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FINITE GROUPS WHICH ACT FREELY ON SPHERES

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We will study the problem: Let G be a finite group which acts freely (and topologically) on the sphere S^{2t-1}. Can G act freely and orthogonally on S^{2t-1}?

The result of T. Petrie [5] shows that the answer is no for t odd prime. As is easily seen, the answer is yes for $t=1$. The problem for $t=2$ is unsolved at present (see [2], [3], [4]). In this note it will be shown that the answer is yes for $t=4$, and also for $t=2^{\nu}$ ($\nu \geqq 3$) if G is solvable.

1. Preliminary theorems

By J. Milnor [3] we have

(1.1) *If G is a group which acts freely on S^n, then G satisfies the following properties*:

i) *Any element of order 2 in G belongs to the center of G.*

ii) *G has at most one element of order 2.*

The following (1.2) and (1.3) are shown in [1].

(1.2) *If G acts freely on S^n, the cohomology of G has period $n+1$.*

(1.3) *For a finite group G, the following two conditions are equivalent*:

i) *G has periodic cohomology.*

ii) *Every abelian subgroup of G is cyclic.*

A complete classification of finite groups satisfying the condition ii) of (1.3) is known by H. Zassenhaus [11] and M. Suzuki [6]. For future reference we reproduce it below after J. Wolf [10] and C.B. Thomas-C.T.C. Wall [8].

(1.4) *Let G be a finite group satisfying the condition* ii) *of* (1.3). *If G is solvable, it is one of the following groups*:

Type	Generators	Relations	Conditions	Order
I	A, B	$A^m=B^n=1$, $BAB^{-1}=A^r$	$m\geqq 1$, $n\geqq 1$, $(n(r-1), m)=1$, $r^n\equiv 1\ (m)$	mn
II	A, B, R	As in I; also $R^2=B^{n/2}$, $RAR^{-1}=A^l$, $RBR^{-1}=B^k$	As in I; also $l^2\equiv r^{k-1}\equiv 1\ (m)$, $n=2^u v$, $u\geqq 2$, $k\equiv -1\ (2^u)$, $k^2\equiv 1\ (n)$	$2\,mn$
III	A, B, P, Q	As in I; also $P^4=1$, $P^2=Q^2=(PQ)^2$, $AP=PA$, $AQ=QA$, $BPB^{-1}=Q$, $BQB^{-1}=PQ$	As in I; also $n\equiv 1\ (2)$, $n\equiv 0\ (3)$	$8\,mn$
IV	A, B, P, Q, R	As in III; also $R^2=P^2$, $RPR^{-1}=QP$ $RQR^{-1}=Q^{-1}$, $RAR^{-1}=A^l$, $RBR^{-1}=B^k$	As in III; also $k^2\equiv 1\ (n)$, $k\equiv -1\ (3)$, $r^{k-1}\equiv l^2\equiv 1\ (m)$	$16\,mn$

If G is non-solvable, it is one of the following groups.

V. *$G=K\times SL(2, p)$, where p is a prime $\geqq 5$, and K is a group of type* I *and order prime to* $|SL(2, p)|=p(p^2-1)$.

VI. *G is generated by a group of type* V *and an element S such that*

$$S^2=-1\in SL(2, p)\,,\quad SAS^{-1}=A^{-1},$$
$$SBS^{-1}=B,\quad SLS^{-1}=\theta(L)\ \ (L\in SL(2, p))\,.$$

Here, $SL(2, p)$ denotes the multiplicative group of 2×2 matrices of determinant 1 with entries in the field $\boldsymbol{Z}_p$, and θ is an automorphism of $SL(2, p)$ given by

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\to\begin{pmatrix}1 & 0\\ -\omega & 1\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\to\begin{pmatrix}0 & \omega^{-1}\\ -\omega & 0\end{pmatrix},$$

ω being a generator of the multiplicative group in $\boldsymbol{Z}_p$.

Let G be any finite group, and p a prime. Then the *p-period* of G is defined to be the least positive integer q such that the Tate cohomology groups $\hat{H}^i(G; A)$ and $\hat{H}^{i+q}(G; A)$ have isomorphic p-primary components for all i and all A. The *period* of G is the least common multiple of all the p-periods. R.G. Swan [7] gave a method to calculate the p-period as follows:

(1.5) (i) *If a 2-Sylow subgroup of a finite group G is cyclic, the 2-period of G is 2. If a 2-Sylow subgroup of G is a generalized quaternion group, the 2-peirod of G is 4.*

(ii) *Suppose p is odd and a p-Sylow subgroup G_p of G is cyclic. Let Φ_p denote the group of automorphisms of G_p induced by inner automorphisms of G. Then the p-period of G is $2|\Phi_p|$.*

If $N(G_p)$, $C(G_p)$ denote the normalizer and centralizer of G_p, it holds $\Phi_p \cong N(G_p)/C(G_p)$. From this we have the following (see [8]).

(1.6) *If a 3-Sylow subgroup of G is cyclic, the 3-period of G divides 4.*

We shall next consider free orthogonal actions on S^n. A representation ρ of a group G is said to be *fixed point free* if $1 \neq g \in G$ implies that $\rho(g)$ does not have $+1$ for an eigenvalue.

With the notations of (1.4), let d denote the order of r in the multiplicative group of residues modulo m of integers prime to m. Modifying the work of G. Vincent [9], J. Wolf proves the following (1.7), (1.8) in [10].

(1.7) *For a finite group G, the following two conditions are equivalent:*

i) *G has a fixed point free complex representation.*

ii) *G is of type* I, II, III, IV, V *for $p=5$, or* VI *for $p=5$, with the additional condition: n/d is divisible by every prime divisor of d.*

(1.8) *Let G be a finite group satisfying the conditions in* (1.7). *Then each fixed point free, irreducible complex representation of G has the degree $\delta(G)$ which is given as follows:*

Type	I	II	III	IV	V	VI
$\delta(G)$	d	d or $2d$	$2d$	$2d$ or $4d$	$2d$	$4d$

If $|G|>2$, G acts freely and orthogonally on S^{2q-1} if and only if q is divisible by $\delta(G)$.

REMARK. Wolf states in 7.2.18 of [10] that $\delta(G)=2d$ for G of type II. This mistake is revised in the errata sheet of [10].

2. Finite groups acting freely on $S^{2^\nu-1}$

We shall consider the following conditions for a finite group G:

(A_ν) *G can act freely and orthogonally on $S^{2^\nu-1}$.*

(B_ν) *G can act freely on $S^{2^\nu-1}$.*

(C_ν) *G has the cohomology of period 2^ν and has at most one element of order 2.*

$(A_\nu) \Rightarrow (B_\nu)$ is trivial, and $(B_\nu) \Rightarrow (C_\nu)$ holds by (1.2) and (1.3). We shall study whether $(C_\nu) \Rightarrow (A_\nu)$ holds.

Let G be a finite group satisfying (C_ν). Then, by (1.3) and (1.4), G is of type I, II, III, IV, V or VI. We shall retain the notations in §1.

Case 1: $m \neq 1$.

It follows from the conditions of type I that m is odd. Put $m=\Pi p_i^{c_i}$, where $\{p_i\}$ are distinct odd primes and $c_i \geqq 1$. Then the subgroup generated by $A_i=A^{m/m_i}$ $(m_i=p_i^{c_i})$ is a p_i Sylow-subgroup of G. Let d_i denote the order of r in the multiplicative group of residues modulo m_i of integers prime to m_i. It follows that

$$B^j A_i B^{-j} = A_i^{r^j} \qquad (j=0, 1, \cdots, d_i-1)$$

are distinct. Therefore, by (1.5) the p_i-period of G is a multiple of $2d_i$. Let d' denote the least common multiple of $\{d_i\}$. Then it follows that d divides d', and that $2d'$ divides the period of G. Thus 2^ν is a multiple of $2d$, and so d is a divisor of $2^{\nu-1}$. Since $m=1$ is equivalent to $d=1$, we have

$$d = 2^\alpha \quad \text{with} \quad \alpha = 1, 2, \cdots, \nu-1 .$$

Since n is a multiple of d, n is even. Therefore G can not be of type III, IV, V or VI. If G is of type II and $d=2^\alpha$ with $\alpha \geqq 2$, the conditions on k yield a contradiction. Thus G is of type I with $d=2^\alpha$ $(\alpha=1, 2, \cdots, \nu-1)$, or of type II with $d=2$.

Since the order of $B^{n/2}$ is 2, by (1.1) we have

$$B^{n/2}AB^{-n/2} = A .$$

Since $BAB^{-1}=A^r$, we have also

$$B^{n/2}AB^{-n/2} = A^{r^{n/2}}.$$

Hence $r^{n/2} \equiv 1(m)$, and $n/2$ is a multiple of $d=2^\alpha$. This shows that n/d is divisible by every prime divisor of d. Therefore it follows from (1.7) and (1.8) that G has a fixed point free complex representation whose degree is 2^α if G is of type I with $d=2^\alpha$, and is 2 or 4 if G is of type II with $d=2$. Thus if $\nu \geqq 3$, G acts freely and orthogonally on $S^{2^\nu-1}$. If $\nu=2$, so does G of type I with $d=2$. However (1.8) shows that some groups G of type II with $d=2$ can not act freely and orthogonally on S^3.

Case 2: $m=1$, G is solvable.

In this case we have $d=1$. Therefore it follows from (1.7) and (1.8) that G has a fixed point free complex representation whose degree is 1, 2 or 4. Thus if $\nu \geqq 3$, G acts freely and orthogonally on $S^{2^\nu-1}$. If $\nu=2$, so does G of type I, II, or III. However (1.8) shows that some groups G of type IV can not act freely and orthogonally on S^3.

Case 3: $m=1$, G is non-solvable.

For

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, p)$$

we have

$$X^i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \qquad (i = 0, 1, \cdots, p-1).$$

Therefore X generates a cyclic group of order p. If we observe the order of G, it follows that this cyclic group is a p-Sylow subgroup of G. For

$$Y_i = \begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix}, \quad Z_i = \begin{pmatrix} 0 & -\omega^i \\ \omega^{-i} & 0 \end{pmatrix}$$

we have

$$Y_i X Y_i^{-1} = \begin{pmatrix} 1 & \omega^{2i} \\ 0 & 1 \end{pmatrix},$$

$$Z_i S X S^{-1} Z_i^{-1} = \begin{pmatrix} 1 & \omega^{2i+1} \\ 0 & 1 \end{pmatrix}.$$

Therefore it follows from (1.5) that 2^ν is a multiple of $p-1$ if G is of type V, and that 2^ν is a multiple of $2(p-1)$ if G is of type VI. Thus G is of the following type $\mathrm{V}_\alpha^*(2 \leqq \alpha \leqq \nu)$ or $\mathrm{VI}_\alpha^*(2 \leqq \alpha \leqq \nu-1)$.

V_α^*. $G = \boldsymbol{Z}_n \times SL(2, p)$, where p is a prime of the form $2^\alpha + 1$, and $(n, p(p^2-1)) = 1$.

VI_α^*. G is generated by a group of type V_α^* and an element S satisfying the conditions in VI.

In particular, if $\nu=2$, G is of type V_2^* and it acts freely and orthogonally on S^3 by (1.7) and (1.8). If $\nu=3$, G is of type V_2^* or VI_2^*, and it acts freely and orthogonally on S^7 by (1.7) and (1.8). If $\nu=4$, G is of type V_2^*, V_4^* or VI_2^*. The groups of type V_2^* or VI_2^* act freely and orthogonally on S^{15}, but (1.7) shows that the groups of type V_4^* can not do so.

REMARK. A prime of the form $2^\alpha+1$ is called the *Fermat number*, and α is known to be of a power 2^β. But the converse is not true; for example $2^{32}+1$ is divisible by 641.

Summing up the above arguments, we have proved the following two theorems.

(2.1) **Theorem.** *The conditions* (A_3), (B_3), (C_3) *are mutually equivalent for any finite group* G, *and the following is a list of all finite groups satisfying these conditions*:

1) *The groups of types* I, II, III, IV *with* $d=1$.
2) *The groups of type* I *with* $d=2^\alpha$ *and* $n \equiv 0$ *mod* $2^{\alpha+1}$ $(\alpha=1, 2)$.
3) *The groups of type* II *with* $d=2$.
4) *The groups of types* V, VI *with* $d=1$ *and* $p=5$.

(2.2) **Theorem.** *If $\nu \geqq 3$, the conditions (A_ν), (B_ν), (C_ν) are mutually equivalent for any finite solvable group G, and the following is a list of all finite solvable groups satisfying these conditions*:

1) *The groups of types* I, II, III, IV *with* $d=1$.
2) *The groups of type* I *with* $d=2^\alpha$ *and* $n \equiv 0$ *mod* $2^{\alpha+1}$ $(\alpha=1, 2, \cdots, \nu-1)$.
3) *The groups of type* II *with* $d=2$.

For $\nu=4$ we have also

(2.3) **Theorem.** *The following two conditions for a finite group G are equivalent*:

i) *G satisfies the condition (C_4) but does not satisfy (A_4).*
ii) $G=\boldsymbol{Z}_n \times SL(2, 17)$ *with* $(n, 2\cdot 3\cdot 17)=1$.

Proof. It has been proved in the arguments above that i) implies ii) and the groups of type V_4^* do not satisfy (A_4). It is easily seen that the groups of type V_4^* has only one element of order 2. We shall prove that each group G of type V_4^* has period 16.

If $UXU^{-1}=X^i$ for some $U \in SL(2, p)$, then it is easily seen that i is an even power of ω. Therefore it follows that the p-period of $SL(2, p)$ is $p-1$. This shows that the 17-period of G is 16. By (1.5) and (1.6), the 2-period and the 3-period of G divide 4. If q is a prime dividing n, the q-period of G is 2. Since $|G|=2^5\cdot 3^2\cdot 17\cdot n$, it holds that the period of G is 16.

Here is a problem: Can $SL(2, 17)$ act freely on the sphere S^{16t-1}?

In his study on finite groups acting freely on S^3, Milnor [3] introduces the finite groups presented as follows:

(1) $D'(2^t(2s+1))=\{A, B;\ A^{2s+1}=B^{2^t}=1, BAB^{-1}=A^{-1}\}$, where $s \geqq 1$, $t \geqq 1$.

(2) $Q(8t, s_1, s_2)=\{A, B, R;\ A^{s_1 s_2}=1,\ R^2=B^{2t},\ BAB^{-1}=A^{-1},\ RAR^{-1}=A^l, RBR^{-1}=B^{-1}\}$, where $8t$, s_1, s_2 are pairwise relatively prime positive integers, and $l \equiv -1$ mod s_1, $l \equiv +1$ mod s_2.

(3) $T'(8\cdot 3^t)=\{B, P, Q;\ B^{3^t}=1,\ P^2=Q^2=(PQ)^2,\ BPB^{-1}=Q,\ BQB^{-1}=PQ\}$, where $t \geqq 1$.

(4) $O'(48t)=\{B, P, Q, R;\ B^{3t}=1, P^2=Q^2=R^2=(PQ)^2, BPB^{-1}=Q, BQB^{-1}=PQ,\ RPR^{-1}=QP,\ RQR^{-1}=Q^{-1},\ RBR^{-1}=B^{-1}\}$, where t is a positive odd integer.

These groups are generalizations of the binary polyhedral groups. In fact, the binary dihedral group $Q(4n)$ is $D'(4(2s+1))$ if $n=2s+1$ and is $Q(8t, 1, 1)$ if $n=2t$; the binary tetrahedral group T^* and the binary octahedral group O^* are $T'(24)$, $O'(48)$ respectively.

We shall generalize the group $D'(2^t(2s+1))$ as follows:

$(1)_\alpha$ $D^{(\alpha)}(2^t(2s+1))=\{A, B;\ A^{2s+1}=B^{2^t}=1,\ BAB^{-1}=A^r\}$, where s, t, α are positive integers, $\alpha \leqq t$, and $r^{2^{\alpha-1}} \equiv -1$ mod $2s+1$. Note that $D^{(1)}(2^t(2s+1))=D'(2^t(2s+1))$.

(2.4) **Theorem.** *The following is a list of all finite solvable groups which act freely (and orthogonally) on* $S^{2\nu-1}$ $(\nu\geqq 3)$.

(1) *The groups* 1, $Q(8t, s_1, s_2)$, $T'(8\cdot 3^t)$, $O'(48t)$.

(2) *The groups* $D^{(\alpha)}(2^t(2s+1))$ *with* $t\geqq\alpha+1$, *where* $\alpha=1, 2, \cdots, \nu-1$.

(3) *The direct product of any of these groups with a cyclic group of relatively prime order.*

Proof. If G is of type I with $d=1$, we have $G=\boldsymbol{Z}_n$.

Let G be of type II with $d=1$. It follows that there are B_i of order n_i $(i=1, 2)$ such that

$$\{B\}=\{B_1\}\times\{B_2\}\,,\quad n=n_1n_2\,,\quad (2n_1, n_2)=1\,,$$
$$RB_1R^{-1}=B_1^{-1}\,,\quad RB_2R^{-1}=B_2$$

(see p. 203 of [10]). Then $n_1=4t$, and G is the product of $\boldsymbol{Z}_{n_2}=\{B_2\}$ and $Q(8t)=\{B_1, R\}$.

Let G be of type III with $d=1$. Put $n=3^tn'$, $(n', 3)=1$. Then G is the product of $\boldsymbol{Z}_{n'}=\{B^{3^t}\}$ and $T'(8\cdot 3^t)=\{B^{n'}, P, Q\}$.

Let G be of type IV with $d=1$. Then $n_1=3t$ with t odd, and G is the product of $\boldsymbol{Z}_{n_2}=\{B_2\}$ and $O'(48t)=\{B_1, P, Q, R\}$.

Let G be of type I with $d=2^\alpha$ and $n\equiv 0 \bmod 2^{\alpha+1}(\alpha\geqq 1)$. Put $n=2^tn'$, $(2, n')=1$, $m=2s+1$. Then $t\geqq\alpha+1$ and G is the product of $\boldsymbol{Z}_{n'}=\{B^{2^t}\}$ and $D^{(\alpha)}(2^t(2s+1))=\{A, B^{n'}\}$.

Let G be of type II with $d=2$. Then $n=4t$, $m>1$, and there exist positive integers s_1, s_2 such that $m=s_1s_2$, $l\equiv -1 \bmod s_1$, $l\equiv +1 \bmod s_2$. G is the product of $\boldsymbol{Z}_{n_2}=\{B_2\}$ and $Q(8t, s_1, s_2)=\{A, B_1, R\}$.

Consequently the desired result is only a restatement of Theorem (2.2).

From (2.1) and (2.4), we have

(2.5) **Theorem.** *The following is a list of all finite groups which act freely (and orthogonally) on* S^7.

(1) *The groups* 1, $Q(8t, s_1, s_2)$, $T'(8\cdot 3^t)$, $O'(48t)$.

(2) *The groups* $D^{(\alpha)}(2^t(2s+1))$ *with* $t\geqq\alpha+1$, *where* $\alpha=1, 2$.

(3) *The binary icosahedral group* $I^*=SL(2, 5)$.

(4) *The group generated by* $SL(2, 5)$ *and* S, *where* $S^2=-1\in SL(2, 5)$ *and* $SLS^{-1}=\theta(L)$ $(L\in SL(2, 5))$.

(5) *The direct product of any of these groups with a cyclic group of relatively prime order.*

REMARK. A necessary and sufficient condition for G of type II (or IV) to have $\delta(G)=2d$ (or $4d$) is given in [10] (see the errata sheet of [10]). If we use these results, the above arguments for $\nu=2$ yield theorems 2 and 3 of [3].

3. Finite groups acting freely on S^{2n-1} (n: odd prime)

Let $Z_{m,n}$ be a group of type I with m odd, n odd prime and $d=n$.
By the arguments similar to §2 but simpler, we have

(3.1) **Theorem.** *Let n be an odd prime. Then the following two conditions for a finite group G are equivalent*:

i) *G has cohomology of period* $2n$, *has at most one element of degree* 2, *and can not act freely and orthogonally on* S^{2n-1}.

ii) *G is of type* $\boldsymbol{Z}_h \times Z_{m,n}$ *with* $(h, mn)=1$ *and* $h \geqq 1$.

REMARK. It is known by T. Petrie [5] that the group $Z_{m,n}$ can act freely on S^{2n-1}. Here is a problem: If $h>1$, can the group $\boldsymbol{Z}_h \times Z_{m,n}$ act freely on S^{2n-1}? (R. Lee states in a letter to the author that if h, m are odd primes the problem has an affirmative answer.)

REMARK. Consideration of groups satisfying the condition i) of (3.1) for $n=6$ yields the following problem: Can the group $SL(2, 7)$ act freely on S^{11}?

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