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ON MULTIPLY TRANSITIVE PERMUTATION GROUPS III

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Introduction

The purpose of the present note is to prove the following theorem.

Theorem 1. *Let p be an odd prime $\geqslant 11$. Then there exists no permutation group G on a set $\Omega=\{1, 2, \cdots, n\}$ which satisfies the following two conditions:*
(i) *G is $(p+1)$-ply transitive, and $n \equiv p \pmod{p^2}$, and*
(ii) *the order of $G_{1,2,\cdots,p+1}$, the stabilizer of $p+1$ points of Ω in G, is not divisible by p.*

This Theorem 1 is a kind of (but not full even in the case of $p \geqslant 11$) generalization of the result (Theorem 1) in [1]. In the case of $p \geqslant 11$, the Main Theorem in [2] (i.e., the determination of $2p$-ply transitive permutation groups whose stabilizer of $2p$ points is of order prime to p) is also completed alternatively by combining this Theorem 1 with the result of Miyamoto [6].

The brief outline of the proof of Theorem 1 is as follows. The proof will be done by the way of contradiction. First we will show that the symmetric group S_p is not involved in the group $GL(p-3, p)$, if $p \geqslant 11$ (Theorem A). This is proved by the similar argument as in [1, §1], by exploiting the (ordinary, modular and projective) representation theories of the symmetric groups. Next, we will restrict the structure of the Sylow p subgroup P_0 of $G_{1,2,\cdots,p}$. That is, if $|P_0| > p^{p+1}$ then we have $|Z(P_0)| > p$, and moreover we can lead a contradiction by using the well known theorem of Burnside on fusion of elements in the center of a sylow p subgroup and by using a consequence (Theorem B) of Theorem A. If $|P_0| \leqslant p^{p+1}$, then we can show (also by using Theorem A) that we have only one possibility for the structure of P_0, namely, P_0 is isomorphic to the extraspecial p group of order p^{p+1} and of exponent p. Finally, we exclude this remaining case and complete the proof of Theorem 1. This is done by considering the fusion of p elements in P_0. The proof of this final step was provided by T. Yoshida.

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the final step of proof of Theorem 1, and for giving the permission to mention his proof here.

Our notation follows that of [1].

1. S_p is not involved in $GL(p-3, p)$ when $p\geqslant 11$, and a consequence of it

The purpose of this section is to prove the following Theorem A and a corollary of it (Theorem B).

Theorem A. *Let p be an odd prime $\geqslant 11$. Then S_p is not involved in $GL(p-3, p)$.*

The proof of Theorem A will be done by quite the same argument as in the proof of Theorem A in [1]. Theorem A will be proved through the following lemmas.

Lemma 1. *Let $p\geqslant 7$. Then S_p is not a subgroup of $GL(p-3, K)$, where K is an algebraically closed field of characteristic p.*

Lemma 1′. *Let $p\geqslant 7$. Then A_p is not a subgroup of $GL(p-3, K)$, where K is an algbraically closed field of characteristic p.*

These assertions are verified as in Lemma 2 in [1], by using a result of Nakayama [8] on modular representations of symmetric groups. That is,

(1) The degree of any p-modular irreducible representation of S_p (or A_p) over K which is contained in a p-block of defect 0 is more than $p-3$.

Because, the degree is divisible by p (and of course $\geqslant p$), and so it is obviously more than $p-3$.

(2) The degree of any not 1 dimensional p-modular irreducible representation of S_p (or A_p) over K which is contained in a p-block of defect 1 is more than $p-3$.

S_p contains just one (hence principal) p-block B_0 of defect 1 (cf. [8]). Moreover, B_0 consists of p ordinary irreducible representations $T_{0,r}(0\leqslant r\leqslant p-1)$, where $T_{0,r}$ is the representation associated with the Young diagram of type $[1^{p-r}, 1^r]$. Moreover, $T_{0,r}$ and $T_{0,r+1}$ have just one (p-modular) irreducible representation (say, let us denote it by $\phi_{0,r}(0\leqslant r\leqslant p-2)$) of S_p over K in common, and $T_{0,r}$ and $T_{0,s}$ with $s>r+1$ has no p-modular irreducible representation in common (see Nakayama [8]). Thus, the Brauer graph associated with the p-block B_0 is a tree without branches and the nodes are arranged in natural order on r. Thus, we can explicitly calculate the values of $|\phi_{0,r}|$, the degree of $\phi_{0,r}$. Namely, we obtain that $|\phi_{0,r}|={}_{p-2}C_r$, where ${}_{p-2}C_r$ denotes the number of r elements subsets among $p-2$ elements. Since $\phi_{0,r}$ $(0\leqslant r\leqslant p-2)$ are the only p-modular irreducible representations (over K) which are contained in a p block of defect 1, we obtain the assertion for the case of S_p. While, we can see that A_p contains just one (hence principal) p-block B_0 of defect 1, and that B_0 contains $(p+1)/2$ ordinary irreducible representations: $T_{0,r}$ (with $0\leqslant r\leqslant(p-3)/2$ which is the restriction of $T_{0,r}$ to A_p) and p-conjugate two representations (which are obtained by restricting $T_{0,(p-1)/2}$ to A_p), and that B_0 contains $(p-1)/2$ (p-modular) irreducible representations of A_p over K whose degrees are $|\phi_{0,r}|$ $(0\leqslant r\leqslant(p-3)/2)$. Thus, we also obtain the assertion for the case of A_p. Thus, we have completed the proof of Lemma 1 and Lemma 1′.

Lemma 2. *Let $p\geqslant 11$. Then S_p is not a subgroup of $PGL(p-3, K)$, where K is an algebraically closed field of characteristic p.*

Lemma 2′. *Let $p\geqslant 11$. Then A_p is not a subgroup of $PGL(p-3, K)$, where K is an algebraically closed field of characteristic p.*

These assertions are verified similarly as in Lemma 3 in [1] by using the theory of projective representations of the symmetric groups due to Schur [9]. These two lemmas are proved through the following steps (1), (2) and (3). Now let T_p be a representation group of S_p over K, and T_p' the commutator subgroup of index 2 of T_p (see Schur [9] and cf. [1]). Notice that if $p\geqslant 11$, then T_p' is a representation group of A_p over K (cf. Schur [9], and Yamazaki [10]).

(1) The degree of any (p-modular) irreducible representation of A_p over K which is contained in a p-block of defect 0 is more than $p-3$.

Because, they are divisible by p.

(2) Let us assume that $p\geqslant 11$. Then the degree of any ordinary irreducible representation of T_p' of the second kind which is not divisible by p is divisible by $2^{[(p-2)/2]}$. Moreover $2^{[(p-2)/2]}>p-3$ (since $p\geqslant 11$).

By Schur [9], the degree of any ordinary irreducible representation of T_p' of the second kind is one of the following numbers $f_{\nu_1,\nu_2,\dots,\nu_m}$ and $\frac{1}{2}\bar{f}_{\nu_1,\nu_2,\dots,\nu_m}$, where

$$f_{\nu_1,\nu_2,\dots,\nu_m}=2^{[(p-m)/2]}\cdot g_{\nu_1,\nu_2,\dots,\nu_m}\ \text{(when } p-m \text{ is odd)}$$

and

$$\bar{f}_{\nu_1,\nu_2,\dots,\nu_m}=2^{[(p-m)/2]}\cdot g_{\nu_1,\nu_2,\dots,\nu_m}\ \text{(when } p-m \text{ is even)}$$

with

$$g_{\nu_1,\nu_2,\dots,\nu_m}=\frac{p!}{\nu_1!\nu_2!\cdots\nu_m!}\prod_{\alpha>\beta}\frac{\nu_\alpha-\nu_\beta}{\nu_\alpha+\nu_\beta},$$

where $\nu_1+\nu_2+\cdots+\nu_m=p$ and $\nu_1>\nu_2>\cdots>\nu_m>0$. In order that $f_{\nu_1,\nu_2,\dots,\nu_m}$ or $\frac{1}{2}\bar{f}_{\nu_1,\nu_2,\dots,\nu_m}$ is not divisible by p, we obtain that $m\leqslant 2$. Thus, we obtain the first assertion immediately, because $g_{\nu_1\nu_2,\dots,\nu_m}$ is always an integer (see [9]).

(3) Completion of the proof of Lemma 2 and 2′.

Now, for $p\geqslant 11$ the proof of Lemma 2 and Lemma 2′ is completed by quite the same argument as in the proof of step (3) of Lemma 3 in [1]. That is, as in [1] (cf. Dornhoff, Group Representation Theory Part B, §68, Dekker, 1972), we obtain that if ϕ is a p-modular irreducible representation of T_p (or T_p') over K which is contained in a p-block of defect 1, then either ϕ is contained in the principal p-block B_0 (and ϕ is of the first kind and of degree more than $p-3$ by Lemma 1), or ϕ is contained in a block such that ordinary irreducible representations contained in it are all of the second kind (and $|\phi|$ is divisible by $2^{[(p-2)/2]}$ by step (2), because the Brauer graphs are tree).

Thus, we have completed the proof of Lemma 2 and Lemma 2′.

REMARK. A_7 and S_7 are actually subgroups of $PGL(4, 7)$ (cf. H. Mitchell, The subgroups of the quaternary abelian linear group, Trans. A.M.S. 15(1914), 379–395). Also, A_5 and S_5 are clearly subgroups of $PGL(2, 5)$. The fact that our proof of Theorem 1 is valid only for $p \geqslant 11$ is owing to this fact.

Lemma 3. *Let $p \geqslant 11$. Then A_p is not involved in a finite subgroup of $GL(p-3, K)$, where K is an algebraically closed field of characteristic p.*

The proof of Lemma 3 is quite the same as that of Lemma 4 in [1]. We proceed as in Lemma 4 in [1]. Let l be the smallest integer no more than $p-3$ such that A_p is involved in a finite subgroup X of $GL(l, K)$. Moreover, let us take X to be of the least order among them. Then as in Lemma 4 in [1], we obtain the following assertions (1), (2) and (3).

(1) X is primitive (as a subgroup of $GL(l, K)$), and contains a normal q subgroup Q which is not contained in $Z(GL(l, K))$.

(2) $p \neq q$, and Q does not contain any characteristic abelian subgroup of rank $\geqslant 2$. Moreover Q is the central product of Q_1 and Q_2, where Q_1 is either 1 or extraspecial q group (say of order q_1^{2r+1}), and Q_2 is either cyclic (of order $\neq q$) or $q=2$ and one of dihedral, generalized quaternion and semidihedral of order $\geqslant 2^4$.

(3) A_p is not involved in $C_{GL(l,K)}(Q)$.

Now, by combining a result of Jordan [5, Chap., IV page 56 (3)] (cf. also [1]) with a result of Cebysev about the distribution of primes (which asserts that there is a prime between $\frac{1}{2}n$ and n for any positive integer $n \geqslant 3$), we obtain the following assertion (4).

(4) A_p is not involved in $GL((p-3)/2, q)$, where q is an arbitrary prime different from p. Moreover, if $p=11$ and $p=13$, then A_p is not involved in $GL(7, q)$ (when $p=13$, or $p=11$ and $q=2$) and not involved in $GL(5, q)$ (when $p=11$ and $q \neq 2$).

(5) A_p is not involved in $\mathrm{Aut}(Q)$.

For the proof of (5), cf. step (6) of Lemma 4 in [1]. Namely we may assume that $Q_1 \neq 1$. Since any faithful absolutely irreducible representation of Q over K is of degree q^r (or q^{r+1}), we obtain that $p-3 \geqslant l \geqslant q^r$ (or q^{r+1}). Now, we obtain that $(p-3)/2 \geqslant 2r$ when $p \geqslant 17$. Moreover, $7 \geqslant 2r$ (when $p=13$, or $p=11$ and $q=2$), and $5 \geqslant 2r$ (when $p=11$ and $q \neq 2$). While, from the structure of Q, we obtain (as step (6) of Lemma 4 in [1]) that A_p must be involved in $GL(2r, q)$. But, this is a contradiction.

Thus, we have completed the proof of Lemma 3.

Thus, Theorem A was completely proved.

Next, by using the similar idea which proved Theorem A, we also obtain the following theorems, which will be essentially used later in the proof of Theorem 1.

Theorem B. *Let p be an odd prime $\geqslant 11$. Let V be the vector space of dimension l over the field of p elements. Let $GL(l, p)$ act on V as the automorphism group. Then there exists no pair of subgroups $\bar{L}$ and $\bar{N}$ of $GL(l, p)$ which satisfies the following four conditions, where l is one of p, $p-1$ and $p-2$.*

(i) *$\bar{N}$ is a normal subgroup of $\bar{L}$,*
(ii) *$\bar{L}/\bar{N}$ is isomorphic to the symmetric group S_p,*
(iii) *$\bar{N}$ is of order prime to p, and*
(iv) *$\bar{L}$ is transitive on V^*, the set of the non zero elements of V.*

Proof. Proof will be done by way of contradiction. We use the following notation: $\bar{\bar{L}}$ and $\bar{\bar{N}}$ denote respectively the homomorphic images of the groups $\bar{L}$ and $\bar{N}$ regarded as the permutation groups on $\bar{V}$, the set of the points of the associated projective space of V.

(1) First let us assume that $l=p$. Let q be a prime different from p, and let $\bar{Q}$ be a (nontrivial) Sylow q subgroup of $\bar{N}$. First we will show that q divides p^2-1. (Next we will show that $\bar{N}=1$.) Let us assume that q does not divide p^2-1. Then, for any nonidentity element x in $\bar{Q}$, $C_{GL(p, p)}(x)$ does not involve A_p for $p \geqslant 11$, according to Theorem A that A_p is not involved in $GL(p-3, K)$ where K is an algebraically closed field of characteristic p. (Notice that $C_{GL(p,p)}(x)$ is of the form $\prod_{i=1}^{k} GL(l_i, p^{a_i})$ with $\sum_{i=1}^{k} l_i a_i = p$ and that $a_i \geqslant 3$ for some i and so any l_i is $\leqslant p-3$.) By the theorem of Sylow (Frattini argument), S_p must be involved in $N_{GL(p, p)}(\bar{Q})$. Let $\bar{Q}_0$ be a characteristic subgroup of $\bar{Q}$. Then, S_p must be also involved in $N_{GL(p, p)}(\bar{Q})$. Now, we may assume without loss of generality that $\bar{Q}_0$ is a minimal characteristic subgroup of $\bar{Q}$. Then $\bar{Q}_0$ is an elementary abelian subgroup of order, say, q^r. Since $q^r \leqslant p$, we obtain that $r \leqslant (p-1)/2$ (because of $p \geqslant 11$). But A_p is not involved in $GL\left(\frac{p-1}{2}, q\right)$, as is immediately verified from the Lemma of Jordan (cf. [1]). Therefore, we have proved that A_p is not involved in $\mathrm{Aut}(\bar{Q}_0)$ and in $C_{GL(p, p)}(\bar{Q}_0)$. Thus S_p is not involved in $N_{GL(p, p)}(\bar{Q}_0)$, and this is a contradiction. Thus we have proved that q divides p^2-1. By the assumption (iv), $\bar{L}$ is transitive on V^*, and hence $\bar{\bar{L}}$ is transitive on $\bar{V}$, the points of the associated projective space of V. Therefore,

a normal subgroup $\bar{\bar{N}}$ of $\bar{\bar{L}}$ must be half-transitive on $\bar{V}(|\bar{V}|=(p^p-1)/(p-1))$. But, the $G.C.D.$ of p^2-1 and $(p^p-1)/(p-1)$ is 1. Therefore, $\bar{\bar{N}}$ must be 1. Since $|\bar{V}|>|S_p|=|\bar{\bar{L}}|$, this is a contradiction.

(2) Secondary let us assume that $l=p-1$. Then the same argument as in the proof of step (1) shows that if q is a prime which divides the order of $\bar{N}$, then q must divide $p-1$. Moreover, since the $G.C.D.$ of $p-1$ and $(p^{p-1}-1)/(p-1)$ is 2, $\bar{\bar{N}}$ must be divisible by 2. (Otherwise, $\bar{\bar{N}}=1$ and the same argument as above shows that $|S_p|<|\bar{V}|=(p^{p-1}-1)/(p-1)$, and this is a contradiction.) Let $\bar{Q}$ be a Sylow 2 subgroup of $\bar{N}$. Then, from the fact that $\bar{\bar{Q}}$ $(\neq 1)$ is half-transitive on $\bar{V}$, we obtain that $C_{GL(p-1,p)}(\bar{Q})$ does not involve A_p. Now first let us assume that $\bar{Q}$ is an irreducible subgroup of $GL(p-1,p)$. Then, if $\bar{Q}$ contains an abelian characteristic subgroup $\bar{Q}_0$ of rank $\geqslant 2$, then we easily obtain that S_p is not involved in $N_{GL(p-1,p)}(\bar{Q}_0)$, and this is a contradiction. Therefore, we obtain that $\bar{Q}$ must be a central product of $\bar{Q}_1$ (=an extraspecial 2 group of order 2^{2r+1}, $r\geqslant 0$) and $\bar{Q}_2$ (= either cyclic, or one of dihedral, generalized quaternion and semidihedral of order $\geqslant 2^4$). Therefore, we obtain that for $p\geqslant 11$ S_p is not involved in $\mathrm{Aut}(\bar{Q})$ and hence not involved in $N_{GL(p-1,p)}(\bar{Q})$, and this is a contradiction. Next, let us assume that $\bar{Q}$ is not an irreducible subgroup of $GL(p-1,p)$. In this case we may assume that the irreducible components of the subgroup $\bar{Q}$ in $GL(p-1,p)$ are two and of degree 1 and $p-2$ respectively.) In other cases we easily have the contradiction.) Therefore, we have the situation that S_p is involved in some $N_{GL(p-2,p)}(R)$, where R is an irreducible 2 subgroup of $GL(p-2,p)$. But, the same argument as before show that this is also impossible.

(3) Finally let us assume that $l=p-2$. Then the same argument as in the proof of steps (1) and (2) shows that must $\bar{N}$ be contained in $Z(GL(p-2,p))$. Therefore, since $|\bar{L}|\leqslant(p-1)\cdot p!<p^{p-2}-1=|V^*|$ for $p\geqslant 11$, we have a contradiction to the assumption (iv).

Thus we have completed the proof of Theorem B.

2. Proof of Theorem 1.

Now, we will start the proof of Theorem 1. Let p be an odd prime $\geqslant 11$, and let G satisfy the assumptions of Theorem 1. Then, we will derive a contradiction.

Let a be an element of G of order p such that

$$a=(1,2,\cdots,p)\cdots$$

and $|I(a)|=p$. Then there exists a Sylow p subgroup P_0 of $G_{1,2,\cdots,p}$ which is fixed by the element a. From the assumption, P_0 is of order $\geqslant p^2$. Let $P=\langle a, P_0\rangle$ be the group generated by the element a and the subgroup P_0. Then P is a Sylow p subgroup of G.

Now, we have the following fundamental Proposition.

Proposition 1. $|C_{P_0}(a)|=p$. *Consequently, P is a p group of maximal class (in the sense of Blackburn).*

Proof. We obtain that $|C_{P_0}(a)|=p$, from the semiregularity of P_0 on $\Omega-I(P_0)$. Therefore, P is a p group of maximal class. (cf. [4, Kapital III, Satz 14.23.])

We will divide the proof of Theorem 1 into the following two cases.

Case 1 $|P|>p^{p+1}$,
Case 2 $|P|\leqslant p^{p+1}$.

2.1. Proof of Theorem 1 for Case 1

We first prove the following proposition.

Proposition 2. *Let the Case* 1 *hold. Then we obtain the following assertions.*
(i) *P is nonexceptional (in the sense of* [4, *Kapital* III], *i.e., P is of degree of commutativity >0 in terms of Blackburn). Therefore, we obtain that $[\gamma_i(P), \gamma_j(P)] \leqslant \gamma_{i+j+1}(P)$ for all i and j such that $i+j>2$. (Here, $\gamma_2(P)=[P, P]$ and $\gamma_i(P) =[\gamma_{i-1}(P), P]$ for $i\geqslant 2$. For the definition of $\gamma_1(P)$, see* [4, *Kapital* III, *Definition* 14.3]).
(ii) $P_0=\gamma_1(P)$.
(iii) $|\Omega_1(Z(P_0))|\geqslant p^2$.

Proof. (i) This assertion is due to Blackburn. (cf. Huppert [4, Kapital III, Hauptsatz 14.6 and Hauptsatz 14.7].)
(ii) If $P_0\neq\gamma_1(P)$, then we obtain that P_0 is also of maximal class by [4, Kapital III, Satz 14.22]. But then, since S_p is not involved in the automorphism group of a p group of maximal class (since $p\geqslant 7$), we obtain that $C_G(P_0)^{\Omega-I(P_0)}\geqslant A^{\Omega-I(P_0)}$, because there exists the series of characteristic subgroups $\gamma_i(P_0)$, $i=1, 2, \cdots$. Thus, we obtain that $|Z(P)|\geqslant p^2$, and this contradicts Proposition 1 that P is of maximal class.
(iii) Since $P_0=\gamma_1(P)$ is a regular p group and since $\Omega_1(P_0)=\gamma_{m-p+1}(P)\geqslant\gamma_{m-2}(P)$ (cf. [4, Kapital III, Satz 14.16]), $\gamma_{m-2}(P)$ is an elementary abelian p group of order p^2, and moreover $\gamma_{m-2}(P)$ is contained in $Z(P_0)$, because of the assertion (i). Thus we have proved (iii), and completed the proof of Proposition 2.

Now, let us set $L=N_G(P_0)$ and $N=N_{G_{1,2,\cdots,p}}(P_0)$. Then, we obtain that N is a normal subgroup of L and that L/N is isomorphic to S_p. Now, L and N act naturally on $\Omega_1(Z(P_0))$. Let $\bar{L}$ and $\bar{N}$ be the homomorphic images of L and N in $\mathrm{Aut}(\Omega_1(Z(P_0))\cong GL(l, p)(l\geqslant 2)$. Since there exists the element a (defined at the beginning of this section) in L such that a acts nontrivially on $\Omega_1(Z(P_0))$, we obtain the following proposition.

Proposition 3. *Let the Case* 1 *hold. Then the pair of subgroups* $\bar{L}$ *and* $\bar{N}$ *in* $GL(l, p) \simeq Aut(\Omega_1(Z(P))$ *satisfies the four conditions given in Theorem B, namely,*
(i) $\bar{N}$ *is a normal subgroup of* $\bar{L}$,
(ii) *the order of is* $\bar{N}$ *not divisible by* p,
(iii) $\bar{L}/\bar{N}$ *is isomorphic to* S_p, *and*
(iv) $\bar{N}$ *is transitive on* $(\Omega_1(Z(P_0)))^*$.
Moreover, l *is one of* p, $p-1$ *and* $p-2$.

Proof. The assertions of (i) and (ii) are immediate. The assertion (iii) is due to the fact that the element a acts nontrivially on $\Omega_1(Z(P_0))$. This also asserts that $l \geqslant p-2$, because of Theorem A. Moreover, we obtain, by Lemma of Ito in Nagao [7], that $l \leqslant p$, because of $|C_{\Omega_1(Z(P_0))}(a)|=p$. The assertion (iv) is proved as follows. Since G_1 is p-ply transitive on $\Omega-\{1\}$, and since $G_{1,2,\cdots,p+1}$ is of order prime to p, any element of order p in G_1 (hence any element in $(\Omega_1(Z(P_0)))^*$) are conjugate in G_1, because of a result of Nagao (see [7, Lemma 1.1]). Since P_0 is a Sylow p subgroup of G_1, a result of Burnside (cf. Gorenstein [3, Theorem 7.1.1]) shows that any two elements of $(\Omega_1(Z(P_0)))^*$ are already conjugate in $N_{G_1}(P_0)$ (hence in $N_G(P_0)=L$). Therefore, $\bar{L}$ acts transitively on $(\Omega_1(Z(P_0)))^*$, and so we have the assertion of (iv).

Now, if Case 1 hold, then Proposition 3 contradicts Theorem B. Thus, we have shown the impossibility of Case 1.

2.2. Proof of Theorem 1 for Case 2.

In this subsection, we always assume that $p \geqslant 11$ and that $|P|=p^m \leqslant p^{p+1}$. We first show that we may assume that either P is of order $\leqslant p^5 (\leqslant p^{p-2})$, or $|Z(P_0)|=p$. Otherwise, $Z(P_0) \geqslant p^2$. And so by [4, Kapital III, Hilfsatz 14.14] $\gamma_2(P)$ is of exponent p, and so we obtain that $\Omega_1(Z(P_0))$ is of rank $\geqslant 2$. Thus, we have a contradiction, as we have already proved in Case 1. Therefore, in the following, we may assume that one of the following two cases hold.
(a) $|P| \leqslant p^5$ and $|\Omega_1(Z(P_0))|=p$,
(b) $|Z(P_0)|=p$. (In this case, P is an exceptional group, and so P is of order p^m with m being even and $m \leqslant p+1$.)

Proposition 4. *Let us assume that Case* 2 *hold. Then one of the following two cases holds.*
(i) P_0 *is a nonabelian extraspecial p-group of order* p^p *and of exponent* p.
(ii) P_0 *contains a series of characteristic subgroups* P_i *such that* $P_0 > P_1 > P_2 > \cdots > P_k = 1$ *and* P_i/P_{i+1} *are elementary abelian* p *groups of rank at most* $p-3$ $(i=0, 1, \cdots, k-1)$.

Proof. If the case (a) holds, then the second assertion (ii) clearly holds. Let us assume that the case (b) hold in the following. If $m < p+1$, then we clearly obtain the assertion (ii), because of $m \leqslant p-1$. Thus we may assume that $m=p+1$ in the following. The existence of the critical subgroup C in P_0 (cf. Gorenstein [3, Theorem 5.3.11]) proves the validity of our assertion. For, if $C=P_0$ then P_0 is extraspecial. If it is of exponent p, then we have the assertion (i). If it is not of exponent p, then we have the assertion (ii) by using the fact that $\gamma_2(P)$ $(=\Omega_1(P_0))$ is not abelian. In the following, we assume that $C\neq P_0$. Clearly we have $C\neq\gamma_{m-1}(P)$, and $C\neq\gamma_{m-2}(P)$, because otherwise $C_{P_0}(C)\not\leq Z(C)$ since $\gamma_{m/2}(P)$ is abelian. Therefore, in order that (ii) does not hold, we have $|P_0: C|=p$. While, $\Phi(P_0)$, the Frattini subgroup of P_0, is a characteristic subgroup of P_0 and we may assume that $\Phi(P_0)\neq\gamma_{m-1}(P)$, because otherwise P_0 becomes extraspecial and this case is already excluded. Since $|P_0: \Phi(P_0)|\geqslant p^2$, we have the chain of characteristic subgroups of P_0 such that $P_0>C>\Phi(P_0)>Z(P_0)>1$. Since these inclusions are proper and P is of order at most p^p, we have the contradiction. Thus, we have completed the proof of Proposition 4.

Proposition 5. *The case* (ii) *in Proposition* 4 *does not hold.*

Proof. By Theorem A, S_p is not involved in $GL(p-3, p)$. Therefore, S_p is not involved in $N_G(P_0)$, and so we obtain that $C_G(P_0)^{\Omega-I(P_0)}\geqslant A^{\Omega-I(P_0)}$. Therefore, we have $|Z(P)|\geqslant p^2$. But, this contradicts Proposition 1 that P is a p group of maximal class. Thus we have completed the proof of Proposition 5.

Proposition 6. (due to Yoshida [11].) *The case* (i) *in Proposition* 4 *does not hold.*

Proof. As we have remarked before, any elements of order p in P_0 are conjugate in G_1 (hence in G) by a result of Nagao. Let z be an element $(\neq 1)$ in $Z(P)(=\gamma_{p-1}(P_0))$.

(i) We first show that for any $x(\neq 1)$ in P_0 there exists an element g in G such that $x^g=z$ and $C_P(x)^g\leqslant P$. For, there exists an element g' in G such that $x^{g'}=z$ and $C_P(x)^{g'}=P^{g'}\cap C_G(x)^{g'}=P^{g'}\cap C(z)$ is a p subgroup of $C_G(z)$. Since P_G is a Sylow p subgroup of $C_G(z)$, there exists an element h in $C_G(z)$ such that $C_P(x)^{g'h}\leqslant P$. Thus, if we set $g=g'h$ then we obtain the required assertion.

(ii) Let us take x in $P_z-\gamma_{p-2}(P)$. Then we have $C_P(x)\leqslant P_0$ and $|P: C_P(x)|=p^2$. Moreover, since $C_P(x)$ is not abelian (because P_0 contains nonabelian subgroup of index p^2), $[C_P(x), C_P(x)]=Z(P)$. While, for any subgroup D of P which satisfies $|P: D|=p^2$ and $[D, D]=p$, we obtain $D\leqslant P_0$, and so $[D, D]=Z(P)$. Because otherwise, if we denote the natural homomorphism $P\rightarrow\bar{P}=P/Z(P)$ by the bar, then we have that for any $\bar{x}$ $(\neq 1)$ in $\bar{D}-\bar{P}_0$ $|\bar{D}: C_{\bar{D}}(\bar{x})|\leqslant p$, and since $\bar{P}$ is a p group of maximal class, we have $|C_{\bar{P}}(\bar{x})|=p^2$. Therefore, $p^2\geqslant|C_{\bar{D}}(\bar{x})|\geqslant p^{p-3}$, and this is a contradiction because $p\geqslant 7$. Thus, we have proved $D\leqslant P_0$. Since $C_P(x)$ satisfies the hypothesis of the subgroup D by step (i), we have $[C_P(x)^g, C_P(x)^g]=Z(P)$. But, this contradicts the fact that $x^g=z$. Thus, we have completed the proof of Proposition 6.

Thus, we have completed the proof of Theorem 1.

REMARK 1. In [11], T. Yoshida proved more stronger assertion than stated in Proposition 6. That is,

Theorem. (T. Yoshida) *Let $p\geqslant 7$. Let G be a finite group, and P a Sylow p subgroup of G. Let P be a p group of maximal class of order p^{p+1}, and assume that $P_0=\gamma_1(P)$ is a nonabelian extraspecial p group. Then the following assertions hold.*

(i) *If $x\in Z(P)$ and $y\in P_0-Z(P)$, then they are not conjugate in G.*

(ii) *If two elements $x, y\in Z_2(P)(=\gamma_{p-2}(P))$ are conjugate in G, then they are already conjugate in $N_G(P)$.*

(iii) *If two elements $x\in Z_2(P)$ and $y\in P_0-Z_2(P)$ are conjugate in G, then they are already conjugate in $N_G(P_0)$.*

(iv) *If any two elements in P_0 are conjugate in G, then they are already conjugate in $N_G(P_0)$.*

REMARK 2. Proof of Proposition 6 will also be stated as follows. By a result of Nagao, any elements of order p of P_0 must be conjugate in G_1. But this is impossible, because we can easily see that the assertion of Lemma 4.6. in Gorenstein and Harada: A characterization of Janko's two new simple groups, J. Fac. Sci. Univ. Tokyo 16 (1970), 331; also holds for any odd prime p.

REMARK 3. It would be interesting to obtain a similar classification theorem as Theorem 1 for $p=3$, 5 and 7.

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