\rangle y_2$ which fixes $\{1', 2', 3', 4'\}$ pointwise. Thus u consists of exactly two 2-cycles on $I(Q)$ and so $I(u) \cap \Delta = \{1', 2', 3', 4'\}$ by the assumption (*). On the other hand $\langle a, by_1, y_2\rangle \leq C(Q)$. Hence $u \in C(Q)$. Thus $|Q| \leq 4$, which is a contradiction. Next suppose that $(by_1)^{y_2}=aby_1$. Then by the same argument as is used for y_2 we may assume that $y_1' \in C(Q)$ and $(by_1)^{y_1'}=aby_1$. Hence $(by_1)^{y_2 y_1'}=by_1$. Since $y_2 y_1' \in C(Q)$, $y_2 y_1'$ commutes with b. Hence $y_2 y_1'$ commutes with y_1. Thus $y_2 y_1'$ fixes $\{1', 2', 3', 4'\}$. Thus $\langle a, by_1, y_2 y_1'\rangle$ is an abelian group fixing $\{1', 2', 3', 4'\}$. Hence there is an element u $(\neq 1)$ in $\langle a, by_1, y_2 y_1'\rangle$ which fixes $\{1', 2', 3', 4'\}$ pointwise. Thus u consists of two 2-cycles or one 4-cycle and one 2-cycle on $I(Q)$. Hence $|I(u) \cap \Delta| \leq 6$ by the assumption (*). On the other hand $u \in C(Q)$ and $|Q|>4$. Hence $|I(u) \cap \Delta| \geq 8$, which is a contradiction. Thus $y_1 \in C(Q)$. Hence $|Q|=4$ or 2.

(ii) Assume that $|Q|=4$. Then Q is elementary abelian or cyclic.

(ii.i) Assume that Q is elementary abelian. Then we may assume that $Q=\langle a, b\rangle$ and

$$a = (1)\,(2)\cdots(t)\,(1'\ 2')\,(3'\ 4')\,(5'\ 6')\,(7'\ 8')\cdots,$$
$$b = (1)\,(2)\cdots(t)\,(1'\ 3')\,(2'\ 4')\,(5'\ 7')\,(6'\ 8')\cdots.$$

As we have proved above, y_1 fixes at least two Q-orbits in Δ. Hence we may assume that

$$y_1 = (1\ 2)\,(3\ 4)\,(5)\,(6)\cdots(t)\,(1')\,(2')\,(3')\,(4')\,(5'\ 6')\,(7'\ 8')\cdots.$$

Since $\langle Q, y_2\rangle$ and $\langle Q, y_1 y_2\rangle$ are conjugate to $\langle Q, y_1\rangle$, both groups are elementary abelian. Hence $\langle Q, y_1, y_2\rangle$ is elementary abelian. Thus y_2 fixes $\{1', 2', 3', 4'\}$ and $\{5', 6', 7', 8'\}$. Hence Qy_2 has an element which fixes $\{1', 2', 3', 4'\}$ pointwise. We may assume that y_2 fixes $\{1', 2', 3', 4'\}$ pointwise. Thus $I(y_2)=(I(Q)-\{1, 2, 5, 6\})\cup\{1', 2', 3', 4'\}$ since $|(I(Q)-\{1, 2, 5, 6,\})\cup\{1', 2', 3', 4'\}|=t$. Furthermore since $|I(y_1y_2)|\leq t$, $y_2=(5'\ 7')\ (6'\ 8')$ or $(5'\ 8')\ (6'\ 7')$ on $\{5', 6', 7', 8'\}$. Since $\langle Q, y_1'\rangle$ and $\langle Q, y_1y_1'\rangle$ are conjugate to $\langle Q, y_1\rangle$, $\langle Q, y_1, y_1'\rangle$ is elementary abelian and by the similar argument as above we may assume that $y_1'=(1')\ (2')\ (3')\ (4')\ (5'\ 7')\ (6'\ 8')$ or $(1')\ (2')\ (3')\ (4')\ (5'\ 8')\ (6'\ 7')$ on $\{1', 2', \cdots, 8'\}$. Then in either case the order of $(y_2y_1')^2$ is even and $|I((y_2y_1')^2)|\geq t+4$, contrary to the assumption $(*)$. Thus Q is not an elementary abelian group.

(ii.ii) Assume that Q is cyclic. Then we may assume that $Q=\langle b\rangle$, $b^2=a$ and

$$b=(1)\ (2)\cdots(t)\ (1'\ 3'\ 2'\ 4')\ (5'\ 7'\ 6'\ 8')\cdots.$$

As we have proved above, y_1 fixes at least two Q-orbits in Δ. Hence we may assume that

$$y_1=(1\ 2)\ (3\ 4)\ (5)\ (6)\cdots(t)\ (1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\cdots.$$

Then $I(ay_1)\cap\Delta=\{5', 6', 7', 8'\}$. Hence $\langle Q, y_1\rangle$ is semiregular on $\{9', 10', \cdots, n\}$. Since y_2 normalizes $\langle Q, y_1\rangle$, $y_1^{y_2}=y_1$ or ay_1. Suppose that $y_1^{y_2}=y_1$. Then y_2 fixes $\{1', 2', 3', 4'\}$ and $\{5', 6', 7', 8'\}$. Furthermore since $\langle Q, y_2\rangle$ is abelian, $\langle Q, y_2\rangle$ has an element

$$y_2'=(1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8)\cdots(t)\ (1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\cdots.$$

Then $|I(y_1y_2')|\geq t+4$, contrary to the assumption $(*)$. Thus $y_1^{y_2}=ay_1$. Since $\langle Q, y_2\rangle$ is conjugate to $\langle Q, y_1\rangle$, Qy_2 has an involution. Hence we may assume that y_2 is an involution. Furthermore by the same argument as is used for y_2, $y_1^{y_1'}=ay_1$. Thus $y_1^{y_2y_1'}=y_1$. Hence y_2y_1' fixes $\{1', 2', 3', 4'\}$ and $\{5', 6', 7', 8'\}$. Hence Qy_2y_1' has an element u fixing $\{1', 2', 3', 4'\}$ pointwise. Then $I(u^2)$ contains $(I(Q)-\{1, 2, 3, 4\})\cup\{1', 2', 3', 4'\}$ of length t. Hence $I(u^2)=(I(Q)-\{1, 2, 3, 4\})\cup\{1', 2', 3', 4'\}$ by the assumption $(*)$. Hence u is a 4-cycle on $\{5', 6', 7', 8'\}$. Since $u\in C(Q)$, $u=b$ or b^{-1} on $\{5', 6', 7', 8'\}$. Furthermore since $y_1^{y_2}=ay_1$, y_2 interchanges $\{1', 2', 3', 4'\}$ and $\{5', 6', 7', 8'\}$ as a set. Hence $u^{y_2}u=b$ or b^{-1}. This means that $(y_2u)^2=b$ or b^{-1}. Thus y_2u is of order eight. On the other hand since $(y_2u)^{I(a)}=y_1'^{I(a)}$, $\langle Q, y_2u\rangle=\langle Q, y_1'\rangle$. Thus we have a contradiction since $\langle Q, y_1'\rangle$ is conjugate to $\langle Q, y_1\rangle$ which has no element of order eight. Thus Q is not cyclic. Hence $|Q|\neq 4$.

(iii) Assume that $|Q|=2$. Then $Q=\langle a\rangle$. Since $C(a)^{I(a)}=S_t$ or A_t, we treat these cases separately.

(iii.i) Assume that $C(a)^{I(a)}=S_t$. Then $C(a)$ has a 2-element

$$x_1=(1\ 2)\ (3)\ (4)\cdots(t)\cdots.$$

By (2.3) we may assume that $\langle a, x_1, y_1, y_2, \cdots, y_k, y_1'\rangle$ is a 2-group. Then x_1

normalizes $\langle a, y_1\rangle$. Hence $y_1^{x_1}=ay_1$ or y_1.

First suppose that $y_1^{x_1}=ay_1$. Since $x_1^2\in\langle a\rangle$, $x_1^2=1$ or a. Suppose that $x_1^2=1$. Then $\langle a, x_1\rangle$ is an elementary abelian group of order four. On the other hand since $y_1^{x_1}=ay_1$, $(x_1y_1)^2=a$. Thus $\langle x_1y_1\rangle$ is a cyclic group of order four. This is a contradiction since $\langle x_1y_1\rangle$ is conjugate to $\langle a, x_1\rangle$. Suppose that $x_1^2=a$. Then $\langle x_1\rangle$ is a cyclic group of order four. On the other hand since $y_1^{x_1}=ay_1$, $(x_1y_1)^2=1$. Thus $\langle a, x_1y_1\rangle$ is an elementary abelian group of order four. This is a contradiction since $\langle a, x_1y_1\rangle$ is conjugate to $\langle a, x_1\rangle$. Thus $y_1^{x_1}\neq ay_1$.

Next suppose that $y_1^{x_1}=y_1$. Then $\langle a, x_1, y_1\rangle$ is an abelian group of order eight. By (2.14) $|I(ay_1)\cap\Delta|=0$ or 4. Assume that $|I(ay_1)\cap\Delta|=4$. Then we may assume that $I(ay_1)\cap\Delta=\{5', 6', 7', 8'\}$ and

$$y_1=(1\ 2)\ (3\ 4)\ (5)\ (6)\cdots(t)\ (1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\cdots.$$

Then $\langle a, y_1\rangle$ is semiregular on $\{9', 10', \cdots, n\}$. By (2.13) $\langle a, x_1\rangle$ and $\langle a, x_1y_1\rangle$ are semiregular on Δ. Hence $\langle a, x_1, y_1\rangle$ is semiregular on $\{9', 10', \cdots, n\}$. Since $\langle a, y_2\rangle$ and $\langle a, y_1y_2\rangle$ are conjugate to $\langle a, y_1\rangle$, $\langle a, y_2\rangle$ and $\langle a, y_1y_2\rangle$ are elementary abelian. Hence $\langle a, y_1, y_2\rangle$ is elementary abelian. Furthermore since $\langle a, y_2, x_1\rangle$ is conjugate to $\langle a, y_1, x_1\rangle$, $\langle a, y_2, x_1\rangle$ is also abelian. Hence $\langle a, x_1, y_1, y_2\rangle$ is abelian. Since $\langle a, y_2\rangle$ is conjugate to $\langle a, y_1\rangle$ in $C(a)$, $|I(y_2)\cap\Delta|=|I(ay_2)\cap\Delta|=4$. If y_2 has fixed points in $\{9', 10', \cdots, n\}$, then since $y_2\in C(\langle a, x_1, y_1\rangle)$ y_2 fixes at least eight points in $\{9', 10', \cdots, n\}$, contrary to the assumption (*). Similarly ay_2 has no fixed point in $\{9', 10', \cdots, n\}$. Thus y_2 or ay_2 fixes $\{1', 2', 3', 4'\}$ pointwise. Hence y_2 or $ay_2=(1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')$ on $\{1', 2', \cdots, 8'\}$. Thus $|I(y_1y_2)|$ or $|I(ay_1y_2)|\geq t+4$, contrary to the assumption (*).

Hence $|I(ay_1)\cap\Delta|=0$. Then $\langle a, x_1, y_1\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$. Since $\langle a, y_i\rangle$ and $\langle a, y_iy_j\rangle$, $i\neq j$ and $1\leq i, j\leq k$, are conjugate to $\langle a, y_1\rangle$, $\langle a, y_i\rangle$ and $\langle a, y_iy_j\rangle$ are elementary abelian. Hence $\langle a, y_1, y_2, \cdots, y_k\rangle$ is elementary abelian. Furthermore since $\langle a, x_1, y_i\rangle$, $2\leq i\leq k$, is conjugate to $\langle a, x_1, y_1\rangle$, $\langle a, x_1, y_i\rangle$ is abelian. Thus $\langle a, x_1, y_1, y_2, \cdots, y_k\rangle$ is abelian. Hence y_i fixes $\{1', 2', 3', 4'\}$, $1\leq i\leq k$. Since $\langle a, y_i\rangle$, $2\leq i\leq k$, is conjugate to $\langle a, y_1\rangle$, y_i or ay_i has fixed points in Δ. Hence we may assume that y_i has fixed points in Δ. Since $y_i\in C(\langle a, x_1, y_1\rangle)$ and $\langle a, x_1, y_1\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$, if y_i has fixed points in $\Delta-\{1', 2', 3', 4'\}$, then y_i fixes at least eight points of $\Delta-\{1', 2', 3', 4'\}$, contrary to the assumption (*). Hence y_i fixes $\{1', 2', 3', 4'\}$ pointwise.

Assume that $\langle a, x_1, y_1, y_2, \cdots, y_i\rangle$, $i\geq 1$, is semiregular on $\Delta-\{1', 2', 3', 4'\}$. If $\langle a, x_1, y_1, y_2, \cdots, y_{i+1}\rangle$ is not semiregular on $\Delta-\{1', 2', 3', 4'\}$, then $\langle a, x_1, y_1, y_2, \cdots, y_{i+1}\rangle$ has an element y' ($\neq 1$) fixing a $\langle a, x_1, y_1, y_2, \cdots, y_i\rangle$-orbit of length 2^{i+2} pointwise. Then since y' consists of at most $i+2$ 2-cycles on $I(a)$ and $i\geq 1$, $|I(y')|\geq t-2(i+1)+2^{i+2}>t$, contrary to the assumption (*). Thus $\langle a, x_1, y_1, y_2, \cdots, y_{i+1}\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$ and this implies by induction that

$\langle a, x_1, y_1, y_2, \cdots, y_k\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$.

Furthermore y_1' fixes $\{1', 2', 3', 4'\}$. Suppose that $\langle a, x_1, y_1, y_2, \cdots, y_k, y_1'\rangle$ is not semiregular on $\Delta-\{1', 2', 3', 4'\}$. Then there is an element y' in $\langle a, x_1, y_1, y_2, \cdots, y_k\rangle y_1'$ which has fixed points in $\Delta-\{1', 2', 3', 4'\}$. Then $y'^{I(a)}$ is of order four or two. If $y'^{I(a)}$ is of order four, then $\langle a, y'^2\rangle=\langle a, y_1\rangle$ and y'^2 has fixed points in $\Delta-\{1', 2', 3', 4'\}$, which is a contradiction. Thus $y'^{I(a)}$ is of order two. Then y' is (1 3) (2 4) or (1 4) (2 3) on $\{1, 2, 3, 4\}$. Hence $y'\in\langle a, y_1', x_1y_2, x_1y_3, \cdots, x_1y_k\rangle$ or $\langle a, y_1y_1', x_1y_2, x_1y_2, \cdots, x_1y_k\rangle$. Thus $\langle a, y_1', x_1y_2, x_1y_3, \cdots, x_1y_k\rangle$ or $\langle a, y_1y_1', x_1y_2, x_1y_3, \cdots, x_1y_k\rangle$ is semiregular on neither $\{1', 2', 3', 4'\}$ nor $\Delta-\{1', 2', 3', 4'\}$. This is a contradiction since $\langle a, y_1', x_1y_2, x_1y_3, \cdots, x_1y_k\rangle$ and $\langle a, y_1y_1', x_1y_2, x_1y_3, \cdots, x_1y_k\rangle$ are conjugate to $\langle a, y_1, x_1y_2, x_1y_3, \cdots, x_1y_k\rangle$ in $C(a)$ which is semiregular on $\Delta-\{1', 2', 3', 4'\}$. Thus $\langle a, x_1, y_1, y_2, \cdots, y_k, y_1'\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$.

On the other hand $\langle a, y_1\rangle$ normalizes $G_{5'6'7'8'}$, which is even order. Hence there is an involution u in $G_{5'6'7'8'}$ commuting with a and y_1. Since $C(a)^{I(a)}=S_t$, $\langle a, y_1, u\rangle$ is conjugate to a subgroup of $\langle a, x_1, y_1, y_2, \cdots, y_k, y_1'\rangle$ in $C(a)$. This is a contradiction since for any point of $\{1', 2', \cdots, 8'\}$ of length eight $\langle a, y_1, u\rangle$ has an element $(\neq 1)$ fixing this point. Thus $C(a)^{I(a)}\neq S_t$.

(iii.ii) Assume that $C(a)^{I(a)}=A_t$. Since $\langle a, y_1y_2\rangle$, $\langle a, y_1y_3\rangle$ and $\langle a, y_2y_3\rangle$ are conjugate to $\langle a, y_1\rangle$, these groups are elementary abelian. Hence $\langle a, y_1, y_2, y_3\rangle$ is elementary abelian. Since $I(y_1)\cap\Delta=\{1', 2', 3', 4'\}$, y_2 and y_3 fix $\{1', 2', 3', 4'\}$. Thus y_2 and y_3 are $(1')(2')(3')(4')$, $(1'\,2')(3'\,4')$, $(1'\,3')(2'\,4')$, $(1'\,4')(2'\,3')$, $(1')(2')(3'\,4')$ or $(1'\,2')(3')(4')$ on $\{1', 2', 3', 4'\}$. Furthermore by (2.14) $|I(ay_1)\cap\Delta|=0$ or 4.

Assume that $|I(ay_1)\cap\Delta|=4$. Then we may assume that

$$\begin{aligned} a &= (1)\,(2)\cdots(t)\,(1'\,2')\,(3'\,4')\cdots(n-1\;n)\,,\\ y_1 &= (1\,2)\,(3\,4)\,(5)\,(6)\cdots(t)\,(1')\,(2')\,(3')\,(4')\,(5'\,6')\,(7'\,8')\,(9'\,11')\\ &\quad (10'\,12')\,(13'\,15')\,(14'\,16')\cdots. \end{aligned}$$

Suppose that $y_2=(1')(2')(3')(4')$ on $\{1', 2', 3', 4'\}$. The proof in the case $y_2=(1'\,2')(3'\,4')\cdots$ is similar since if $y_2=(1'\,2')(3'\,4')\cdots$ then $ay_2=(1')(2')(3')(4')\cdots$. Since $\langle a, y_2\rangle$ and $\langle a, y_1y_2\rangle$ are conjugate to $\langle a, y_1\rangle$, any element of $\langle a, y_1y_2\rangle-\langle a\rangle$ has four fixed points in Δ. Hence we may assume that

$$\begin{aligned} y_2 &= (1\,2)\,(3)\,(4)\,(5\,6)\,(7)\,(8)\cdots(t)\,(1')\,(2')\,(3')\,(4')\,(5'\,7')\,(6'\,8')\,(9'\,10')\\ &\quad (11'\,12')\,(13'\,16')\,(14'\,15')\cdots. \end{aligned}$$

Thus $\langle a, y_1, y_2\rangle$ has two orbits of length two and three orbits of length four in Δ. The remaining $\langle a, y_1, y_2\rangle$-orbits are of length eight in Δ. Since $\langle a, y_3\rangle$ is conjugate to $\langle a, y_1\rangle$, y_3 has four fixed points in Δ. Since $\langle a, y_1, y_2, y_3\rangle$ is abelian, y_3 fixes $\{1', 2', 3', 4'\}$ or one of the $\langle a, y_1, y_2\rangle$-orbits of length four pointwise. Moreover y_3 fixes the $\langle a, y_1, y_2\rangle$-orbits of length four setwise. Thus y_3 fixes $\{1', 2', 3', 4'\}$ pointwise or has no fixed point in $\{1', 2', 3', 4'\}$. First suppose that y_3 fixes $\{1', 2', 3', 4'\}$ pointwise. Then $\langle y_1, y_2, y_3\rangle$ fixes $\{1', 2', 3', 4'\}$ pointwise, and $\{5', 6', 7', 8'\}$ and $\{9', 10' 11', 12'\}$ are $\langle y_1, y_2, y_3\rangle$-orbits of length four. Hence $\langle y_1, y_2, y_3\rangle$ has exactly one element y' $(\neq 1)$ fixing $\{5', 6', 7', 8'\}$ pointwise. Thus $I(y')\cap\Delta\supseteq\{1', 2', \cdots, 8'\}$. Hence $y'=y_1y_2y_3$ by the assumption (*). Similarly $\langle y_1, y_2, y_3\rangle$ has exactly one element $(\neq 1)$ fixing $\{9', 10', 11', 12'\}$ pointwise, which is also $y_1y_2y_3$. Thus $|I(y_1y_2y_3)|\geq t+4$, contrary to the assumption (*). Thus y_3 does not fix $\{1', 2', 3', 4'\}$ pointwise. Similarly $y_3\neq(1'\ 2')\ (3'\ 4')\cdots$ since if $y_3=(1'\ 2')\ (3'\ 4')\cdots$ then $ay_3=(1')\ (2')\ (3')\ (4')\cdots$. Next suppose that $y_3=(1'\ 3')\ (2'\ 4')\cdots$ or $(1'\ 4')\ (2'\ 3')\cdots$. Since $\langle a, y_1, y_3\rangle$ is conjugate to $\langle a, y_1, y_2\rangle$, $\langle a, y_1, y_3\rangle$ has exactly two orbits of length two in Δ. Hence y_3 fixes $\{5', 6'\}$ and $\{7', 8'\}$. Then $\langle a, y_1, y_2y_3\rangle$ has no orbit of length two in Δ. On the other hand $C(a)$ has a 2-element

$$y' = (1)\ (2)\ (3)\ (4)\ (5\ 7)\ (6\ 8)\ (9)\ (10)\cdots(t)\cdots .$$

By (2.3) we may assume that $\langle a, y_1, y_2y_3, y'\rangle$ is a 2-group. Since $\langle a, y_1, y'\rangle$ is conjugate to $\langle a, y_1, y_2y_3\rangle$ in $C(a)$, $\langle a, y_1, y'\rangle$ has no orbit of length two in Δ. Hence $y'=(1'\ 3')\ (2'\ 4')$ or $(1'\ 4')\ (2'\ 3')$ on $\{1', 2', 3', 4'\}$. Then $\langle a, y_1, y_2y_3y'\rangle$ has two orbits $\{1', 2'\}$ and $\{3', 4'\}$ of length two in Δ. This is a contradiction since $\langle a, y_1, y_2y_3y'\rangle$ is conjugate to $\langle a, y_1, y_2y_3\rangle$ in $C(a)$. Thus $y_2\neq(1')\ (2')\ (3')\ (4')\cdots$ and so $y_2\neq(1'\ 2')\ (3'\ 4')\cdots$.

Suppose that $y_2=(1')\ (2')\ (3'\ 4')$ on $\{1', 2', 3', 4'\}$. The proof in the case $y_2=(1'\ 2')\ (3')\ (4')$ on $\{1', 2', 3', 4'\}$ is similar since if $y_2=(1'\ 2')\ (3')\ (4')\cdots$ then $ay_2=(1')\ (2')\ (3'\ 4')\cdots$. Since $\langle a, y_1 y_2\rangle$ is elementary abelian and $|I(y_2)\cap\Delta|=4$, we may assume that

$$y_2 = (1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8)\cdots(t)\ (1')\ (2')\ (3'\ 4')\ (5')\ (6')\ (7'\ 8')\cdots .$$

Since $\langle a, y_1, y_2, y_3\rangle$ is elementary abelian, y_3 fixes $\{1', 2'\}$. $\{3', 4'\}$. $\{5', 6'\}$ and $\{7', 8'\}$. Furthermore $|I(y_3)\cap\Delta|=4$ and $|I(y_2y_3)\cap\Delta|=4$. Hence we may assume that

$$y_3 = (1\ 2)\ (3)\ (4)\ (5)\ (6)\ (7\ 8)\ (9)\ (10)\cdots(t)\ (1')\ (2')\ (3'\ 4')\ (5'\ 6')\ (7')\ (8')\cdots .$$

Then

$$y_1y_2y_3 = (1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9)\ (10)\cdots(t)\ (1')\ (2')\cdots(8')\cdots .$$

Thus $\langle a, y_1, y_2y_3\rangle$ has exactly one involution $y_1y_2y_3$ fixing four $\langle a, y_1\rangle$-orbits of length two pointwise. On the other hand $C(a)$ has a 2-element

$$y' = (1)\ (2)\ (3)\ (4)\ (5\ 7)\ (6\ 8)\ (9)\ (10)\cdots(t)\cdots .$$

By (2.3) we may assume that $\langle a, y_1, y_2y_3, y'\rangle$ is a 2-group. Since $\langle a, y_1, y'\rangle$ is conjugate to $\langle a, y_1, y_2y_3\rangle$ in $C(a)$, $\langle a, y_1, y'\rangle$ has exactly one element y'' $(\neq 1)$ fixing four $\langle a, y_1\rangle$-orbits of length two pointwise.

Then

$$y'' = (12)(34)(57)(68)(9)(10)\cdots(t)(1')(2')\cdots(8')\cdots.$$

Thus $|I(y_1y_2y_3y'')|\geq t+4$, contrary to the assumption (*). Hence $y_2 \neq (1')(2')(3'4')\cdots$ and so $y_2 \neq (1'2')(3')(4')\cdots$.

Suppose that $y_2=(1'3')(2'4')$ on $\{1', 2', 3', 4'\}$. The proof in the case $y_2=(1'4')(2'3')$ on $\{1', 2', 3', 4'\}$ is similar since if $y_2=(1'4')(2'3')\cdots$ then $ay_2=(1'3')(2'4')\cdots$. Since $I(ay_1)\cap\Delta=\{5', 6', 7', 8'\}$, if y_2 or y_3 has fixed points in $\{5', 6', 7'. 8'\}$, then by the same argument as above we have a contradiction. Hence we may assume that

$$y_2 = (12)(3)(4)(56)(7)(8)\cdots(t)(1'3')(2'4')(5'7')(6'8')\cdots.$$

Similarly y_3 or ay_3 is $(1'3')(2'4')$ on $\{1', 2', 3', 4'\}$. Hence we may assume that $y_3=(1'3')(2'4')$ on $\{1', 2', 3', 4'\}$. Furthermore y_3 is $(5'7')(6'8')$ or $(5'8')(6'7')$ on $\{5', 6', 7', 8'\}$. Since $|I(y_2y_3)|\leq t$,

$$y_3 = (12)(3)(4)(5)(6)(78)(9)(10)\cdots(t)(1'3')(2'4')(5'8')(6'7')\cdots,$$

and so

$$y_1y_2y_3 = (12)(34)(56)(78)(9)(10)\cdots(t)(1')(2')\cdots(8')\cdots.$$

Hence by the same argument as in the case $y_2=(1')(2')(3'4')\cdots$, we have a contradiction. Thus $y_2\neq(1'3')(2'4')\cdots$ and so $y_2\neq(1'4')(2'3')\cdots$. Hence $|I(ay_1)\cap\Delta|\neq 4$.

Thus $|I(ay_1)\cap\Delta|=0$. Then we may assume that

$$y_1 = (12)(34)(5)(6)\cdots(t)(1')(2')(3')(4')(5'7')(6'8')\cdots.$$

Since $\langle a, y_2\rangle$ is conjugate to $\langle a, y_1\rangle$ in $C(a)$, either y_2 or ay_2 has four fixed points in Δ. Hence we may assume that y_2 has four fixed points in Δ. Then y_2 fixes $\{1', 2', 3', 4'\}$ or one of the $\langle a, y_1\rangle$-orbits of length four pointwise.

First suppose that y_2 fixes $\{1', 2', 3', 4'\}$ pointwise. Since $\langle a, y_2\rangle$ and $\langle a, y_1y_2\rangle$ are conjugate to $\langle a, y_1\rangle$ in $C(a)$, $\langle a, y_2\rangle$ and $\langle a, y_1y_2\rangle$ are semiregular on $\Delta-\{1', 2', 3', 4'\}$. Hence $\langle a, y_1, y_2\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$. Since $\langle a, y_i\rangle$ and $\langle a, y_iy_j\rangle$. $i\neq i$ and $1\leq i, j\leq k$, are conjugate to $\langle a, y_1\rangle$, $\langle a, y_i\rangle$ and $\langle a, y_iy_j\rangle$ are elementary abelian. Hence $\langle a, y_1, y_2, \cdots, y_k\rangle$ is elementary abelian. Moreover y_i or ay_i, $3\leq i\leq k$, has four fixed points in Δ. Hence we may assume that y_i has fixed points in Δ. Since $y_i\in C(\langle a, y_1, y_2\rangle)$ and $\langle a, y_1, y_2\rangle$ is of order eight and semiregular on $\Delta-\{1', 2', 3', 4'\}$, y_i fixes $\{1', 2', 3', 4'\}$ pointwise.

Now we show that $\langle a, y_1, y_2, \cdots, y_k\rangle$ is semiregular on $\Delta-\{1', 2', 3', 4'\}$. Suppose that $\langle a, y_1, y_2, y_3\rangle$ is not semiregular on $\Delta-\{1', 2', 3', 4'\}$. Then there is exactly one element y' $(\neq 1)$ in $\langle a, y_1, y_2, y_3\rangle$ fixing a $\langle a, y_1, y_2\rangle$-orbit Δ' in $\Delta-\{1', 2', 3', 4'\}$ pointwise. Since $|\Delta'|=8$, $|I(y')\cap I(a)|\leq t-8$. Hence

$y' = y_1y_2y_3$ or $ay_1y_2y_3$. If $y' = y_1y_2y_3$, then $I(y')$ contains $(I(a) - \{1, 2, \cdots, 8\}) \cup \{1', 2', 3', 4'\} \cup \Delta'$ of length $t+4$, contrary to the assumption (*). Thus $y' = ay_1y_2y_3$ and $I(y') = (I(a) - \{1, 2, \cdots, 8\}) \cup \Delta'$ since $|(I(a) - \{1, 2, \cdots, 8\}) \cup \Delta'| = t$. Furthermore this shows that $\langle a, y_1, y_2, y_3 \rangle$ is semiregular on $\Delta - (\{1', 2', 3', 4'\} \cup \Delta')$. Hence $\langle a, y_1, y_2, y_3 \rangle$ has two orbits $\{1', 2'\}$ and $\{3', 4'\}$ of length two and two orbits of length four whose uion is Δ' in Δ, and the remaining orbits in Δ are of length eight. On the other hand $C(a)$ has a 2-element

$$y'' = (1)\,(2)\,(3)\,(4)\,(5\,7)\,(6\,8)\,(9)\,(10) \cdots (t) \cdots.$$

By (2.3) we may assume that $\langle a, y_1, y_2, y_3, y'' \rangle$ is a 2-group. Then y'' normalizes $\langle a, y_1, y_2, y_3 \rangle$ and so y'' fixes $\{1', 2', 3', 4'\}$ and Δ'. Since $\langle a, y_1, y'' \rangle$ is conjugate to $\langle a, y_1, y_2y_3 \rangle$ in $C(a)$, $\langle a, y_1, y'' \rangle$ is elementary abelian and has two orbits $\{1', 2'\}$ and $\{3', 4'\}$ of length two and two orbtis of length four in Δ. Hence we may assume that y'' fixes $\{1', 2', 3', 4'\}$ pointwise and ay_1y'' has eight fixed points in $\Delta - \{1', 2', 3', 4'\}$. Furthermore since y'' fixes Δ', ay_1y'' fixes Δ' pointwise or $\langle a, y_1, y'' \rangle$ is regular on Δ'. If ay_1y'' fixes Δ' pointwise, then $I(ay_1y_2y_3 \cdot ay_1y'') = I(y_2y_3y'')$ contains $(I(a) - \{5, 6, 7, 8\}) \cup \{1', 2', 3', 4'\} \cup \Delta'$ of length $t+8$, contrary to the assumption (*). Thus $\langle a, y_1, y'' \rangle$ is regular on Δ'. On the other hand $\langle a, y_2, y_3 \rangle$ is elementary abelian and regular on Δ'. Hence $\langle a, y_2, y_3 \rangle$ has an element u such that $u^{\Delta'} = y''^{\Delta'}$. Thus $uy'' \in \langle a, y_2, y_3, y'' \rangle$ and $I(uy'')$ contains Δ' of length eight. Hence $|I(uy'') \cap I(a)| \leq t-8$. This is a contradiction since any element of $\langle a, y_2, y_3, y'' \rangle$ fixes at least $t-6$ points of $I(a)$. Thus $\langle a, y_1, y_2, y_3 \rangle$ is semiregular on $\Delta - \{1', 2', 3', 4'\}$. Hence by (2.6) $\langle a, y_1, y_2, \cdots, y_k \rangle$ is semiregular on $\Delta - \{1', 2', 3', 4'\}$.

Since y_1' normalizes $\langle a, y_1, y_2, \cdots, y_k \rangle$, y_1' fixes $\{1', 2', 3', 4'\}$. Suppose that $\langle a, y_1, y_2, \cdots, y_k, y_1' \rangle$ is not semiregular on $\Delta - \{1', 2', 3', 4'\}$. Then there is an element y' in $\langle a, y_1, y_2, \cdots, y_k \rangle y_1'$ which has fixed points in $\Delta - \{1', 2', 3', 4'\}$. Then $y'^{I(a)}$ is of order four or two. If $y'^{I(a)}$ is of order four, then $\langle a, y'^2 \rangle = \langle a, y_1 \rangle$ and y'^2 has fixed points in $\Delta - \{1', 2', 3', 4'\}$, which is a contradiction. Hence $y'^{I(a)}$ is of order two. Thus y' is $(1\,3)\,(2\,4)$ or $(1\,4)\,(2\,3)$ on $\{1, 2, 3, 4\}$. Hence $y' \in \langle a, y_1', y_2y_3, y_2y_4, \cdots, y_2y_k \rangle$ or $\langle a, y_1y_1', y_2y_3, y_2y_4, \cdots, y_2y_k \rangle$. Thus $\langle a, y_1', y_2y_3, y_2y_4, \cdots, y_2y_k \rangle$ or $\langle a, y_1y_1', y_2y_3, y_2y_4, \cdots, y_2y_k \rangle$ is semiregular on neither the orbit $\{1', 2', 3', 4'\}$ of length four nor $\Delta - \{1', 2', 3', 4'\}$. This is a contradiction since these groups are conjugate to $\langle a, y_1, y_2y_3, y_2y_4, \cdots, y_2y_k \rangle$ in $C(a)$ which is semiregular on $\Delta - \{1', 2', 3', 4'\}$. Thus $\langle a, y_1, y_2, \cdots, y_k, y_1' \rangle$ is semiregular on $\Delta - \{1', 2', 3', 4'\}$.

On the other hand $\langle a, y_1 \rangle$ normalizes $G_{5'6'7'8'}$, which is of even order. Hence there is an involution u in $G_{5'6'7'8'}$ commuting with a and y_1. Then $\langle a, y_1, u \rangle$ is conjugate to a subgroup of $\langle a, y_1, y_2, \cdots, y_k, y_1' \rangle$ in $C(a)$. This is a contradiction since for any point of $\{1', 2', \cdots, 8'\}$ of length eight $\langle a, y_1, u \rangle$ has an element $(\neq 1)$ fixing this point. Thus $y_2 \neq (1')\,(2')\,(3')\,(4') \cdots$.

Next suppose that y_2 fixes a $\langle a, y_1\rangle$-orbit of length four pointwise. Then we may assume that y_2 fixes $\{5', 6', 7', 8'\}$ pointwise and

$$y_2 = (1\,2)\,(3)\,(4)\,(5\,6)\,(7)\,(8)\cdots(t)\,(1'3')\,(2'\,4')\,(5')\,(6')\,(7')\,(8')\cdots.$$

Sinee $\langle a, y_1, y_3\rangle$ is conjugate to $\langle a, y_1, y_2\rangle$, y_3 or ay_3 is $(1'3')\,(2'4')$ on $\{1', 2', 3', 4'\}$. Hence we may assume that $y_3=(1'3')\,(2'4')\cdots$. Since $\langle a, y_2, y_3\rangle$ is conjugate to $\langle a, y_1, y_2\rangle$, y_3 is $(5'7')\,(6'8')$ or $(5'8')\,(6'7')$ on $\{5', 6', 7', 8'\}$. On the other hand $C(a)$ has a 2-element

$$y_2' = (1)\,(2)\,(3)\,(4)\,(5\,7)\,(6\,8)\,(9)\,(10)\cdots(t)\cdots.$$

By (2.3) we may assume that $\langle a, y_1, y_2, y_3, y_1', y_2'\rangle$ is a 2-group. Since $\langle a, y_1'\rangle$ and $\langle a, y_2'\rangle$ are conjugate to $\langle a, y_1\rangle$, $\langle a, y_1'\rangle$ and $\langle a, y_2'\rangle$ are elementary abelian. Since $\langle a, y_2y_3, y_1'\rangle$ and $\langle a, y_1, y_2'\rangle$ are conjugate to $\langle a, y_1, y_2y_3\rangle$ and $I(y_1)\cap\Delta=I(y_2y_3)\cap\Delta=\{1', 2', 3', 4'\}$, y_i' or ay_i', $i=1, 2$, fixes $\{1', 2', 3', 4'\}$ pointwise. Hence we may assume that y_1' and y_2' fix $\{1', 2', 3', 4'\}$ pointwise. Thus y_1, y_2y_3, y_1' and y_2' fix $\{1', 2', 3', 4'\}$ pointwise. Hence $\langle a, y_1, y_2y_3, y_1', y_2'\rangle$ is elementary abelian.

If y_1' or y_2' fixes $\{5', 6', 7', 8'\}$, then $(y_2y_1')^2$ or $(y_2y_2')^2$ is of order two and fixes $(I(a)-\{1, 2, 3, 4\})\cup\{1', 2', \cdots, 8'\}$ of length $t+4$ pointwise, contrary to the assumption (*). Thus $\{5', 6', 7', 8'\}^{y_i'}\neq\{5', 6', 7', 8'\}$, $i=1, 2$.

Since $y_3=(5'7')\,(6'8')\cdots$ or $(5'8')\,(6'7')\cdots$, first suppose that $y_3=(5'7')\,(6'8')\cdots$. Then $I(y_1y_2y_3)\cap\Delta=\{1', 2', \cdots, 8'\}$. Since $I(y_1')\cap\Delta=\{1', 2', 3', 4'\}$ and y_1' commutes with $y_1y_2y_3$, y_1' fixes $\{5', 6', 7', 8'\}$, which is a contradiction. Next suppose that $y_3=(5'8')\,(6'7')\cdots$. Since $\{5', 6', 7', 8'\}^{y_1'}\neq\{5', 6', 7', 8'\}$, we may assume that $\{5', 6', 7', 8'\}^{y_1'}=\{9', 10', 11', 12'\}$, where $\{9', 10', 11', 12'\}$ is a $\langle a, y_1\rangle$-orbit. Since $ay_1y_2y_3$ fixes $\{5', 6', 7', 8'\}$ pointwise and commutes with y_1', $ay_1y_1y_2$ fixes $\{9', 10', 11', 12'\}$ pointwise. Then $I(ay_1y_2y_3)\cap\Delta=\{5', 6', \cdots, 12'\}$ since $|I(ay_1y_2y_3)|\leq t$. Furthermore y_2' commutes with $ay_1y_2y_3$. Hence $\{5', 6', 7', 8'\}^{y_2'}=\{9', 10', 11', 12'\}$. Thus $\{5', 6', \cdots, 12'\}$ is a $\langle y_1, y_2y_3, y_1', y_2'\rangle$-orbit of length eight. Since the order of $\langle y_1, y_2y_3, y_1', y_2'\rangle$ is sixteen, there is an element y' $(\neq 1)$ in $\langle y_1, y_2y_3, y_1', y_2'\rangle$ fixing $\{5', 6', \cdots, 12'\}$ pointwsie. Moreover since $I(\langle y_1, y_2y_3, y_1', y_2'\rangle)\supseteq\{1', 2', 3', 4'\}$, $I(y')\supseteq\{1', 2', 3', 4'\}$ and so $|I(y')\cap\Delta|\geq 12$. This contradicts the assumption (*) since $y'^{I(a)}$ is an involution consisting of at most four 2-cycles. Thus $C(Q)^{I(Q)}\not\geq A_t$.

Case 2. $C(Q)^{I(Q)}=1$.[1)]

(i) Since $|I(y_1)\cap\Delta|=4$, $I(y_1)\cap\Delta$ is contained in one or two Q-orbits in Δ. If $I(y_1)\cap\Delta$ is contained in two Q-orbits, then y_1 fixes exactly two points of a Q-orbit. Then by (2.12) $C(Q)^{I(Q)}\geq A_t$, which is a contradiction. Thus $I(y_1)\cap\Delta$ is contained in one Q-orbit.

1) The proof in this case is due to the suggestion of Dr. E. Bannai. The proof was first more complicated.

(ii) Let $\Phi(Q)$ be the Frattini subgroup of Q. Then since y_1 is an automorphism of Q and $\Phi(Q)$ by conjugation, y_1 induces an automorphism of $Q/\Phi(Q)$, which we denote by y_1^*. For an element a of Q, $a^{-1}a^{y_1}$ is in $\Phi(Q)$ if and only if the image in $Q/\Phi(Q)$ of a is in $C_{Q/\Phi(Q)}(y_1^*)$. Hence the number of elements a in Q such that $a^{-1}a^{y_1}$ is in $\Phi(Q)$ is $|C_{Q/\Phi(Q)}(y_1^*)|\cdot|\Phi(Q)|$. On the other hand for elements a and b of Q, ab^{-1} is in $C_Q(y_1)$ if and only if $a^{-1}a^{y_1}=b^{-1}b^{y_1}$. Hence the number of elemenets a in Q such that $a^{-1}a^{y_1}$ is in $\Phi(Q)$ is at most $|C_Q(y_1)|\cdot|\Phi(Q)|=4\cdot|\Phi(Q)|$. Thus $4\cdot|\Phi(Q)|\geqq|C_{Q/\Phi(Q)}(y_1^*)|\cdot|\Phi(Q)|$ and so $4\geqq|C_{Q/\Phi(Q)}(y_1^*)|$. Since $Q/\Phi(Q)$ is elemtary abelian, $|Q/\Phi(Q)|\leqq(2^2)^2=2^4$ by Lemma of [6]. Thus the automorphism group of $Q/\Phi(Q)$ is contained in $GL(4, 2)$. Furthermore if an element of odd order in $N(Q)$ acts trivially on $Q/\Phi(Q)$ by conjugation, then this element belongs to $C(Q)$ ([1], Theorem 5.1.4). Since $C(Q)^{I(Q)}=1$ and $N(Q)^{I(Q)}=S_t$ or A_t, $N(Q)^{I(Q)}$ is involved in the automorphism group of $Q/\Phi(Q)$ and so in $GL(4, 2)$. Thus $N(Q)^{I(Q)}=S_6$ or A_8.

(iii) Suppose that $N(Q)^{I(Q)}=S_6$. Let H be the normal subgroup of G consisting of all even permutations of G. Then for any point i of Ω, H_i is normal in G_i. Since G_i is 3-fold transitive on $\Omega-\{i\}$ and $|\Omega-\{i\}|$ is odd, H_i is 3-fold transitive on $\Omega-\{i\}$ by a theorem of Wagner [15]. Hence H is 4-fold transitive on Ω. Let x be a 2-element of $N_G(Q)$ such that

$$x=(1)\,(2)\,(3)\,(4)\,(5\,6)\cdots.$$

Then x has no fixed point in Δ by (2.13). Hence the number of Q-orbits in Δ is even and so $Q\leq H$. If x is an odd permutation, then $x\notin N_H(Q)$. Hence Q is a Sylow 2-subgroup of H_{1234} and $|I(Q)|=6$, which is a contradiction by [12]. Thus x is an even per- mutation. Hence x^{Δ} is an odd permutation. On the other hand since x has no fixed point in Δ and $x^2\in Q$, every cycle of x in Δ has the same length and $\bar{x}$ consists of 2-cycles. Thus x consists of cycles of length $2|Q|$ in Δ since x^{Δ} is an odd permutation. Thus $|x|=2|Q|$. Hence $|x^2|=|Q|$. Since $x^2\in Q$, $Q=\langle x^2\rangle$. Hence the automorphism group of Q is a 2-group. This is a con-tradiction since $N(Q)^{I(Q)}=S$ and $N(Q)^{I(Q)}$ is involved in the automorphism group of Q. Thus $N(Q)^{I(Q)}\neq S_6$.

(v) Suppose that $N(Q)^{I(Q)}=A_8$.

(v. i) $y_1^{I(Q)}$ is an involution consisting of exactly two 2-cycles. Hence by (2.8) y_1 fixes at most four Q-orbits in Δ. Furthermore we have proved that y_1 fixes at least two Q-orhits in Δ. Thus y_1 fixes two, three or four Q-orbits in Δ.

(v. ii) Suppose that y_1 fixes exactly four Q-orbits in Δ. Then by (2.8) every element of Qy_1 is an involution. Since $\langle Q, y_2\rangle$ and $\langle Q, y_1y_2\rangle$ are conjugate to $\langle Q, y_1\rangle$, every element of Qy_2 and Qy_1y_2 is an involution. In particular y_1, y_2 and y_1y_2 are involutions. Hence y_1 and y_2 commute. Let u be any element of Q. Then uy_1 and $uy_1\cdot y_2$ are also involutions. Hence y_2 commutes with uy_1 and so commutes with u. Thus $y_1 \in C(Q)$, which is a contradiction since $C(Q)^{I(Q)}=1$.

(v. iii) Suppose that y_1 fixes exactly three Q-orbits in Δ. Then by (2.8) there are at least $\frac{3}{4}|Q|$ involutions in Qy_1. Since y_2 normalizes $\langle Q, y_1\rangle$, y_2 fixes at least one $\langle Q, y_1\rangle$-orbit of length $|Q|$. Then for a point i of the $\langle Q, y_1, y_2\rangle$-orbit of length $|Q|$ Qy_1 and Qy_2 have elements fixing i. Hence we may assume that y_1 and y_2 fix i. Then $y_1^2=y_2^2=1$ and $y_1y_2=y_2y_1$. Let T be a set of elements u in Q such that both uy_1 and uy_1y_2 are involutions. Since $\langle Q, y_1y_2\rangle$ is conjugate to $\langle Q, y_1\rangle$, there are at least $\frac{3}{4}|Q|$ involutions in Qy_1y_2. Hence $|T| \geq \frac{1}{2}|Q|$. Since y_2 is an involution, y_2 commutes with uy_1, where $u \in T$. Furthermore y_2 commutes with y_1. Hence y_2 commutes with u. On the other hand $|I(y_2) \cap \Delta|=4$. Hence y_2 commutes with exactly four elements of Q. Thus $|T| \leq 4$. Hence $4 \geq |T| \geq \frac{1}{2}|Q|$ and so $8 \geq |Q|$. Then the automorphism group of Q is a 2-group, S_3, S_4 or $SL(3,2)$ (see [3]). Since $N(Q)^{I(Q)}=A_8$ and $N(Q)^{I(Q)}$ is involved in the automorphism group of Q, we have a contradiction.

(v. iv) Thus y_1 fixes exactly two Q-orbits in Δ. Then any 2-element of $N(Q)$ which is an involution consisting of exactly two 2-cycles on $I(Q)$ fixes two Q-orbits in Δ. Set $\bar{\Delta}=\{\Delta_1, \Delta_2, \cdots, \Delta_r\}$, where $\Delta=\Delta_1 \cup \Delta_2 \cdots \cup \Delta_r$ and Δ_i, $1 \leq i \leq r$, is a Q-orbit. Then we may assume that

$$\bar{y}_1 = (\Delta_1)(\Delta_2)(\Delta_3\Delta_4)(\Delta_5\Delta_6)\cdots$$

and y_1 fixes four points $1'$, $2'$, $3'$, $4'$ of Δ_1.

(v. v) Since y_2 normalizes $\langle Q, y_1\rangle$, $\bar{y}_2$ fixes $\{\Delta_1, \Delta_2\}$, Assume that $\bar{y}_2=(\Delta_1\Delta_2)\cdots$. Since $\langle Q, y_2\rangle$ and $\langle Q, y_1y_2\rangle$ are conjugate to $\langle Q, y_1\rangle$, y_2 and y_1y_2 fix exactly two Q-orbits in Δ. Since $\bar{y}_1=(\Delta_1)(\Delta_2)(\Delta_3\Delta_4)(\Delta_5\Delta_6)\cdots$ and $\bar{y}_2$ commutes with $\bar{y}_1$, we may assume that

$$\bar{y}_2 = (\Delta_1\Delta_2)(\Delta_3)(\Delta_4)(\Delta_5\Delta_6)\cdots .$$

Then $\langle \bar{y}_1, \bar{y}_2\rangle$ is semiregular on $\{\Delta_7, \Delta_8\cdots\}$. Since $\langle \bar{y}_1, \bar{y}_2, \bar{y}_3\rangle$ is elementary abelian, $\bar{y}_3$ fixes $\{\Delta_1, \Delta_2\}$, $\{\Delta_3, \Delta_4\}$ and $\{\Delta_5, \Delta_6\}$. Furthermore since $\langle Q, y_1y_3\rangle$ and $\langle Q, y_2y_3\rangle$ are conjugate to $\langle Q, y_1\rangle$, y_1y_3 and y_2y_3 fix exactly two Q-orbits in Δ. Hence

$$\bar{y}_3 = (\Delta_1\Delta_2)(\Delta_3\Delta_4)(\Delta_5)(\Delta_6)\cdots .$$

Since $\bar{y}_2\bar{y}_3$ fixes Δ_1, there is an element in Qy_2y_3 fixing $1'$ of Δ_1. Hence we may assume that y_2y_3 fixes $1'$. Then $I((y_2y_3)^2)$ and $I((y_2y_3)^{y_1}\cdot y_2y_3)$ contains $I(Q) \cup \{1'\}$ of length $t+1$. Hence by the assumption (*) $(y_2y_3)^2=1$ and $y_1\cdot y_2y_3= y_2y_3\cdot y_1$. Let T be a set of elements u of Q such that both y_2y_3u and $y_1y_2y_3u$ are involutions. Since $\bar{y}_2\bar{y}_3$ fixes Δ_1 and Δ_2, by (2.8) there are at least $\frac{|Q|}{2}$ involutions in y_2y_3Q having fixed points in Δ. Furthermore since $\bar{y}_1\bar{y}_2\bar{y}_3$ fixes $\{\Delta_1, \Delta_2, \ldots, \Delta_6\}$ pointwise and $y_1y_2y_3$ consists of four 2-cycles on $I(Q)$, by (2.8) at least $\frac{3}{4}|Q|$ involutions of $y_1y_2y_3Q$ have fixed points in Δ. Hence $|T|\geq \frac{1}{4}|Q|$. Since for any element u of T y_2y_3u and $y_1\cdot y_2y_3u$ are involutions, y_1 commutes with y_2y_3u. Furthermore y_1 commutes with y_2y_3. Hence y_1 commutes with u. Since $|I(y_1)\cap\Delta|=4$, y_1 commutes with exactly four elemenets of Q. Hence $|T|\leq 4$. Thus $\frac{1}{4}|Q|\leq 4$ and so $|Q|\leq 16$. Since $C(Q)^{I(Q)}=1$, $N(Q)^{I(Q)}=A_t$ is involutved in the automorphism group of Q. Hence Q is an elementary abelian group of order sixteen (see [3]). As we have seen above, at least $\frac{3}{4}|Q|$ elements of $y_1y_2y_3Q$ are involutions. Then since $y_1y_2y_3$ is an involution and Q is elementary abelian, $y_1y_2y_3$ commutes with at least $\frac{3}{4}|Q|$ elements of Q. Hence $y_1y_2y_3$ centralizes Q. This is a contradiction since $C(Q)^{I(Q)}=1$. Thus we may assume that $\bar{y}_2=(\Delta_1)(\Delta_2)(\Delta_3\Delta_5)(\Delta_4\Delta_6)\cdots$. Similarly $\bar{y}_3$ fixes $\{\Delta_1, \Delta_2\}$ pointwise.

Suppose that $\langle\bar{y}_1, \bar{y}_2, \bar{y}_3\rangle$ is not semiregular on $\bar{\Delta}-\{\Delta_1, \Delta_2\}$. Then we may assume that $\bar{y}_3$ fixes $\{\Delta_3, \Delta_4, \Delta_5, \Delta_6\}$. Then $\bar{y}_1\bar{y}_2\bar{y}_3$ fixes $\{\Delta_1, \Delta_2, \cdots, \Delta_6\}$ pointwise. Hence by the same argument as above we have a contradiction. Thus $\langle\bar{y}_1, \bar{y}_2, \bar{y}_3\rangle$ is semiregular on $\bar{\Delta}-\{\Delta_1, \Delta_2\}$.

Since $\langle Q, y_1'\rangle$ is conjugate to $\langle Q, y_1\rangle$, y_1' fixes exactly two Q-orbits in Δ. Since $\langle\bar{y}_1, \bar{y}_2\bar{y}_3, \bar{y}_1'\rangle$ is abelian and $\langle\bar{y}_1, \bar{y}_2\bar{y}_3\rangle$ is semiregular on $\bar{\Delta}-\{\Delta_1, \Delta_2\}$, $\bar{y}_1'$ fixes Δ_1 and Δ_2.

Suppose that $\langle\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_1'\rangle$ is not semiregular on $\bar{\Delta}-\{\Delta_1, \Delta_2\}$. Then there is an element y' in $\langle Q, y_1, y_2, y_3\rangle y_1'$ such that $\bar{y}'$ has fixed points in $\bar{\Delta}$ other than Δ_1 and Δ_2. Then $y'^{I(Q)}$ is of order four or two. If $y'^{I(Q)}$ is of order four, then $\bar{y}'^2=\bar{y}_1$. This is a contradiction since $\bar{y}_1$ has no fixed point in $\bar{\Delta}-\{\Delta_1. \Delta_2\}$. If $y'^{I(Q)}$ is of order two, then $y'^{I(Q)}$ has exactly two or four 2-cycles. Hence $\langle Q, y'\rangle$ is conjugate to $\langle Q, y_1\rangle$ or $\langle Q, y_1y_2y_3\rangle$. This is a contradiction since $\bar{y}_1$ and $\bar{y}_1\bar{y}_2\bar{y}_3$ have exactly two fixed points Δ_1 and Δ_2. Thus $\langle\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_1'\rangle$ is semiregular on $\bar{\Delta}-\{\Delta_1, \Delta_2\}$.

Since $\bar{y}_2$, $\bar{y}_3$ and $\bar{y}_1'$ fix Δ_1, Qy_2, Qy_3 and Qy_1' have elements fixing $1'$ of Δ_1. Hence we may assume that y_2, y_3 and y_1' fix $1'$. Then $\langle y_1, y_2, y_3\rangle$ and $\langle y_1, y_2y_3, y_1'\rangle$ are elementary abelian. Since $I(y_1)\cap\Delta=\{1', 2', 3', 4'\}$, $\langle y_1, y_2, y_3, y_1'\rangle$ fixes $\{1', 2', 3', 4'\}$. Set $R=C_Q(y_1)$. Then R is of order four and has an orbit $\{1', 2', 3', 4'\}$. Hence $\langle y_1, y_2, y_3, y_1'\rangle$ normalizes R. Since $y_1\notin C(Q)$, $|Q|>4$. Hence the number of the R-orbit in Δ_1 is even. Since $\langle y_1, y_2, y_3, y_1'\rangle$ fixes the R-orbit $\{1', 2', 3', 4'\}$ in Δ_1, we may assume that $\langle y_1, y_2, y_3, y_1'\rangle$ fixes one more R-orbit $\{5', 6', 7', 8'\}$ in Δ_1.

(v. vi) Let a be an involution R commuting with y_1, y_2 and y_3. Then $\langle a, y_1\rangle$-orbits in $\Delta-(\Delta_1\cup\Delta_2)$ are of length four. Let $\{i_1, i_2, i_3, i_4\}$ be any $\langle a, y_1\rangle$-orbit in $\Delta-(\Delta_1\cup\Delta_2)$. Then $\langle a, y_1\rangle$ normalizes $G_{i_1 i_2 i_3 i_4}$. Hence there is an involution u in $G_{i_1 i_2 i_3 i_4}$ commuting with a and y_1. Then $\langle y_1, u\rangle$ normalizes $G_{I(Q)}$ and so a Sylow 2-subgroup Q' of $G_{I(Q)}$. Since $N(Q)^{I(Q)}=A_8$, $\langle Q', y_1, u\rangle$ is conjugate to a subgroup of $\langle Q, y_1, y_2, y_3, y_1'\rangle$ in $N(G_{I(Q)})$. Hence y_1 fixes exactly two Q'-orbits Δ_1' and Δ_2' in Δ and $\{i_1, i_2, i_3, i_4\}$ is contained in Δ_1' or Δ_2'. Furthermore since $\langle Q', y_1\rangle$ is conjugate to $\langle Q, y_1\rangle$ in $\langle Q, Q', y_1\rangle$, there is an element v in $\langle Q, Q', y'\rangle$ such that $\langle Q', y_1\rangle^v=\langle Q, y_1\rangle$. Then $(\Delta_1'\cup\Delta_2')^v=\Delta_1\cup\Delta_2$. Since $v^{I(Q)}$ or $(y_1 v)^{I(Q)}=1$ and $\langle Q', y_1\rangle^{y_1 v}=\langle Q, y_1\rangle$, we may assume that $v^{I(Q)}=1$. Then $v\in G_{I(Q)}$ and $(\Delta_1'\cup\Delta_2')^v=\Delta_1\cup\Delta_2$. Thus $\{i_1, i_2, i_3, i_4\}$ is contained in a $G_{I(Q)}$-orbit which contains Δ_1 or Δ_2. Since $\{i_1, i_2, i_3, i_4\}$ is any $\langle a, y_1\rangle$-orbit in $\Delta-(\Delta_1\cup\Delta_2)$, any $\langle a, y_1\rangle$-orbit in $\Delta-(\Delta_1\cup\Delta_2)$ is contained in the $G_{I(Q)}$-orbit which contains Δ_1 or Δ_2. Hence $G_{I(Q)}$ is transitive or has two orbits Γ_1 and Γ_2 on Δ, where $\Gamma_1\supseteq\Delta_1$ and $\Gamma_2\supseteq\Delta_2$.

Since y_1 fixes exactly two Q-robits in Δ, the number of Q-orbits in Δ is even. Hence $|\Delta|$ is divisible by $2|\Delta_1|=2|Q|$. If $G_{I(Q)}$ is transitive on Δ, then the order of $G_{I(Q)}$ is divisible by $2|Q|$. This is a contradiction since Q is a Sylow 2-subgroup of $G_{I(Q)}$. Hence $G_{I(Q)}$ has two orbits Γ_1 and Γ_2 on Δ.

Since $y_1\notin C(Q)$, $|Q|>4$. Hence $\langle Q, y_1, y_1'\rangle$ is a Sylow 2-subgroup of G_{5678}. Since G is 4-fold transitive, any Sylow 2-subgroup P of a stabilizer of four points in G is conjugate to $\langle Q, y_1, y_1'\rangle$ and so has exactly one orbit of length four. Furthermore a stabilizer of a point of this orbit of length four in P is conjugate to Q.

We may assume that

$$y_1=(1\ 2)\ (3\ 4)\ (5)\ (6)\ (7)\ (8)\ (1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\cdots,$$
$$a=(1)\ (2)\cdots(8)\ (1'\ 2')\ (3'\ 4')\cdots.$$

Since y_2 and y_3 fix $1'$ and commute with a and y_1, y_2 and y_3 are $(1')\ (2')\ (3')\ (4')$ or $(1')\ (2')\ (3'\ 4')$ on $\{1', 2', 3', 4'\}$.

Assume that $y_2=(1')\ (2')\ (3')\ (4')$ on $\{1', 2', 3', 4'\}$. Since $|I(y_1y_2)|\leq t$, we may assume that

$$y_2=(1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8)\ (1')\ (2')\ (3')\ (4')\ (5'\ 7')\ (6'\ 8')\cdots.$$

Thus $\langle y_1, y_2\rangle$ is semiregular on $\{5', 6', \cdots, n\}$. Suppose that y_3 has fixed points in $\{5', 6', \cdots, n\}$. Since $\langle y_1, y_2, y_3\rangle$ is abelian, y_3 has at least four fixed points in $\{5', 6', \cdots, n\}$. This is a contradiction since $I(y_3)\supset\{1'\}$ and $|I(y_3)|\leq 8$. Hence y_3 fixes $\{1', 2', 3', 4'\}$ pointwise. Since $\langle y_1, y_2, y_3\rangle$ fixes the R-orbit $\{5', 6', 7', 8'\}$, there is an element $(\neq 1)$ in $\langle y_1, y_2, y_3\rangle$ fixing $\{5', 6', 7', 8'\}$ pointwise. Since $I(\langle y_1, y_2, y_3\rangle)\supseteq\{1', 2', 3', 4'\}$, this element is $y_1y_2y_3$. Hence

$$y_3=(1\ 2)\ (3)\ (4)\ (5)\ (6)\ (7\ 8)\ (1')\ (2')\ (3')\ (4')\ (5'\ 8')\ (6'\ 7')\cdots.$$

Then $\langle y_1, y_2, y_3\rangle$ normalizes $G_{1\,2\,1'\,2'}$. Hence as we have seen above, $\langle y_1, y_2, y_3\rangle$ normalizes a 2-subgroup Q'' of $G_{1\,2\,1'\,2'}$ which is conjugate to Q. Then $|I(Q'')|=8$ and $N(Q'')^{I(Q'')}=A_8$. Hence $y_1^{I(Q'')}$, $y_2^{I(Q'')}$ and $y_3^{I(Q'')}$ are even permutations. Since y_1, y_2 and y_3 are $(1\ 2)(1')(2')$ on $\{1, 2, 1', 2'\}$, y_1, y_2 and y_3 have exactly one more 2-cycle other than $(1\ 2)$ in $I(Q'')$. This is impossible. Hence $y_2 \neq (1')(2')(3')(4')\cdots$. Similarly $y_3 \neq (1')(2')(3')(4')\cdots$.

Thus y_2 and y_3 are $(1')(2')(3'\ 4')$ on $\{1', 2', 3', 4'\}$. Since $|R|=4$, R is cyclic or elementary abelian. First assume that R is cyclic. Then $R=\langle b\rangle$ and

$$b = (1)(2)\cdots(8)(1'\ 3'\ 2'\ 4')(5'\ 7'\ 6'\ 8')\cdots .$$

Then $\langle R, y_1\rangle$ is semiregular on $\{9', 10', \cdots, n\}$. Since $\langle a, y_1, y_2\rangle$ is abelian, if y_2 has fixed points in $\{9', 10', \cdots, n\}$, then y_2 fixes at least four points of $\{9', 10', \cdots, n\}$. This is a contradiction since $I(y_2)$ contains $\{3, 4, 7, 8\}\cup\{1'\}$ of length five. Thus y_2 has no fixed points in $\{9', 10', \cdots, n\}$. Similalry y_3 has no fixed points in $\{9', 10', \cdots, n\}$. Hence y_2 and y_3 have exactly two fixed points in $\{5', 6', 7', 8'\}$. Next assume that R is elementary abeliain. Then $R=\langle a, b'\rangle$ and

$$b' = (1)(2)\cdots(8)(1'\ 3')(2'\ 4')\cdots .$$

Then $b'y_2$ and $b'y_3$ are of order four and so 4-cycle on $\{5', 6', 7', 8'\}$. Hence y_2 and y_3 have exactly two fixed points in $\{5', 6', 7', 8'\}$. Thus in both cases we may assume that

$$\begin{aligned}
a &= (1)(2)\cdots(8)(1'\ 2')(3'\ 4')(5'\ 6')(7'\ 8')\cdots .\\
y_2 &= (1\ 2)(3)(4)(5\ 6)(7)(8)(1')(2')(3'\ 4')(5')(6')(7'\ 8')\cdots ,\\
y_3 &= (1\ 2)(3)(4)(5)(6)(7\ 8)(1')(2')(3'\ 4')(5'\ 6')(7')(8')\cdots .
\end{aligned}$$

Since $\langle a, y_1, y_2, y_3\rangle$ normalizes $G_{1\,2\,1'\,2'}$, as we have seen above $\langle a, y_1, y_2, y_3\rangle$ normalizes a 2-subgroup Q'' of $G_{1\,2\,1'\,2'}$ which is conjugate to Q. Then $|I(Q'')|=8$ and $N(Q'')^{I(Q'')}=A_8$. Hence $a^{I(Q'')}$, $y_1^{I(Q'')}$, $y_2^{I(Q'')}$ and $y_3^{I(Q'')}$ are even permutations. Since $a=(1)(2)(1'\ 2')$ and $y_i=(1\ 2)(1')(2')$, $i=1, 2, 3$, on $\{1, 2, 1', 2'\}$, a and y_i have exactly one more 2-cycle other than $(1'\ 2')$ and $(1\ 2)$ respectively in $I(Q'')$. Since the lengths of $\langle a, y_1, y_2, y_3\rangle$-orbits in $\{9', 10', \cdots, n\}$ are at elast eight, $|I(Q'')\cap\{9', 10', \cdots, n\}|=0$. Hence $I(Q'')=\{1, 2, 3, 4, 1', 2', 3', 4'\}$, $\{1, 2, 5, 6, 1', 2', 5', 6'\}$, or $\{1, 2, 7, 8, 1', 2', 7', 8'\}$.

First assume that $I(Q'')=\{1, 2, 3, 4, 1', 2', 3', 4'\}$. Then a Sylow 2-subgroup of $G_{1\,2\,3\,4}$ containing Q or Q'' has exactly one orbit $\{5, 6, 7, 8\}$ or $\{1', 2', 3', 4'\}$ of lengh four respectively. Since Sylow 2-subgroups of $G_{1\,2\,3\,4}$ are conjugate, $\{5, 6, 7, 8\}$ and $\{1', 2', 3', 4'\}$ are contained in th same $G_{1\,2\,3\,4}$-orbit. Since $\Gamma_1 \supset \{1', 2', 3', 4'\}$, $\{5, 6, 7, 8\}$ and Γ_1 are contained in the same $G_{1\,2\,3\,4}$-orbit. By (2.11) G is not 5-fold transitive. Hence $G_{1\,2\,3\,4}$ has two orbits $\{5, 6, 7, 8\}\cup\Gamma_1$ and Γ_2 on $\Omega-\{1, 2, 3, 4\}$.

Next assume that $I(Q'')=\{1, 2, 5, 6, 1', 2', 5', 6'\}$. Then by the same argument as above G_{1256} has two orbits $\{3, 4, 7, 8\} \cup \Gamma_1$ and Γ_2. Since $N(Q)^{I(Q)} = A_8$, there is an element $z=(1)(2)(3\ 5)(4\ 6)(7)(8)\cdots$. Then $G_{1234}=(G_{1256})^z$ has two orbits $\{5, 6, 7, 8\} \cup \Gamma_1^z$ and Γ_2^z. Since Γ_1 and Γ_2 are $G_{I(Q)}$-robits, $\Gamma_1^z = \Gamma_1$ or Γ_2. On the other hand G is 4-fold transitive on Ω. Hence G_{1278} has two orbits $\{3, 4, 5, 6\} \cup \Gamma_i$ and Γ_j, where $\{i, j\}=\{1, 2\}$. Since $z \in G_{1278}$, z fixes Γ_1 and Γ_2. Hence G_{1234} has two orbits $\{5, 6, 7, 8\} \cup \Gamma_1$ and Γ_2. Similarly if $I(Q'')=\{1, 2, 7, 8, 1', 2', 7', 8'\}$, then G_{1234} has two orbits $\{5, 6, 7, 8\} \cup \Gamma_1$ and Γ_2. Thus in any case G_{1234} has the two orbits $\{5, 6, 7, 8\} \cup \Gamma_1$ and Γ_2.

On the other hand Δ_2 is contained in Γ_2 and fixed by y_1. Hence there is an element in Qy_1 fixing four points of Δ_2. Then by the same argument as above $\{5, 6, 7, 8\}$ and Γ_2 are contained in the same G_{1234}-orbit. Thus G_{1234} is transitive on $\Omega - \{1, 2, 3, 4\}$, contrary to (2.11). Thus $N(Q)^{I(Q)} \neq A_8$. Hence we complete the proof of (2.15).

2.16. $N(Q)^{I(Q)} \neq S_t$.

Proof. Suppose by way of contradiction that $N(Q)^{I(Q)}=S_t$. Then by (2.4) $N(Q)$ has the 2-group $\langle Q, x_1, x_2, \cdots, x_k \rangle$. Now we show that $\langle Q, x_1, x_2, \cdots, x_k \rangle$ is semiregular on Δ. By (2.13) and (2.15) $\langle Q, x_1, x_2 \rangle$ is semiregular on Δ.

Suppose that $\langle Q, x_1, x_2, x_3 \rangle$ is not semiregular on Δ. Then x_3 fixes *a* $\langle Q, x_1, x_2 \rangle$-orbit Δ' of length $4|Q|$ in Δ. Then by (2.13) and (2.15) $\bar{x}_1\bar{x}_2\bar{x}_3$ fixes Q-orbits in Δ'. Furthermore $\langle \bar{x}_1, \bar{x}_2, \bar{x}_3 \rangle$ is abelian and $\langle \bar{x}_1, \bar{x}_2 \rangle$ is semiregular on $\bar{\Delta}$. Hence $\bar{x}_1\bar{x}_2\bar{x}_3$ fixes four Q-orbits in Δ'. By (2.8) $\bar{x}_1\bar{x}_2\bar{x}_3$ fixes at most six Q-orbits in $\bar{\Delta}$. Hence $\bar{x}_1\bar{x}_2\bar{x}_3$ does not fix any Q-orbit in $\Delta - \Delta'$. Hence $\langle Q, x_1, x_2, x_3 \rangle$ is semiregular on $\Delta - \Delta'$. Since $N(Q)^{I(Q)}=S_t$, $N(Q)$ has a 2-element

$$y_1' = (1\ 3)(2\ 4)(5)(6)\cdots(t)\cdots.$$

By (2.3) we may assume that $\langle Q, x_1, x_2, x_3, y_1' \rangle$ is a 2-group. Then y_1' normalizes $\langle Q, x_1, x_2, x_3 \rangle$. Hence y_1' fixes the $\langle Q, x_1, x_2, x_3 \rangle$-orbit Δ'. Thus Δ' is *a* $\langle Q, x_1, x_2, y_1' \rangle$-orbit. Hence $\langle Q, x_1, x_2, y_1' \rangle$ has an element x $(\neq 1)$ fixing a point of Δ'. Then by (2,13) and (2.15) $x^{I(Q)}$ is of order four and has exactly one 4-cycle $(1\ 3\ 2\ 4)$ or $(1\ 4\ 2\ 3)$. Hence $(x^2)^{I(Q)}=(1\ 2)(3\ 4)$ and has fixed points in Δ, contrary to (2.15). Thus $\langle Q, x_1, x_2, x_3 \rangle$ is semiregular on Δ.

Suppose that $\langle Q, x_1, x_2, x_3, x_4 \rangle$ is not semiregular on Δ. Then x_4 fixes *a* $\langle Q, x_1, x_2, x_3 \rangle$-orbit Δ' of length $8|Q|$ in Δ. Since $\langle \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \rangle$ is abelian and $\langle \bar{x}_1, \bar{x}_2, \bar{x}_3 \rangle$ is semiregular on $\bar{\Delta}$, by (2.8) $\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4$ fixes exactly eight Q-orbits in Δ, whose union is Δ'. Thus $\langle Q, x_1, x_2, x_3, x_4 \rangle$ is semiregular on $\Delta - \Delta'$. Since $N(Q)^{I(Q)}=S_t$, $N(Q)$ has a 2-element

$$y_1' = (1\ 3)(2\ 4)(5)(6)\cdots(t)\cdots.$$

By (2.3) we may assume that $\langle Q, x_1, x_2, x_3, x_4, y_1' \rangle$ is a 2-group. Then y_1' normalizes $\langle Q, x_1, x_2, x_3, x_4 \rangle$. Hence y_1' fixes Δ'. Then Δ' is *a* $\langle Q, x_1, x_2, x_3, y_1' \rangle$-

orbit. Hence there is an element x in $\langle Q, x_1, x_2, x_3\rangle y_1'$ fixing a point of Δ'. Since $\langle Q, x\rangle$ is not conjugate to any subgroup of $\langle Q, x_1, x_2, x_3\rangle$, $x^{I(Q)}$ is of order four and has exactly one 4-cycle (1 3 2 4) or (1 4 2 3). Hence $(x^2)^{I(Q)}=(1\ 2)\ (3\ 4)$ and x^2 has fixed points in Δ, contrary to (2.15). Thus $\langle Q, x_1, x_2, x_3, x_4\rangle$ is semiregular on Δ. Hence by (2.9) $\langle Q, x_1, x_2, \cdots, x_k\rangle$ is semiregular on Δ.

On the other hand Q has an involution $a=(1)\ (2)\cdots(t)\ (i\,j)\cdots$. Then a normalizes $G_{1\,2\,i\,j}$ and so commutes with an involution u of $G_{1\,2\,i\,j}$. Then u normalizes $G_{I(Q)}$. Hence u normalizes a Sylow 2-subgroup Q' of $G_{I(Q)}$. Since Q' is conjugate to Q in $G_{I(Q)}$ and $N(Q)^{I(Q)}=S_t$, $\langle Q', u\rangle$ is conjugate to a subgroup of $\langle Q, x_1, x_2, \cdots, x_k\rangle$ in $N(G_{I(Q)})$. Hence $\langle Q, x_1, x_2, \cdots, x_k\rangle$ has an element $(\neq 1)$ which has fixed points in Δ. This is a contradiction. Thus $N(Q)^{I(Q)}\neq S_t$.

2.17. *We show that $N(Q)^{I(Q)}\neq A_t$ and complete the proof of the theorem.*

Proof. Suppose by way of contradiction that $N(Q)^{I(Q)}=A_t$. First suppose that $t=8$ or 9. Let $a=(1)\ (2)\cdots(t)\ (i\,j)\cdots$ be an involution of Q. Then a normalizes $G_{1\,2\,i\,j}$ and so commutes with an involution u of $G_{1\,2\,i\,j}$. Since $N(Q)^{I(Q)}=N(G_{I(Q)})^{I(Q)}=A_8$ or A_9 and $|I(u)|\leq t$, $u^{I(Q)}$ consists of exactly two 2-cycles. This contradicts (2.15) since $|I(u)\cap\Delta|\neq 0$.

Thus $t\geq 10$. Then by (2.4) $N(Q)$ has the 2-group $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$, $k\geq 4$. Now we show that $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ is semiregular on Δ. By (2.15) $\langle Q, y_1, y_2\rangle$ is semiregular on Δ.

Let y be any element of $\langle Q, y_1, y_2, y_1'\rangle - Q$. Then $y^{I(Q)}$ is of order two or four. If $y^{I(Q)}$ is of order two, then $y^{I(Q)}$ consists of exactly two 2-cycles. Hence by (2.15) y is semiregular on Δ. If $y^{I(Q)}$ is of order four, then $(y^2)^{I(Q)}=y_1{}^{I(Q)}$. Hence y is semiregular on Δ. Thus $\langle Q, y_1, y_2, y_1'\rangle$ is semiregular on Δ.

Suppose that $\langle Q, y_1, y_2, y_3\rangle$ is not semiregular on Δ. Then by (2.15) $\bar{y}_1\bar{y}_2\bar{y}_3$ has fixed points in $\bar{\Delta}$. Since $(y_1y_2y_3)^{I(Q)}$ is an involution consisting of exactly four 2-cycles $\bar{y}_1\bar{y}_2\bar{y}_3$ fixes at most eight Q-orbits by (2.8). On the other hand $\langle\bar{y}_1, \bar{y}_2, \bar{y}_3\rangle$ is abelian and $\langle\bar{y}_1, \bar{y}_2\rangle$ is a semiregular group of order four. Hence $\bar{y}_1\bar{y}_2\bar{y}_3$ fixes four or eight Q-orbits. Thus y_3 fixes one or two $\langle Q, y_1, y_2\rangle$-orbits in Δ.

Assume that y_3 fixes exctly one $\langle Q, y_1, y_2\rangle$-orbit Γ in Δ. Then since y_1' normalizes $\langle Q, y_1, y_2, y_3\rangle$, y_1' fixes Γ. Hence Γ is also a $\langle Q, y_1, y_2, y_1'\rangle$-orbit. This is a contradiction since $\langle Q, y_1, y_2, y_1'\rangle$ is semiregular on Δ. Thus y_3 fixes exactly two $\langle Q, y_1, y_2\rangle$-orbits in Δ, say Γ_1 and Γ_2. Hence by (2.8) any element of $Qy_1y_2y_3$ is an involution and has exactly eight fixed points in Δ.

Suppose that $\Gamma_1=\Delta_1\cup\Delta_2\cup\Delta_3\cup\Delta_4$ and $\Gamma_2=\Delta_5\cup\Delta_6\cup\Delta_7\cup\Delta_8$, where Δ_i, $1\leq i\leq 8$, is a Q-orbit. Set $\bar{\Gamma}_1=\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ and $\bar{\Gamma}_2=\{\Delta_5, \Delta_6, \Delta_7, \Delta_8\}$. Then we may assume that

$$\bar{y}_1 = (\Delta_1\ \Delta_2)(\Delta_3\ \Delta_4)(\Delta_5\ \Delta_6)(\Delta_7\ \Delta_8)\cdots,$$
$$\bar{y}_2 = (\Delta_1\ \Delta_3)(\Delta_2\ \Delta_4)(\Delta_5\ \Delta_7)(\Delta_6\ \Delta_8)\cdots,$$
$$\bar{y}_3 = (\Delta_1\ \Delta_4)(\Delta_2\ \Delta_3)(\Delta_5\ \Delta_8)(\Delta_6\ \Delta_7)\cdots.$$

Since y_i, $i\geq 4$, normalizes $\langle Q, y_1, y_2, y_3\rangle$, $\Gamma_1^{y_i}=\Gamma_1$ or Γ_2. Suppose that $\Gamma_1^{y_i}=\Gamma_1$. Then Γ_1 is a $\langle Q, y_1, y_2, y_i\rangle$-orbit. Hence $y_1y_2y_i$ fixes a Q-orbit in Γ_1 by (2.15). Since $\bar{y}_1\bar{y}_2\bar{y}_3$ is the identity on $\bar{\Gamma}_1$, $\bar{y}_1\bar{y}_2\bar{y}_3$. $\bar{y}_1\bar{y}_2\bar{y}_i=\bar{y}_3\bar{y}_i$ fixes a Q-orbit in Γ_1, contrary to (2.15). Thus $\Gamma_1^{y_i}=\Gamma_2$.

Suppose that $t\geq 12$. Then $N(Q)$ has y_4 and y_5. Since $\langle\bar{y}_1, \bar{y}_2, \bar{y}_4\rangle$ is elementary abelian and $\Gamma_1^{y_4}=\Gamma_2$, we may assume that

$$\bar{y}_4 = (\Delta_1\ \Delta_5)(\Delta_2\ \Delta_6)(\Delta_3\ \Delta_7)(\Delta_4\ \Delta_8)\cdots.$$

Furthemore since $\Gamma_1^{y_5}=\Gamma_2$, $\bar{\Gamma}_1\cup\bar{\Gamma}_2$ is a $\langle\bar{y}_1, \bar{y}_2, \bar{y}_4, \bar{y}_5\rangle$-orbit of length eight. Hence $\langle\bar{y}_1, \bar{y}_2\rangle\bar{y}_4\bar{y}_5$ has an element fixing $\bar{\Gamma}_1\cup\bar{\Gamma}_2$ pointwise. Thus we may assume that $\bar{y}_1\bar{y}_4\bar{y}_5$ fixes $\bar{\Gamma}_1\cup\bar{\Gamma}_2$ pointwise and so

$$\bar{y}_5 = (\Delta_1\ \Delta_6)(\Delta_2\ \Delta_5)(\Delta_3\ \Delta_8)(\Delta_4\ \Delta_7)\cdots.$$

On the other hand $N(Q)$ has 2-elements

$$y_4' = (1)(2)(3\ 4)(5)(6)(7)(8)(9\ 11)(10)(12)(13)\cdots(t)\cdots,$$
$$y_5' = (1)(2)(3\ 4)(5)(6)(7)(8)(9)(11)(10\ 12)(13)(14)\cdots(t)\cdots.$$

By (2.3) we may assume that $\langle Q, y_1, y_2, y_3, y_4', y_5'\rangle$ is a 2-group. Then by the same argument as above $\Gamma_1^{y_4'}=\Gamma_1^{y_5'}=\Gamma_2$. If $\bar{y}_i'=(\Delta_1\ \Delta_5)\cdots$, $i=4$, 5, then $(y_4y_i')^3$ has the same form as y_1 on $I(Q)$ and fixes Δ_1, which is a contradiction. Similarly $\bar{y}_i'\neq(\Delta_1\ \Delta_6)\cdots$, $i=4$, 5, since $(\bar{y}_5\bar{y}_i')^3=\bar{y}_1$. Hence we may assume that

$$\bar{y}_4' = (\Delta_1\ \Delta_7)(\Delta_2\ \Delta_8)(\Delta_3\ \Delta_5)(\Delta_4\ \Delta_6)\cdots,$$
$$\bar{y}_5' = (\Delta_1\ \Delta_8)(\Delta_2\ \Delta_7)(\Delta_3\ \Delta_6)(\Delta_4\ \Delta_5)\cdots.$$

Then $y_4y_5y_4'y_5'$ consists of exactly two 2-cycles on $I(Q)$ and fixes Δ_1, contrary to (2.15).

Thus $t=10$ or 11. Assume that $t=10$. The proof in the case $t=11$ is similar. Since $\langle Q, y_1, y_2, y_1'\rangle$ is semiregular on Δ, the lengths of $\langle Q, y_1, y_2, y_1'\rangle$-orbits on Δ are $8|Q|$. On the other hand $\langle Q, y_1, y_2, y_1'\rangle$ fixes 7, 8, 9, 10 and has two orbits $\{1, 2, 3, 4\}$ and $\{5, 6\}$ on $I(Q)$. Hence $\langle Q, y_1, y_2, y_1'\rangle$ is a Sylow 2-group of $G_{7\,8\,9\,10}$. Furthemore in $\langle Q, y_1, y_2, y_1'\rangle$ any element fixing ten points belongs to Q. Since G is 4-fold transitive, this shows that any element fixing ten points is conjugate to an element of Q. Set $z_1=y_1y_2y_3$. By what we have proved above every element of Qz_1 is an involution. Hence for any element u of Q $u^{z_1}=u^{-1}$. Furthermore $N(Q)$ has a 2-element

$$z_2 = (1\ 3)(2\ 4)(5\ 7)(6\ 8)(9)(10)\cdots.$$

By (2.3) we may assume that $\langle Q, z_1, z_2\rangle$ is a 2-group. Since $\langle Q, z_2\rangle$ and $\langle Q, z_1z_2\rangle$ are conujgate to $\langle Q, z_1\rangle$, every element of Qz_2 and Qz_1z_2 is an involution. Hence for any element u of Q $u^{z_2}=u^{-1}$ and $u^{z_1 z_2}=u^{-1}$. On the other hand $(u^{z_1})^{z_2}=(u^{-1})^{z_2}=u$. Hence $u=u^{-1}$. Thus Q is elementary abelian and $z_1, z_2 \in C(Q)$. Then since $N(Q)^{I(Q)}=A_{10}$ and $C(Q)^{I(Q)}$ is a normal subgroup $(\neq 1)$, $N(Q)^{I(Q)}=C(Q)^{I(Q)}$. In particular since Q is abelian, every 2-element of $N(Q)$ belongs to $C(Q)$.

Since $y_1^2 \in Q$, the order of y_1 is two or four. Suppose that y_1 is of order two. Then for any 2-cycle $(i\,j)$ of y_1 in Δ y_1 normalizes $G_{1\,2\,i\,j}$. Hence y_1 normalizes a 2-subgroup Q' of $G_{1\,2\,i\,j}$ which is conjugate to Q. Since $N(Q')^{I(Q')}=A_{10}$, y_1 consist of exactly two or four 2-cycles on $I(Q')$. Suppose that y_1 consists of exactly four 2-cycles on $I(Q')$. Then $\langle Q', y_1\rangle$ is conjugate to $\langle Q, z_1\rangle$. Then $|I(y_1)|=10$, which is a contradiction. Thus y_1 consists of exactly two 2-cycles on $I(Q')$. Then $I(Q')=\{i, j, 1, 2, 5, 6, \cdots, 10\}$. Then Q and Q' are contained in $G_{7\,8\,9\,10}$ and so conjugate in $G_{7\,8\,9\,10}$. Thus $G_{7\,8\,9\,10}$ has an element which takes $\{1, 2, i, j\}$ into $\{1, 2, \cdots, 6\}$. Since $\{1, 2, \cdots, 6\}$ is contained in a $G_{7\,8\,9\,10}$-orbit and $(i\,j)$ is any 2-cycle of y_1 in Δ, $G_{7\,8\,9\,10}$ is transitive on $\Omega-\{7, 8, 9, 10\}$, contrary to (2.11). Thus y_1 is of order four. Hence every involution of $N(Q)-Q$ consists of exactly four 2-cycles on $I(Q)$ and every involution of G fixes exactly ten points.

$C(Q)$ has an involution

$$z_3=(1\ 3)(2\ 4)(5\ 6)(7)(8)(9\ 10)\cdots .$$

By (2.3) we may assume that $\langle Q, z_1, z_3\rangle$ is a 2-group. Then since $z_1 z_3$ consists of exactly four 2-cycles on $I(Q)$, $z_1 z_3$ is of order two. Hence $z_1 z_3=z_3 z_1$. Since $I(z_1)\neq I(z_3)$ and any Sylow 2-subgroup of $G_{I(z_1)}$ is conjugate to Q, z_3 fixes exactly two points of $I(z_1)$. Hence $|I(z_1)\cap I(z_3)\cap \Delta|=2$. Then since Q is semiregular on Δ and $\langle z_1, z_3\rangle < C(Q)$, $|Q|=2$. Set $Q=\langle a\rangle$.

Since $\langle a, y_3 y_4\rangle$ is conjugate to $\langle a, y_1\rangle$, $y_3 y_4$ is of order four and $(y_3 y_4)^2=a$. Let $(i\,j\,k\,l)$ be any 4-cycle of $y_3 y_4$ in Δ. Then $y_3 y_4$ normalizes $G_{i\,j\,k\,l}$. Hence $y_3 y_4$ commutes with an involution z of $G_{i\,j\,k\,l}$. Since z commutes with $(y_3 y_4)^2=a$, z fixes $I(a)$. Thus $y_3 y_4 z$ is of order four and $(y_3 y_4 z)^{I(a)}$ is of order two. Hence $y_3 y_4 z$ consists of exactly two 2-cycles on $I(a)$. Then since $(y_3 y_4)^{I(a)}=(7\ 8)(9\ 10)$ and $z^{I(a)}$ consists of exactly four 2-cycles, z has 2-cycles $(7\ 8)$ and $(9\ 10)$. Hence $y_3 y_4 z \in G_{7\,8\,9\,10}$. Furthermore $y_3 y_4 z$ is $(i\,j\,k\,l)$ on $\{i, j, k, l\}$. Hence $\{i, j, k, l\}$ is contained in a $G_{7\,8\,9\,10}$-orbit. Set $z_4=y_1 y_3 y_4$. Then z_4 has 2-cycles $(7\ 8)$ and $(9\ 10)$. Since $C(a)^{I(a)}{}_{7\,8\,9\,10}=A_6$, $C(a)$ has an involution z' which is conjugate to z under $C(a)_{7\,8\,9\,10}$ and has the same form as z_4 on $I(a)$. Then $\langle a, z'\rangle$ and $\langle a, z_4\rangle$ are Sylow 2-subgroups of $\langle a, z_4, z'\rangle$ and $\langle a, z_4\rangle^{I(a)}=\langle a, z'\rangle^{I(a)}$. Hence $\langle a, z'\rangle$ is conjugate to $\langle a, z_4\rangle$ under $\langle a, z_4, z'\rangle_{I(a)}$ and so z' is conjugate to z_4 or az_4 under $\langle a, z_4, z'\rangle_{I(a)}$. Thus z is conjugate to z_4 or az_4 under $C(a)_{7\,8\,9\,10}$. Since $I(z)\cap\Delta\subset\{i, j, k, l\}$, there is an element in $C(a)_{7\,8\,9\,10}$ which takes $\{i, j, k, l\}$ into $I(z_4)\cap\Delta$ or $I(az_4)\cap\Delta$. On the other hand $z_4^{y_1'}=z_4 a$.

Hence $(I(z_4)\cap\Delta)^{y_1'}=I(az_4)\cap\Delta$. Thus $C(a)_{78910}$ has an element taking $\{i, j, k, l\}$ into $I(z_4)\cap\Delta$. Furthermore $y_1'y_2$ is of order eight and commutes with z_4. Hence $y_1'y_2$ consists of a 8-cycle on $I(z_4)\cap\Delta$. Thus $I(z_4)\cap\Delta$ is contained in a $C(a)_{78910}$-orbit. Since $(i\,j\,k\,l)$ is any 4-cycle of y_3y_4 in Δ, Δ is cntained in a $C(a)_{78910}$-orbit and so in a G_{78910}-orbit. By (2.11) G_{78910} is intransitive on $\Omega-\{7, 8, 9, 10\}$. Hence G_{78910} has exactly two orbits $\{1, 2, \cdots, 6\}$ and Δ on $\Omega-\{7, 8, 9, 10\}$. Since G is 4-fold transitive, any four points i_1, i_2, i_3, i_4 of Ω uniquely determine a subset $\Delta(i_1, i_2, i_3, i_4)$ of Ω which is the $G_{i_1 i_2 i_3 i_4}$-orbit of lengt six.

For a 2-cycle (11 12) of a and any two points i_1, i_2 of $\{1, 2, \cdots, 10\}$ four points $11, 12, i_1, i_2$ uniquenly determine $\Delta(11, 12, i_1, i_2)$, on which a consists of exactly three 2-cycles. Conversely for any 2-cycle $(j_1 j_2)$ of a in $\Delta-\{11, 12\}$ four points $11, 12, j_1, j_2$ uniquely determine $\Delta(11, 12, j_1, j_2)$ and a fixes exactly two points of $\Delta(11, 12, j_1, j_2)$ which are contained in $\{1, 2, \cdots, 10\}$. Hence the number of 2-cycles of a in $\Delta-\{11, 12\}$ is $\binom{10}{2}\cdot 3=135$. Hence $n=12+135\cdot 2=282$. On the other hand for any point i of $\Omega-\{1, 2, 3\}$ four points $1, 2, 3, i$ uniquely determine $\Delta(1, 2, 3, i)$. Hence $282-3\equiv 0 \pmod 7$, which is a contradiction. (In the case $t=11$ for any two points i_1, i_2 of $\{1, 2, \cdots, 11\}$ $|\{1, 2, \cdots, 11\}\cap\Delta(11, 12, i_1, i_2)|=3$. Hence $\binom{11}{2}\equiv 0 \pmod 3$, which is a contradiction.) Thus $\langle Q, y_1, y_2, y_3\rangle$ is semiregular on Δ.

Let y' be any element of $\langle Q, y_1, y_2, y_3, y_4, y_1'\rangle - Q$. Then $y'^{I(Q)}$ is of order two or four. If $y'^{I(Q)}$ is of order two, then $y'^{I(Q)}$ consists of two or four 2-cycles. Hence $\langle Q, y'\rangle$ is conjugate to a subgroup of $\langle Q, y_1, y_2, y_3\rangle$ in $N(Q)$. Hence y' is semiregular on Δ. If $y'^{I(Q)}$ is of order four, then $(y'^2)^{I(Q)}=y_1^{I(Q)}$. Hence y' is semiregular on Δ. Thus $\langle Q, y_1, y_2, y_3, y_4, y_1'\rangle$ is semiregular on Δ. Hence by (2.10) $\langle Q, y_1, y_2, \cdots, y_k, y_1'\rangle$ is semireglar on Δ.

Let x be any 2-element of $N(G_{I(Q)})$. Then x normalizes a Sylow 2-subgroup Q' of $G_{I(Q)}$. Since Q is a Sylow 2-subgroup of $G_{I(Q)}$ and $N(Q)^{I(Q)}=A_t$, $\langle Q', x\rangle$ is cnjugate to a subgroup of $\langle Q, y_1, y_2, \cdots, y_k\rangle$. Hence x is semiregular on Δ. On the other hand Q has an involution $a=(1)(2)\cdots(t)(i\,j)\cdots$. Then a normalizes G_{12ij}, and so commutes with an involution u of G_{12ij}. Then $u\in N(G_{I(Q)})$ and $|I(u)\cap\Delta|\neq 0$, which is a contradiction. Thus $N(Q)^{I(Q)}\neq A_t$.

Thus we complete the proof of the theorem.

3. Proof of the lemma

In this section we assume that G is a permutation group as in Lemma. Suppose by way of contradiction that there is a 2-group Q in G such that $|I(Q)|=12$ and $N(Q)^{I(Q)}=M_{12}$. Let $\bar{Q}$ be a Sylow 2-subgroup of $G_{I(Q)}$. Since $N(\bar{Q})^{I(\bar{Q})}=N(G_{I(Q)})^{I(Q)}\geq N(Q)^{I(Q)}=M_{12}$, $N(\bar{Q})^{I(\bar{Q})}=S_{12}$, A_{12} or M_{12}. If $N(\bar{Q})^{I(\bar{Q})}$

$=S_{12}$, or A_{12}, then by Thereom $G=S_{14}$ or A_{16}. Hence $N(Q)^{I(Q)}=S_{12}$, which is a contradiction. Thus $N(\bar{Q})^{I(\bar{Q})}=M_{12}$. Hence we may assume that Q is a Sylow 2-subgroup of $G_{I(Q)}$.

Set $I(Q)=\{1, 2, \cdots, 12\}$ and $\Delta=\Omega-I(Q)$. Then $n\geq 35$ ([2], p. 80) and so $|\Delta|\geq 23$.

Since $N(Q)^{I(Q)}=M_{12}$, we may assume that $N(Q)$ has 2-element

$$x_1=(1)(2)(3)(4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)\cdots,$$
$$y_1=(1)(2)(3)(4)(5\ 7\ 6\ 8)(9\ 11\ 10\ 12)\cdots,$$
$$y_2=(1)(2)(3)(4)(5\ 10\ 6\ 9)(7\ 11\ 8\ 12)\cdots,$$

and $\langle Q, x_1, y_1, y_2\rangle$ is a 2-group (see (2.3)). Then $\langle Q, y_1^2\rangle=\langle Q, y_2^2\rangle=\langle Q, y_1\rangle$. Since Q is a normal subgroup of $\langle Q, y_1, y_2\rangle$, Q has a central involution a of $\langle Q, y_1, y_2\rangle$. Then we may assume that

$$a=(1)(2)\cdots(12)(13\ 14)(15\ 16)\cdots(n-1\ n).$$

3.1. *First we show hat $\langle Q, y_1, y_2\rangle$ has at least one orbit of length eight in Δ on which $\langle Q, y_1, y_2\rangle$ is a quaternion group.*

Proof. Suppose by way of contradiction that $\langle Q, y_1, y_2\rangle$ has no orbit of length eight in Δ on which $\langle Q, y_1, y_2\rangle$ is a quaternion group. Then $\{5, 6, \cdots, 12\}$ is the unique $\langle Q, y_1, y_2\rangle$-orbit of length eight and on which $\langle Q, y_1, y_2\rangle$ is a quaternion group.

(i) We show that $\langle Q, y_1, y_2\rangle$ is a Sylow 2-subgroup of G_{1234} and Q is a characteristic subgroup of $\langle Q, y_1, y_2\rangle$. Let x be any 2-element of $N(\langle Q, y_1, y_2\rangle)_{1234}$. Then x fixes $\{5, 6, \cdots, 12\}$ and so $I(Q)$. Hence $x\in N(Q)$. Since $(N(Q)_{1234})^{I(Q)}=\langle y_1, y_2\rangle^{I(Q)}$, $x^{I(Q)}\in\langle y_1, y_2\rangle^{I(Q)}$. Hence there is an element x' in $\langle Q, y_1, y_2\rangle$ such that $x'^{I(Q)}=x^{I(Q)}$. Hence $(x'^{-1}x)^{I(Q)}=1$ and so $x'^{-1}x\in Q$. Thus $x\in\langle Q, x'\rangle\leq\langle Q, y_1, y_2\rangle$. This shows that $\langle Q, y_1, y_2\rangle$ is a Sylow 2-subgroup of G_{1234}. Furthermore since any automorphism of $\langle Q, y_1, y_2\rangle$ fixes $I(Q)$ and $\langle Q, y_1, y_2\rangle_{I(Q)}=Q$, Q is a characteristic subgroup of $\langle Q, y_1, y_2\rangle$.

(ii) Let i, j, k, l be any four points of Ω and X be a 2-group such that $X\leq N(G_{ijkl})$. Then we show that G_{ijkl} has an involution x such that $X\leq C(x)$, $|I(x)|=12$ and $C(x)^{I(x)}\leq M_{12}$. Since $X\leq N(G_{ijkl})$, X normalizes a Sylow 2-subgroup P' of G_{ijkl}. Since G is 4-fold transitive, P' is conjugate to $\langle Q\ y_1, y_2\rangle$. Hence P' has a characteristic subgroup Q' which is conjugate to Q. Then $X\leq N(Q')$. Hence there is an involution x in Q' such that $X\leq C(x)$. Since $|I(Q')|=12$ and $N(Q')^{I(Q')}=M_{12}$, $|I(x')|=12$ and $C(x)^{I(x)}\leq M_{12}$. We remark that if x is the unique involution of Q' then $C(x)^{I(x)}=M_{12}$.

(iii) We show that Q is a cyclic or generalized quaternion group and $C(Q)^{I(Q)}=N(Q)^{I(Q)}$. Suppose by way of contradiction that Q has an involution b other than a. Then since a is a central involution of Q, we may assume that

$$b=(1)(2)\cdots(12)(13\ 15)(14\ 16)(17\ 19)(18\ 20)(21\ 23\ )(22\ 24)\cdots.$$

Then $\langle a, b\rangle \leq N(G_{13\,14\,15\,16})$. Hence by (ii) $G_{13\,14\,15\,16}$ has an involution u such that $\langle a, b\rangle \leq C(u)$, $|I(u)|=12$ and $C(u)^{I(u)} \leq M_{12}$. Then $|I(a)\cap I(u)|=0$ or 4. If $|I(a)\cap I(u)|=4$, then $b^{I(u)}$ fixes the same four points that a fixes and commutes with $a^{I(u)}$. This is a contradiction since $C(u)^{I(u)} \leq M_{12}$. Hence $|I(a)\cap I(u)|=0$. Then we may assume that

$$u=(1\ 3)\ (2\ 4)\ (5\ 7)\ (6\ 8)\ (9\ 11)\ (10\ 12)\ (13)\ (14)\cdots(24)\cdots .$$

Since $\langle a, u\rangle \leq N(G_{1\,3\,13\,14})$, by (ii) $G_{1\,3\,13\,14}$ has an involution v such that $\langle a, u\rangle \leq C(v)$, $|I(v)|=12$ and $C(v)^{I(v)} \leq M_{12}$. Let R be a Sylow 2-subgroup of $\langle a ,b, u, v\rangle$ containing $\langle a, b, u\rangle$. Then $R^{I(Q)}=\langle u, v\rangle^{I(Q)}$. Hence R has an element v' such tha $v'^{I(Q)}=v^{I(Q)}$ and v' is conjugate to v. Since $u\in Z(\langle a, b, u, v\rangle)$, v' fixes $I(u)$. Since v' fixes 1,3 which are not contained in $I(u)$ and $|I(v')|=12$, v' does not fix $I(u)$ pointwise. Furthermore $I(u)$ is a union of of $\langle a, b, u, v\rangle$-orbits and v' is conjugate to v which has fixed points in $I(u)$. Hence v' has fixed points in $I(u)$ and so v' fixes exctly four points of $I(u)$. Since $(bv')^{I(u)}$ is a 2-element of $C(u)^{I(u)} \leq M_{12}$, $(bv')^{I(u)}$ is of order two, four or eight. If $(bv')^{I(u)}$ is of order two, then b commutes with v'. Hence $\langle a, b\rangle^{I(v')}$ is a four group and $|I(\langle a, b\rangle^{I(v')})|=4$. This is a contradiction since M_{12} has no such subgroup. If $(bv')^{I(u)}$ is of order four or eight, then $((bv')^{I(u)})^2$ or $((bv')^{I(u)})^4$ is an invlution fixing four points and so $I((bv')^2)$ or $I((bv')^4)$ contains $\{1, 2, \cdots, 12\}$ and four points of $I(u)$, contrary to the assumption. Thus Q has exactly one involution and so Q is a cyclic or generalized quaternion group. Hence the automorphism group of Q is a 2-group or S_4. Since $N(Q)^{I(Q)}=M_{12}$ and $N(Q)^{I(Q)}/C(Q)^{I(Q)}$ is involved in the automorphism grup of Q, $C(Q)^{I(Q)}=N(Q)^{I(Q)}$.

(iv) Thus a is the unique involution of Q. Since $a\in N(G_{1\,2\,13\,14})$, $G_{1\,2\,13\,14}$ has an involution x such that $ax=xa$, $|I(x)|=12$ and $C(x)^{I(x)}=M_{12}$ by (ii). Then we may assume that $x=x_1$ and

$$x_1=(1)\ (2)\ (3)\ (4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\ (13)\ (14)\cdots(20)\cdots .$$

Since $\langle a, x_1\rangle \leq N(G_{5\,6\,13\,14})$, $G_{5\,6\,13\,14}$ has an involution x_2 such that $\langle a, x_1\rangle \leq C(x_2)$, $|I(x_2)|=12$ and $C(x_2)^{I(x_2)}=M_{12}$ by (ii). Then $\langle x_1, x_2\rangle$ normalizes a Sylow 2-subgroup of $G_{I(Q)}$ containing a. Hence we may assume that $\langle x_1, x_2\rangle$ normalizes Q. Furthermoe since $N(Q)^{I(Q)}=M_{12}$ and $C(x_1)^{I(x_1)}=M_{12}$, we may assume that

$$x_2=(1\ 2)\ (3\ 4)\ (5)\ (6)\ (7)\ (8)\ (9\ 10)\ (11\ 12)\ (13)\ (14)\ (15)\ (16)\ (17\ 18)\ (19\ 20)\cdots$$

or

$$x_2=(1)\ (2)\ (3\ 4)\ (5)\ (6)\ (7\ 8)\ (9\ 11)\ (10\ 12)\ (13)\ (14)\ (15\ 16)\ (17\ 19)\ (18\ 20)\cdots .$$

(v) We show that $x_1, x_2 \notin C(Q)$. Suppose by way of contradiction that $x_1\in C(Q)$. Since $\langle Q, x_2\rangle$ is conjugate to $\langle Q, x_1\rangle$ in $N(Q)$, there is an element

u in Q such that x_2u is conjugate to x_1 in $N(Q)$. Then $x_2u \in C(Q)$ and $|I(x_2u)| = 12$. Hence x_2u commutes with u and so x_2 commutes with u. Since x_2 and x_2u are of order two, $u^2=1$. Hence $u=a$ or 1. Thus $x_2 \in C(Q)$. Since $\langle x_1, x_2\rangle < C(Q)$ and $|I(x_1) \cap I(x_2) \cap \Delta| = 2$ or 4, Q is of order two or four. Thus Q is abelian. Then since $N(Q)^{I(Q)} = C(Q)^{I(Q)}$ by (iii), $y_i \in C(Q)$, $i=1, 2$. Since $y_i^2 \in \langle Q, x_1\rangle$, there is an element u_i in Q such that $y_i^2 = u_ix_1$. Then y_i commutes with u_ix_1. Since y_i commutes with u_i, y_i commutes with x_1. Hence y_i fixes $I(x_1) \cap \Delta$. Furthermore since $x_1 \in C(Q)$, Q fixes $I(x_1) \cap \Delta$. Thus $I(x_1) \cap \Delta$ is a union of $\langle Q,, y_1, y_2\rangle$-orbits.

Suppose that Q is of order four. Since $\langle Q, y_1, y_2\rangle^{I(x_1) \cap \Delta}$ is not a quaternion group and $C(x_1)^{I(x_1)} = M_{12}$, $\langle Q, y_1, y_2\rangle^{I(x_1) \cap \Delta} = Q^{I(x_1) \cap \Delta}$. Hence $|\langle Q, y_1, y_2\rangle_{I(x_1 \cap \Delta)}| = 8$ and so Qy_i, $i=1, 2$, has an element y_i' fixing $I(x_1) \cap \Delta$ pointwise. Then $I(\langle y_1', y_2'\rangle) = I(x_1)$. Since $N(G_{I(x_1)})^{I(x_1)} \geq C(x_1)^{I(x_1)} = M_{12}$, for the four points 1, 2, 3, 4 of $I(x_1)$ a Sylow 2-subgroup of $G_{1\,2\,3\,4}$ cotaining $\langle y_1', y_2'\rangle$ is of order at least $8 \cdot 8$. This is a contradiction since $\langle Q, y_1, y_2\rangle$ is a Sylow 2-subgroup of $G_{1\,2\,3\,4}$ and of order $8 \cdot 4$.

Next suppose that Q is of order two. Then by the same reason as above $\langle Q, y_1, y_2\rangle^{I(x_1) \cap \Delta}$ is a cyclic group of order two or four. Hence $\langle Q, y_1, y_2\rangle$ has an element y which is of order four and fixes $I(x_1) \cap \Delta$ pointwise. Then by the same argument as above $G_{1\,2\,3\,4}$ has a Sylow 2-subgroup containing y and of order at least $8 \cdot 4$. This is a contradiction since $\langle Q, y_1, y_2\rangle$ is a Sylow 2-subgroup of $G_{1\,2\,3\,4}$ and of order $8 \cdot 2$. Thus $x_1 \notin C(Q)$. Similarly $x_2 \notin C(Q)$.

(vi) Since $C(Q)^{I(Q)} = N(Q)^{I(Q)}$ and $x_1 \notin C(Q)$, Q is nonabelian. Hence by (iii) Q is a generalized quaternion group. Moreover there are elements b_1 and b_2 in Q such that b_1x_1 and b_2x_2 belong to $C(Q)$. Then b_ix_i commutes with b_i, $i=1, 2$. Hence x_i commutes with b_i. Thus b_i fixes $I(x_i)$. Since $|I(x_i) \cap I(Q)| = 4$ and $C(x_i)^{I(x_i)} = M_{12}$, b_i fixes exactly four points of $I(x_i)$ and so b_i is of order two or four. If b_i is of order two, then $b_i = a$ since a is the unique involution of Q. This is a contradiction since $x_i \notin C(Q)$. Thus b_i is of order four. Furthermore this shows that $\langle Q, y_1, y_2\rangle$ has exactly one central involution a.

Suppose that Q is of order at least sixteen. Then we may assume that $Q = \langle c, d\rangle$, where $c^4 = d^{2^r} = 1$ and $r \geq 3$. Suppose that $b_1 \in \langle d\rangle$. Then since d commutes with b_1x_1, d commutes with x_1. Then d fixes $I(x_1) \cap \Delta$ of length eight. Since d is of order at least eight, d is of order eight. Thus $d^{I(x_1)}$ has four fixed points and one 8-cycle, which is a contradiction since $C(x_1)^{I(x_1)} = M_{12}$. Thus $b_1 \notin \langle d\rangle$ and so $Q = \langle b_1, d\rangle$. Similarly $Q = \langle b_2, d\rangle$. Hence $d^{b_i} = d^{-1}$, $i=1, 2$, and so $d^{b_ix_i} = (d^{-1})^{x_i}$. On the other hand since $b_ix_i \in C(Q)$. Hence $d^{b_ix_i} = d$. Thus $d^{x_i} = d^{-1}$ and so $d^{x_1x_2} = d$. Since $|I(x_1x_2)| \leq 12$, $|I(x_1x_2) \cap I(Q)| = 4$ and $I(x_1x_2) \cap \Delta \supseteq \{13, 14\}$, $2 \leq |I(x_1x_2) \cap \Delta| \leq 8$. Then since d is of order at least eight, $|I(x_1x_2) \cap \Delta| = 8$ and d is of order eight. Thus $|I(x_1x_2)| = 12$ and $d^{I(x_1x_2)}$ has four fixed points and one 8-cycle. This implies that $C(x_1x_2)^{I(x_1x_2)} \not\leq M_{12}$.

On the other hand for any four points i, j, k, l of $I(x_1x_2)$ let P' be a Sylow 2-subgroup of G_{ijkl} containing x_1x_2. Then since G is 4-fold transitive, P' is conjugate to $\langle Q, y_1, y_2\rangle$. Hence P' has the unique central involution a' which is conjugate to a. Then $P'_{I(a')}$ is conjugate to Q and $C(a')^{I(a')}=M_{12}$. If $x_1x_2=a'$, then $C(x_1x_2)^{I(x_1x_2)}=M_{12}$, which is a contradiction. Hence $x_1x_2\neq a'$. Then since $P'_{I(a')}$ has exactly one involution a', $x_1x_2\notin P'_{I(a')}$. Hence $I(x_1x_2)\cap I(a')=\{i, j, k, l\}$ because $C(a')^{I(a')}=M_{12}$. Thus $a^{I(x_1x_2)}$ fixes exactly four points i, j, k, l. Then by a lemma of Livingstone and Wanger [4] $C(x_1x_2)^{I(x_1x_2)}$ is 4-fold transitive on $I(x_1x_2)$. Since $C(x_1x_2)^{I(x_1x_2)}\neq M_{12}$, $C(x_1x_2)^{I(x_1x_2)}\geq A_{12}$. Then by Theorem $G=S_{14}$ or A_{16}, which is a contradiction.

Thus Q is a quaternion group. Since $C(Q)^{I(Q)}=N(Q)^{I(Q)}$, Qy_1 has an element which belongs to $C(Q)$. Hence we may assume that $y_1\in C(Q)$. Hence $y_1^2(b_1x_1)^{-1}\in C(Q)\cap Q=\langle a\rangle$. Thus $y_1^2=b_1x_1$ or ab_1x_1 and so y_1 is of order eight. Furthermore y_1 commutes with a and b_1. Hence y_1 commutes with x_1. Thus y_1 fixes $I(x_1)$ and so $y_1^{I(x_1)}$ has four fixed points and one 8-cycle. This is a contradiction since $C(x_1)^{I(x_1)}=M_{12}$. Thus we complete the proof of (3.1).

3.2. *Next we show tht Q is of order two and Qx_1 has an involution x_1' such that $|I(x_1')|=12$ and $C(x_1')^{I(x_1')}=M_{12}$.*

Proof. By (3.1) $\langle Q, y_1, y_2\rangle$ has an orbit Γ in Δ such that $|\Gamma|=8$ and $\langle Q, y_1, y_2\rangle^\Gamma$ is a quaternion group. Then Q is a quaternion group or a cyclic group of order four or two. Hence the automorphism group of Q is S_4 or a 2-group. Furthermore $N(Q)^{I(Q)}=M_{12}$ and $N(Q)^{I(Q)}/C(Q)^{I(Q)}$ is involved in the automorphism group of Q. Hence $N(Q)^{I(Q)}=C(Q)^{I(Q)}$.

Suppose that Q is a cyclic group of order four. Then since $N(Q)^{I(Q)}=C(Q)^{I(Q)}$ and Q is abelian, any 2-element of $N(Q)$ is contained in $C(Q)$. Thus $Z(\langle Q, y_1, y_2\rangle)\geq Q$. On the other hand $\langle Q, y_1, y_2\rangle^\Gamma$ is a quaternion group. Hence Q has an element b of order four and $b^\Gamma\notin Z(\langle Q, y_1, y_2\rangle^\Gamma)$, which is a contradiction. Thus the order of Q is not four.

Since $\langle Q, y_1, y_2\rangle^\Gamma$ is a quaternion group and $\langle Q, y_1, y_2\rangle$ is of order at least $8\cdot 2$, $\langle Q, y_1, y_2\rangle_\Gamma$ has an involution, which is contained in Qx_1. Hence we may assume that $x_1\in\langle Q, y_1, y_2\rangle_\Gamma$. Then $x_1\in Z(\langle Q, y_1, y_2\rangle)$ and $|I(x_1)|=12$. Let x be any involution of $\langle Q, y_1, y_2\rangle$ other than a and x_1. Since Q has exactly one involution a, $x\notin Q$. Hence $x\in Qx_1$. Thus $x^{I(Q)}=x_1^{I(Q)}$ and so xx_1 is an involution of Q. Hence $xx_1=a$ and so $x=ax_1$. Thus $\langle Q, y_1, y_2\rangle$ has exactly three involution a, x_1, and ax_1, which are contained in $Z(\langle Q, y_1, y_2\rangle)$.

Assume that $\langle Q, y_1, y_2\rangle$ is a Sylow 2-subgroup of G_{1234}. For any four points, i, j, k, l of $I(x_1)$ let P' be a Sylow 2-subgroup of G_{ijkl} containing x_1. Since G is 4-fold transitive, P' is conjugate to $\langle Q, y_1, y_2\rangle$. Since any involution of $\langle Q, y_1, y_2\rangle$ is contained in the center of $\langle Q, y_1, y_2\rangle$, x_1 is contained in the center of P'. Thus $P'^{I(x_1)}\leq C(x_1)^{I(x_1)}$ and $P'^{I(x_1)}$ fixes exactly four points $i, j,$

k, l. Then by a lemma of Livingstone and Wagner [4] $C(x_1)^{I(x_1)}$ is 4-fold transitive. Since $|I(x_1)|=12$, $C(x_1)^{I(x_1)}=M_{12}$ by Theorem.

Assume that $\langle Q, y_1, y_2\rangle$ is not a Sylow 2-subgroup of G_{1234}. Then $N(\langle Q, y_1, y_2\rangle)_{1234}$ has a 2-element x' such that $x'\notin\langle Q, y_1, y_2\rangle$. If x' fixes $I(Q)$, then $x'^{I(Q)}\in\langle y_1, y_2\rangle^{I(Q)}$ since $N(G_{I(Q)})^{I(Q)}=M_{12}$. Hence there is an element x'' in $\langle Q, y_1, y_2\rangle$ such that $x'^{I(Q)}=x''^{I(Q)}$. Thus $x'x''^{-1}\in Q$ and so $x'\in\langle Q, y_1, y_2\rangle$, which is a contradiction. Thus x' does not fix $I(Q)$. Then $a^{x'}\neq a$. Hence $a^{x'}=x_1$ or ax_1. Since $C(a)^{I(a)}=M_{12}$, $C(x_1)^{I(x_1)}$ or $C(ax_1)^{I(ax_1)}=M_{12}$. Thus Qx_1 has an element x_1', where $x_1'=x_1$ or ax_1, such that $|I(x_1')|=12$ and $C(x_1')^{I(x_1')}=M_{12}$.

Since $N(Q)^{I(Q)}=M_{12}$, we may assume that $N(Q)$ has a 2-element

$$x_2=(1)\,(2)\,(3\ 4)\,(5)\,(6)\,(7\ 8)\,(9\ 12)\,(10\ 11)\cdots$$

and $\langle Q, y_1, y_2, x_2\rangle$ is a 2-group. Then $\langle Q, x_2\rangle$ is conjugate to $\langle Q, x_1\rangle$. Hence we may assume that $|I(x_2)|=12$, $x_2\in C(Q)$, $|I(x_2')|=12$ and $C(x_2')^{I(x_2')}=M_{12}$, where $x_2'=x_2$ or ax_2.

Since $x_2\in N(\langle Q, y_1, y_2\rangle)$, $x_1^{x_2}=x_1$ or a_1x. Suppose that $x_1^{x_2}=ax_1$. If Q is of order two, then $\langle Q, x_1\rangle$ is an elementary abelian group of order four. On the other hand $\langle Q, x_1x_2\rangle$ is conjugate to $\langle Q, x_1\rangle$ and x_1x_2 is of order four, which is a contradiction. Thus Q is a quaternion group. Set $\Gamma'=I(ax_1)\cap\Delta$. Then $(I(x_1)\cap\Delta)^{x_2}=I(ax_1)\cap\Delta$. Hence $|\Gamma'|=8$ and $\langle Q, y_1, y_2\rangle^{\Gamma'}$ is a quaternion group. Since $|\langle Q, y_1, y_2\rangle_\Gamma|=8$, Qy_1 has an element y_1' fixing Γ pointwise. Then $y_1'\in C(Q)$. Since $Q^{\Gamma'}$ is a quaternion group, $y_1'^{\Gamma'}$ is the identity or an involution. Hence $y_1'^2$ is not the identity and fixes $\{1, 2, 3, 4\}\cup\Gamma\cup\Gamma'$ pointwise. This is a contradiction since $|\{1, 2, 3, 4\}\cup\Gamma\cup\Gamma'|=20$. Thus $x_1^{x_2}=x_1$.

Then x_1' and x_2' commute. Since $C(x_1')^{I(x_1')}=M_{12}$, $I(x_2')\cap I(x_1')=\{1, 2, i, j\}$, where $\{i, j\}\subset\Delta$. Thus $\langle x_1', x_2'\rangle$ fixes exactly two points i, j of Δ. Then since $\langle x_1', x_2'\rangle\leq C(Q)$, Q is of order two.

3.3. *Finally we show that* $|Q|\neq 2$ *and complete the proof.*

Proof. By (3.2) $|Q|=2$, and so $Q=\langle a\rangle$ and $\langle a, x_1\rangle$ is an elementary abelian group of order four. Furthermore we may assume that $C(x_1)^{I(x_1)}=M_{12}$ and $I(x_1)=\{1, 2, 3, 4, 13, 14, \cdots, 20\}$. Since $N(Q)^{I(Q)}=C(a)^{I(a)}=M_{12}$ and $C(a)^{I(a)}>\langle y_1, y_2\rangle$, $C(a)$ has 2-elements

$$x_2=(1)\,(2)\,(3\ 4)\,(5)\,(6)\,(7\ 8)\,(9\ 11)\,(10\ 12)\cdots,$$
$$x_3=(1\ 2)\,(3\ 4)\,(5)\,(6)\,(7)\,(8)\,(9\ 10)\,(11\ 12)\cdots.$$

Then we may assume that $\langle a, y_1, y_2, x_2, x_3\rangle$ is a 2-group (see (2.3)). Since $\langle a, x_i\rangle$ is conjuagte to $\langle a, x_1\rangle$ in $C(a)$, $i=2, 3$, we may assume that $|I(x_i)|=12$ and $C(x_i)^{I(x_i)}=M_{12}$. Furthermore since $\langle a, x_ix_j\rangle$, $i\neq j$ and $1\leq i, j\leq 3$, is conjugate to $\langle a, x_1\rangle$ x_ix_j is of order two. Thus x_i and x_j commute and so $\langle a, x_1, x_2, x_3\rangle$ is elementary ableian.

Since $a^{I(x_1)}=(1)(2)(3)(4)(13\ 14)(15\ 16)(17\ 18)(19\ 20)$ and $C(x_1)^{I(x_1)}=M_{12}$, we may assume that $x_2^{I(x_1)}=(1)(2)(3\ 4)(13)(14)(15\ 16)(17\ 19)(18\ 20)$ and $x_3^{I(x_1)}=(1\ 2)(3\ 4)(13)(14)(15)(16)(17\ 18)(19\ 20)$. Since $|I(x_2)|=12$, we may assume that $I(x_2)=\{1, 2, 5, 6, 13, 14, 21, 22, \cdots, 26\}$. Then since $a^{I(x_2)}=(1)(2)(5)(6)(13\ 14)(21\ 22)(23\ 24)(25\ 26)$ and $C(x_2)^{I(x_2)}=M_{12}$, we may assume that $x_1^{I(x_2)}=(1)(2)(5\ 6)(13)(14)(21\ 22)(23\ 25)(24\ 26)$ and $x_3^{I(x_2)}=(1\ 2)(5)(6)(13)(14)(21\ 22)(23\ 26)(25\ 24)$. Since $|I(x_3)|=12$, we may assume that $I(x_3)=\{5, 6, 7, 8, 13, 14, 15, 16, 27, 28, 29, 30\}$. Then since $a^{I(x_3)}=(5)(6)(7)(8)(13\ 14)(15\ 16)(27\ 28)(29\ 30)$ and $C(x_3)^{I(x_3)}=M_{12}$, we may assume that $x_2^{I(x_3)}=(5)(6)(7\ 8)(13)(14)(15\ 16)(27\ 29)(28\ 30)$ and $x_1^{I(x_3)}=(5\ 6)(7\ 8)(13)(14)(15)(16)(27\ 28)(29\ 30)$. Then ax_1x_3 is of order two and $I(ax_1x_3)$ contains $\{9, 10, 11, 12, 17, 18, 19, 20, 23, 24, \cdots, 30\}$ of length sixteen, which is a contradiction. Thus we complete the proof of the lemma.

4. Proof of Corollary 1

In this section we assume that G is a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$ and n is even. Let P be a Sylow 2-subgroup of a stabilizer of four points in G. Then $|I(P)|=4$ by Corollary of [13].

Proof of (1) of Corollary 1. We proceed by way of contradiction. We assume that G is a counter-example to (1) of Corollary 1 of the least possible degree. Then $n\geq 35$ ([2],p.80). Set $I(P)=\{1, 2, 3, 4\}$. Let t be the maximal number of fixed points of involutions of G and Q be a Sylow 2-subgroup of $G_{I(Q)}$ such that $|I(Q)|=t$. For any four points i, j, k, l of $I(Q)$ let P' be a Sylow 2-subgroup of G_{ijkl} containing Q. Since G is 4-fold transitive, P' is conjugate to P. Hence by the assumption $I(P')=I(Z(P'))=\{i, j, k, l\}$. Thus $C(Q)^{I(Q)}\geq Z(P')^{I(Q)}$ and $I(Z(P')^{I(Q)})=\{i, j, k, l\}$. Hence by a lemma of Livingstone and Wagner [4], $C(Q)^{I(Q)}$ is 4-fold transitive on $I(Q)$. If $(C(Q)^{I(Q)})_{ijkl}$ is of odd order, then $|I(Q)|=4$. Hence by a theorem of H. Nagao [10] $G=S_6$, A_8 or M_{12}, which is a contradiction since $n\geq 35$. Hence $(C(Q)^{I(Q)})_{ijkl}$ is of even order. Then $C(Q)^{I(Q)}$ satisfies the assumption of (1) of Corollary 1. Hence by the minimal nature of the degree of G, $C(Q)^{I(Q)}=S_t$, A_t or M_{12}. By Lemma $C(Q)^{I(Q)}\neq M_{12}$. If $C(Q)^{I(Q)}=S_t$ or A_t, then by Theorem $G\geq A_n$, which is a contradiction. Thus we complete the proof.

Proof of (2) of Corollary 1. If $P_i=1$, then by a theorem of H. Nagao [10] $G=S_6$, A_8 or M_{12}. Suppose that there is a point i of $\Omega-I(P)$ such that $P_i\neq 1$. Let t be the maximal number of fixed points of involutions of G. Since P_i is semiregular $(\neq 1)$, we may assume that $|I(P_i)|=t$. For any four points i_1, i_2, i_3, i_4 of $I(P_i)$ let P' be a Sylow 2-subgroup of $G_{i_1 i_2 i_3 i_4}$ containing P_i. Then $N_{P'}(P_i)^{I(P_i)}$ is semiregular $(\neq 1)$ and fixes exactly four points i_1, i_2, i_3, i_4. Hence by a lemma of Livingstone and Wagner [4] $N(P_i)^{I(P_i)}$ is 4-fold transitive on $I(P_i)$

and by a theorem of H. Nagao [10] $N(P_i)^{I(P_i)}=S_6$, A_8 or M_{12}. Hence by Theorem and Lemma, $G=S_8$ or A_{10}. Thus we complete the proof.

5. Proof of Corollary 2

In this section we assume that G is a permutation group as in Corollary 2. We may assume that P is a Sylow 2-subgroup of G_{1234}. Then by a corollary of [13] $|I(P)|=4$, 5 or 7.

Suppose that $|I(P)|=4$. Then n is even. Furthermore since P is transitive on $\Omega-I(P)$, $I(P)=I(Z(P))$. Hence by Corollary 1, $G=S_{2^k+4}$ $(k\geq 1)$, A_{2^k+4} $(k\geq 2)$ or M_{12}.

Next suppose that $|I(P)|=5$. Since P is transitive on $\Omega-I(P)$, by a theorem of H. Nagao [9] G_{1234} is doubly transitive on $\Omega-\{1, 2, 3, 4\}$. Then G_1 satisfies the assumption of Corollary 2 and $|I(P)-\{1\}|=4$. Hence by what we have proved above, G_1 is one of the groups listed above. Hence $G=S_{2^k+5}$ $(k\geq 1)$ or A_{2^k+5} $(k\geq 2)$.

Finally suppose that $|I(P)|=7$. Then by a theorem of [12] $G=M_{23}$. Thus we complete the proof.

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