Fujiwara, D.
Osaka J. Math.
11 (1974), 265–281

AN APPROXIMATE POSITIVE PART OF A SELF-ADJOINT PSEUDO-DIFFERENTIAL OPERATOR I

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(Received September 26, 1973)

1. Introduction

Among many problems concerning pseudo-differential operators, one of the most interesting problem is "to what extent does the symbol function $p(x, \xi)$ describe the spectral properties of an operator $p(x, D)$?" Motivation of this paper comes from this problem.

Actually what we do in this note is the following: Assume that $P=p(x, D)$ is a self-adjoint pseudo-differential operator of class $L^0_{1,0}$ of Hörmander [4]. Then starting from its principal symbol, we explicitly construct self-adjoint operators P^+, P^-, R, F^+ and F^- with the following properties;

(i) $F^+ + F^- = Id.$

(ii) $P = P^+ - P^- + R.$

(iii) P^+, P^- and F^+, F^- are non-negative self-adjoint operators.

(iv) We have the following estimates;

$$|(P^+F^-u, F^\pm v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3},$$
$$|(P^-F^+u, F^\pm v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3},$$
$$|(Ru, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3},$$

for any $u, v \in C_0^\infty(\boldsymbol{R}^n)$.

Theorem I gives more precise statement. Proof is found in §5 and §6.

If the principal symbol does not change sign, the problem has been settled. In fact strong Gårding inequality [3], [6] means that we can take $P^- = 0$, $F^- = 0$ and that R satisfies stronger inequality

$$|(Ru, v)| \leqq C\|u\|_{-1/2}\|v\|_{-1/2}.$$

However our result seems new if the principal symbol changes sign. Difficulty arises at the point of characteristics of the operator $p(x, D)$. The operator F^+ and F^- are closely related to location of characteristics of $p(x, D)$. This is discussed in §7.

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1) As to general theory of pseudo-differential operators. See [1], [2], [5] and [7].

Our method is based on localization of Hörmander in [4]. His terminology will frequently be used.

2. Localization

We treat a pseudo-differential operator $p(x, D)$ defined by

$$(2.1)\qquad p(x, D)u(x) = (2\pi)^{-n}\iint_{R^n\times R^n} p(x, \xi)e^{i\langle x-y\rangle\cdot\xi}u(y)\,dy\,d\xi\,.$$

We assume that the symbol $p(x, \xi)$ is of the form

$$p(x, \xi) = p_0(x, \xi)+p_1(x, \xi)\,,$$

where $p_0(x, \xi)$ is homogeneous of degree 0 with respect to ξ for large $|\xi|$ and $p_1(x, \xi)$ is a function in $S^{-1}_{1,0}(\boldsymbol{R}^n)$ in the sense of Hörmander [4]. We further assume that the principal part $p_0(x, \xi)$ vanishes unless x lies in a bounded domain $\Omega\subset\boldsymbol{R}^n$. (See [4]). We use Hörmander's localization in [4]. Let $g_0=0$, $g_1, g_2, \cdots$, be the unit lattice points in $\boldsymbol{R}^n$. Then $\boldsymbol{R}^n$ is covered by open cubes of side 2 with center at these points. Let $\Theta(x)$ be a non-negative C_0^∞ function which equals 1 in $|x_j|\leqq 1$ and zero outside $|x_j|\leqq \frac{3}{2}$, $1\leqq i\leqq n$. We use

$$(2.2)\qquad \varphi_k(x) = \Theta(x-g_k)/(\sum_{k=0}^{\infty}\Theta(x-g_k)^2)^{1/2} \quad\text{and}$$

$$\dot{\varphi}_k(x) = \varphi_k\Big(\frac{x-g_k}{2}k+g_k\Big)\,.$$

The following properties hold:

$$(2.3)\qquad \sum_k \varphi_k(x)^2 \equiv 1 \quad\text{and}$$

$$(2.4)\qquad \sum_k D^\alpha\varphi_k(x) \leqq C_\alpha\,,$$

where α is an arbitrary multi-index $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$. D^α is the usual notation, *i.e.*, $D^\alpha=\Big(-i\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\cdots\Big(-i\frac{\partial}{\partial x_n}\Big)^{\alpha_n}$.

$$(2.5)\qquad |x-y|\leqq 2\sqrt{n} \qquad \text{if}\quad x, y\in \operatorname{supp}\varphi_k\,.$$

Let

$$(2.6)\qquad \psi_k(\xi) = \varphi_k\Big(\frac{\xi}{|\xi|^{2/3}}\Big)\,,\quad \dot{\psi}_k(\xi) = \dot{\varphi}_k\Big(\frac{\xi}{|\xi|^{2/3}}\Big)\,.$$

Then

$$(2.7)\qquad \sum_k \psi_k(\xi)^2 = 1\,,$$

(2.8) $|\xi|^{3/4|\alpha|}\sum_k |D^\alpha\psi_k(\xi)|^2 \leqq C_\alpha$ for ${}^\forall\alpha$.

(2.9) $|\xi-\eta|\leqq C|\xi|^{2/3}$ if $\xi, \eta\in\text{supp}\,\psi_k$.

(2.10) $\sum_k |\psi_k(\xi)-\psi_k(\eta)|^2 \leqq \dfrac{C|\xi-\eta|^2}{(1+|\xi|^{2/3})(1+|\eta|)^{2/3}}$ for ${}^\forall\xi, \eta\in\boldsymbol{R}^n$.

Functions $\dot{\varphi}_k$ and $\dot{\psi}_k$ are identically one in some neighbourhood of supp φ_k and supp ψ_k respectively. They also have properties (2.4)~(2.10) except (2.7). Note that $\delta_j^2 g_j$ belongs to supp ψ_j if $\delta_j=|g_j|$. We define operator $\psi_j(D)$ by

(2.11) $$\psi_j(D)u(x)=(2\pi)^{-n}\iint_{R^n\times R^n} e^{i(x-y)\cdot\xi}\psi_j(\xi)u(y)\,dy\,d\xi\,.$$

Obviously we have

(2.12) $$\sum_j \varphi_j(D)^2 = Id\,,$$

and

(2.13) $$C\|u\|_s^2\leqq \sum_{j=0}^{\infty}\delta_j^{6s}\|\varphi_j(D)u\|_0^2\leqq C^{-1}\|u\|_s^2\,,$$

where $\|u\|_s$ is Sobolev norm of u of order s in $\boldsymbol{R}^n$.
We set $\varphi_{jk}(x)=\varphi_j(\delta_k x)$ and $\phi_{jk}(x,\xi)=\varphi_{jk}(x)\psi_k(\xi)$. Note that for any multi-indices α, β, we have

(2.14) $$|D_x^\alpha D_\xi^\beta\phi_{jk}(x,\xi)|\leqq C_{\alpha\beta}\delta_k^{|\alpha|}|\xi|^{-2/3|\beta|}\leqq C_{\alpha\beta}|\xi|^{1/3|\alpha|-2/3|\beta|}\,.$$

This means that ϕ_{jk} belongs to class $S^0_{2/3,1/3}$ of Hörmander. It follows from (2.3) and (2.13) that

(2.15) $$C\|u\|_s^2\leqq \sum_{jk}\delta_k^{6s}\|\phi_j(x,D)\,u\|_0^2\leqq C^{-1}\|u\|_s^2\,,$$

and

(2.16) $$\sum_{jk}\phi_{jk}(x,D)^*\phi_{jk}(x,D)=Id\,.$$

For any pair (j, k) of integers we set

(2.17) $$P_{jk}(x,D)=p_0(x^{jk},\xi^k)+\sum_{\nu=1}^{n}p_{0(\nu)}(x^{jk},\xi^k)(x-x^{jk})_\nu + \sum_{\nu=1}^{n}p_0^{(\nu)}(x^{jk},\xi^k)(D-\xi^k)_\nu\,,$$

where ξ^k is a point in supp ψ_k and x^{jk} is a point in supp φ_{jk}. The following proposition is due to Hörmander.

Proposition 2.1. *For any ${}^\forall u$, $v\in D(\boldsymbol{R}^n)$, we have*

(2.18) $$|(p(x,D)u,v)-\sum_{jk}(p_{jk}(x,D)\phi_{jk}(x,D)u,\phi_{jk}(x,D)v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}\,.$$

Proof is found in [4].

3. Spectral decomposition of localized operators

We shall call $P_{jk}(x, D)$ localized operator. $P_{jk}(x, D)$ is an operator of order 1. The spectral decomposition of $P_{jk}(x, D)$ is well known. In fact, after multiplication of $e^{ix\cdot\xi^k}$ and suitable change of coordinates, $P_{jk}(x, D)$ is unitarily transformed to an operator of the form

$$L = \alpha D_1 + b\cdot x\,,$$

where α is a real constant and $b\cdot x$ is Euclidean scalar product of two vectors

$$b = (b_1, b_2, \cdots, b_n) \quad \text{and} \quad x = (x_1, x_2, \cdots, x_n)\,.$$

Let

$$L = \int_{-\infty}^{\infty} \lambda\, dE(\lambda)$$

be spectral decomposition of L. Then the projection operator $E(\lambda)$ is the multiplication of function $Y(\lambda - b\cdot x)$ if $\alpha=0$. Here $Y(t)$, $t\in\boldsymbol{R}$, stands for Heaviside function, that is,

$$Y(t) = \begin{cases} 1 & t\geqq 0 \\ 0 & t<0\,. \end{cases}$$

If $\alpha\neq 0$, we set
$$L' = e^{-i\frac{b_1x_1^2}{2\alpha_1}}\, L\, e^{i\frac{b_1x_1^2}{2\alpha_1}}\,.$$

L' is an operator of the form

$$L' = \alpha D_1 + b'\cdot x'\,,$$

where $b'=(b_2, \cdots, b_n)$ and $x'=(x_2, \cdots, x_n)$.

Taking partial Fourier transform with respect to x_1, we have reduced to the case that $\alpha=0$.

We shall use the following notations:

$$P_{jk}(x, D) = \int_{-\infty}^{\infty} \lambda\, dE_{jk}(\lambda)\,. \tag{3.1}$$

Here $E_{jk}(\lambda)$ is the spectral measure of P_{jk}.

We put $\quad E_{jk}^- = E_{jk}(0) \qquad E_{jk}^+ = I - E_{jk}^-$
$\qquad\quad P_{jk}^+ = P_{jk}E_{jk}^+ \qquad P_{jk}^- = -P_{jk}E_{jk}^-\,.$

4. Statement of Theorem I

We put

$$P^+ = \sum_{jk} \phi_{jk}(x, D)^* P^+_{jk} \phi_{jk}(x, D)\,, \tag{4.1}$$

$$P^- = \sum_{jk} \phi_{jk}(x, D)^* P^-_{jk} \phi_{jk}(x, D)\,, \tag{4.2}$$

$$F^+ = \sum_{jk} \phi_{jk}(x, D)^* E^+_{jk} \phi_{jk}(x, D)\,, \tag{4.3}$$

$$F^- = \sum_{jk} \phi_{jk}(x, D)^* E^-_{jk} \phi_{jk}(x, D)\,. \tag{4.4}$$

Then we have

Theorem I. *Operators P^+, P^-, F^+ and F^- are self-adjoint and satisfy the following properties*:

(4.5) (i) $I = F^+ + F^-$.

(4.6) (ii) $(P^{\pm}u, u) \geqq 0$.

(4.7) (iii) $|(F^-P^+F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.8) $|(F^-P^+F^-u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.9) $|(F^-P^-F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.10) $|(F^+P^-F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.11) (iv) $|([P, F^{\pm}]u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.12) $|([P^{\pm}, F^{\pm}]u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(v) *If we set $R = P - (P^+ - P^-)$ then*

(4.13) $|(Ru, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

Corollary 4.2. *We have*

(4.14) $|(PF^+u, v) - (F^+P^+F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$

(4.15) $|(PF^-u, v) + (F^-P^-F^-u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$.

(4.16) $|(P^+F^-u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$,

(4.17) $|(P^-F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}$,

(4.18) $P = P^+ - P^- + R$.

We shall prove Theorem I in §6.

5. Some lemmas about self-adjoint operators

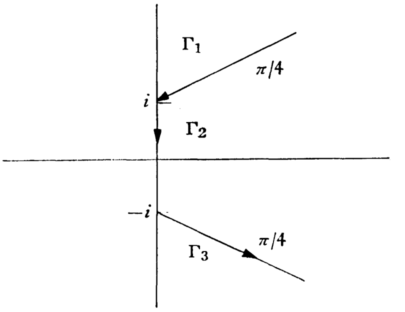
In this section X stands for an abstract Hilbert space.

Lemma 5.1. *Let A be a self-adjoint operator in X and A^+ be its positive part. Then*

$$A^{+}u=\frac{1}{2\pi i}\int_{\Gamma}\left(\lambda(\lambda-A)^{-1}-1-\frac{A}{\lambda+1}\right)ud\lambda$$

provided $u\in D(A^2)$=*domain of* A^2. Γ *is the complex contour as is shown in* fig 1.

fig 1.



Proof. Note that

$$\lambda(\lambda-\sigma)^{-1}-1-\frac{\sigma}{\lambda+1}=\frac{\sigma(\sigma+1)}{(\lambda-\sigma)(\lambda+1)}\,.$$

Integrate this with respect to λ on Γ then we have σ if $\sigma>0$ and 0 if $\sigma<0$. Therefore if we use spectral decomposition of A, then we can prove our Lemma.

Lemma 5.2. *Let A be a self-adjoint operator in X and let B be a bounded linear operator. We assume that operators AB and A^2B are densely defined. We further assume that the communtator $[A, B]$, $[A, [A, B]]$ are bounded.*
Then we have

$$(5.2)\qquad \|[A^{\pm}, B]\|\leqq C(\|B\|+\|[A, B]\|+\|[[A, B], A]\|).$$

Proof. Let $u\in D(A^2)\cap D(A^2B)\cap D(AB)$,

$$2\pi i[A^{+}, B]u=\int_{\Gamma}\left[\left(\lambda(\lambda-A)^{-1}-1-\frac{A}{\lambda+1}\right), B\right]ud\lambda\,.$$

We split Γ into theree parts $\Gamma_1+\Gamma_2+\Gamma_3$. (see fig. 1). Corresponding integrals are denoted by A_1, A_2 and A_3. Obviously $[A^{+}, B]=[A_1, B]+[A_2, B]+[A_3, B]$.

Since $$\int_{\Gamma_2}\frac{1}{\lambda+1}d\lambda=\log(1-i)-\log(1+i)\,,$$

we have

$$(5.3)\qquad \|[A_2, B]\|\leqq 4(\|B\|+\|[A, B]\|)\,.$$

Let us treat

$$2\pi i[A_2+A_3, B]=\int_{\Gamma_1+\Gamma_2}\left[\left[\lambda(\lambda-A)^{-1}-1-\frac{A}{\lambda+1}\right], B\right]d\lambda$$

$$=-[A, B]\int_{\Gamma_1+\Gamma_3}\frac{d\lambda}{\lambda+1}+\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}d\lambda\,.$$

We know

$$\left|\int_{\Gamma_1+\Gamma_3}\frac{d\lambda}{\lambda+1}\right|\leqq\text{const.}$$

On the other hand we have

$$\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}d\lambda$$

$$=\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-2}[A, B]d\lambda+\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-2}[A, [A, B]](\lambda-A)^{-1}d\lambda\,.$$

The last term is majorized by $C\|[A, [A, B]]\|$.

The first is

$$\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-2}d\lambda=\int_{\Gamma_1+\Gamma_3}d\lambda\int_{-\infty}^{\infty}\lambda(\lambda-\sigma)^{-2}dE(\sigma)$$

$$=\int_{\Gamma_1+\Gamma_3}d\lambda\int_{-\infty}^{\infty}\left(\frac{1}{\lambda-\sigma}+\frac{\sigma}{(\lambda-\sigma)^2}\right)dE(\sigma)\,.$$

Since $\left|\int_{\Gamma_1+\Gamma_3}\frac{d\lambda}{\lambda-\sigma}\right|\leqq\text{Const.}$ and $\left|\int\frac{\sigma}{(\lambda-\sigma)^2}d\lambda\right|\leqq2\left|\frac{\sigma}{i+\sigma}\right|\leqq2$,

we have

$$\left\|\int_{\Gamma_1+\Gamma_3}\lambda(\lambda-A)^{-2}[A, B]\,d\lambda\right\|\leqq C\|[A, B]\|\,.$$

We have thus proved our lemma.

Lemma 5.3. *Let A and B be two self-adjoint operators in X. If the commutator $[A, B]$ is bounded, then for any*

$$x\in D(A^2)\cap D(B^2)$$

we have

$$\|(A^+-B^+)x\|\leqq C(\|[A, B]\|\,\|x\|+\sum_{k=1}^{2}\|(A-B)^k x\|+\|x\|+\|[A, B]]\|\,\|(A-B)x\|)\,.$$

Proof. We have to majorize

$$(2\pi i)\,(A^+x-B^+x)=\int_\Gamma\left(\lambda(\lambda-A)^{-1}-\lambda(\lambda-B)^{-1}-\frac{A}{\lambda+1}+\frac{B}{\lambda+1}\right)xd\lambda\,.$$

We decompose Γ as we did in the proof of Lemma 5.2. The integral over Γ_2 is majorized by $C(||x||+||(A-B)x||)$.

Note that

$$\begin{aligned}
&\lambda(\lambda-A)^{-1}-\lambda(\lambda-B)^{-1}x \\
&\quad = -\lambda(\lambda-B)^{-1}(A-B)(\lambda-A)^{-1}x \\
&\quad = -\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}(A-B)x \\
&\qquad -\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x \\
&\quad = -\lambda(\lambda-B)^{-2}\{1+(A-B)(\lambda-A)^{-1}\}(A-B)x \\
&\qquad +\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x \\
&\quad = -\lambda(\lambda-B)^{-2}\{1+(\lambda-A)^{-1}(A-B) \\
&\qquad +(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}\}(A-B)x \\
&\qquad +\lambda(\lambda-B)^{-1}(\lambda-A)^{-1}[A, B](\lambda-A)^{-1}x\,.
\end{aligned}$$

From this we can majorize the integral over $\Gamma_1+\Gamma_3$ by

$$C(||(A-B)x||+||(A-B)^2x||+||[A, B]||\,||x||+||[A, B]||\,||(A-B)x||)\,.$$

We have thus proved our lemma.

6. Proof of Theorem

We start with the propositions which simplify discussions later.

Proposition 6.1. *Let $u\in C_0^\infty(\boldsymbol{R}^n)$ be arbitrary and (j, k) be a pair of indices. Then there is a point $\bar{x}$ satisfying*

(6.1) $\quad |\bar{x}-x^{jk}|\leq\alpha\delta_k\,,$

(6.2) $\quad \int(x_\nu-\bar{x}_\nu)|\phi_{jk}(x, D)u(x)|^2dx = 0$

for $\nu=1, 2, 3, \cdots, n$. Here α is a positive constant independent of u and (j, k).

Proof is found in [3], page 171.
The point $\bar{x}$ can be chosen in supp φ_{jk}.

Proposition 6.2. *There exists a bounded sequence $\{\phi'_{jk}(x, \xi)\}_{jk}$ of symbols in $S^0_{2/3,1/3}$ such that we have*

(6.3) $\quad$ (i) $\;||(D_\nu-\xi_\nu^k)\phi_{jk}(x, D)u||^2\leq C\delta_k^4||\phi'_{jk}(x, D)u||^2$

and

(6.4) $\quad$ (ii) $\;\mathrm{supp}\;\phi'_{jk}\subset\mathrm{supp}\;\phi_{jk}$

for $\nu=1, 2, 3, \cdots, n$.

Proof. We have

$$(6.5)\qquad (D_\nu-\xi_\nu^k)\phi_{jk}(x, D)u=\delta_k^2\phi'_{jk}(x, D)u\,,$$

where

$$(6.6)\qquad \phi'_{jk}(x, \xi)=\delta_k^{-2}\Big(-i\frac{\partial}{\partial x_\nu}\varphi_{jk}(x)\psi_k(\xi)+\varphi_{jk}(x)\psi_k(\xi)(\xi_\nu-\xi_\nu^k)\Big)\,.$$

The sequence $\{\phi'_{jk}(x, \xi)\}_{j,k}$ is bounded in $S^0_{2/3,1/3}$ because of (2.9) and

$$-i\frac{\partial}{\partial x_\nu}\varphi_{jk}(x)=-i\delta_k\Big(\frac{\partial}{\partial x_\nu}\varphi_j\Big)(\delta_k x)\,.$$

Proposition 6.3. *Let* $\{(\dot{x}^{jk}, \xi^{jk})\}_{jk}$ *be another sequence of points. Let* $\dot{P}_{jk}$, $\dot{P}^{\pm}$ *and* $\dot{F}^{\pm}$ *be operators defined by* (2.17), (4.1), (4.2), (4.3) *and* (4.4) *where* (x^{jk}, ξ^k) *is replaced by* $(\dot{x}^{jk}, \xi^{jk})$. *If there exists a constant* $C>0$ *satisfying*

$$(6.7)\qquad |x^{jk}-\dot{x}^{jk}|\leqslant C\delta_k^{-1}\quad \textit{and}\quad |\xi^k-\xi^{jk}|\leqslant C\delta_k^2\,,$$

then we have

$$(6.8)\qquad \|(\dot{P}_{jk}-P_{jk})\phi_{jk}(x, D)u\|^2\leqslant C\delta_k^{-4}\|\phi_{jk}^{(1)}(x, D)u\|^2\,,$$

$$(6.9)\qquad \|(\dot{P}_{jk}-P_{jk})^2\phi_{jk})(x, D)u\|^2\leqslant C\delta_k^{-8}\|\phi_{jk}^{(2)}(x, D)u\|^2\,,$$

$$(6.10)\qquad \|[P_{jk}, \dot{P}_{jk}]\|\leqslant C\delta_k^{-4}\,,$$

$$(6.11)\qquad \|(\dot{P}_{jk}^{\pm}-P_{jk}^{\pm})\phi_{jk}(x, D)u\|\leqslant C\delta_k^{-2}\|\phi_{jk}^{(3)}(x, D)u\|\,,$$

$$(6.12)\qquad |((\dot{P}^{\pm}-P^{\pm})u, u)|\leq C\|u\|_{-1/3}\|v\|_{-1/3}\,.$$

Here, $\{\phi_{jk}^{(l)}\}_{jk}$, $l=1, 2, 3$, *are bounded sequences of symbols in* $S^0_{2/3,1/3}$ *with the property that* supp $\phi_{jk}^{(l)}\subset$ supp ϕ_{jk}.

REMARK 6.4. We require that the point (x^{jk}, ξ^k) lies in supp ϕ_{jk} but we don't require that $(\dot{x}^{jk}, \xi^{jk})$ lies in supp ϕ_{jk}.

Proof. It follows from Taylor's formula that

$$(6.13)\qquad \begin{aligned} P_0(\dot{x}^{jk}, \xi^{jk}) &= P_0(x^{jk}, \xi^k)+\sum_\nu(\dot{x}^{jk}-x^{jk})P_{0(\nu)}(x^{jk}, \xi^k)\\ &\quad+\sum_\nu(\xi^{jk}-\xi^k)_\nu P_0^{(\nu)}(x^{jk}, \xi^k)+R_1\,, \end{aligned}$$

$$(6.14)\qquad P_{0(\nu)}(\dot{x}^{jk}, \xi^{jk})=P_{0(\nu)}(x^{jk}, \xi^k)+R_{2,(\nu)}\quad\text{and}$$

$$(6.15)\qquad P_0^{(\nu)}(\dot{x}^{jk}, \xi^{jk})=P_0^{(\nu)}(x^{jk}, \xi^k)+R_3^{(\nu)}\,.$$

By (6.7) the remainder terms are majorized as

$$(6.16)\qquad |R_1|\leqq C\delta_k^{-2},\ |R_{2(\nu)}|\leqq C\delta_k^{-1},\ |R_3^{(\nu)}|\leqq C\delta_k^{-4}.$$

We have

$$(6.17)\qquad \dot{P}_{jk}(x, D)-P_{jk}(x, D) = R_1+\sum_\nu (x-x^{jk})_\nu R_{2(\nu)}+\sum_\nu (D-\xi^{jk})_\nu R\,.$$

This implies that

$$\begin{aligned}&\|(\dot{P}_{jk}(x, D)-P_{jk}(x, D)\phi_{jk}(x, D)u\|^2\\ &\leqq C(\delta_k^{-4}\|\phi_{jk}(x, D)u\|^2+\delta_k^{-2}\sum_\nu \|(x-x^{jk})_\nu\phi_{jk}(x, D)u\|^2\\ &+\delta_k^{-8}\sum_\nu \|(D-\xi^{jk})_\nu\phi_{jk}(x, D)u\|^2)\\ &\leqq C\delta_k^{-4}\|\phi_{jk}^{(1)}(x, D)u\|^2\,.\end{aligned}$$

This is (6.8).
Similarly

$$(6.18)\qquad \begin{aligned}&\|(\dot{P}_{jk}(x, D)-P_{jk}(x, D))^2\phi_{jk}(x, D)u\|^2\\ &\leqq C\delta_k^{-8}\|\phi_{jk}^{(2)}(x, D)u\|^2\,.\end{aligned}$$

Now

$$(6.19)\qquad \begin{aligned}[P_{jk}, \dot{P}_{jk}] &= [P_{jk}, \dot{P}_{jk}-P_{jk}]\\ &= -[i\sum_\nu R_{2(\nu)}P_0^{(\nu)}(x^{jk}, \xi^k)-\sum_\nu R_3^{(\nu)}P_{0(\nu)}(x^{jk}, \xi^k)]\,.\end{aligned}$$

This proves (6.10).

We apply Lemma 5.3 to operators $A=\delta_k^2 P_{jk}$, and $B=\delta_k^2\dot{P}_{jk}$. Then we have

$$(6.20)\qquad \|(A^+-B^+)\phi_{jk}(x, D)u\|\leqq C\|\phi_{jk}^{(3)}(x, D)u\|\,.$$

This proves that

$$(6.21)\qquad \|(P_{jk}^+-\dot{P}_{jk}^+)\phi_{jk}(x, D)u\|\leqq C\delta_k^{-2}\|\phi_{jk}^{(3)}(x, D)u\|\,.$$

Let v be in $C_0^\infty(\boldsymbol{R}^n)$. Then

$$\begin{aligned}|((P^+-\dot{P}^+)u, v)| &\leqq \sum_{jk} |((P_{jk}^+-\dot{P}_{jk}^+)\phi_{jk}(x, D)u, \phi_{jk}(x, D)v)\\ &\leqq C\sum_{jk}\delta_k^{-2}\|\phi_{jk}^{(3)}(x, D)u\|\,\|\phi_{jk}(x, D)v\|\,.\end{aligned}$$

Take arbitrary positive $t>0$. Then

$$\begin{aligned}|((P^+-\dot{P}^+)u, v)| &\leqq C\sum_{jk}\frac{t^2}{2}\delta_k^{-2}\|\phi_{jk}^{(3)}(x, D)u\|^2+\frac{t^{-2}}{2}\delta_k^{-2}\|\phi_{jk}(x, D)v\|^2\\ &\leqq C\left(\frac{t^2}{2}\|u\|_{-1/3}^2+\frac{t^{-2}}{2}\|v\|_{-1/3}^2\right).\end{aligned}$$

Taking the minimum of this with respect to t, we have

$$|((P^{+}-\dot{P}^{+})u, v)|\leqq C\|u\|_{-1/3}\|v\|_{-1/3}.$$

Next we need bounds for commutators

$$[P_{jk}^{\pm}, \phi_{lm}(x, D)], \quad [E_{jk}^{\pm}, \phi_{lm}(x, D)] \quad \text{etc.}$$

These are needed only when $\operatorname{supp} \phi_{jk} \cap \operatorname{supp} \phi_{lm} \neq \phi$.
We introduce notation

$$I(j, k) = \{(l, m) | \operatorname{supp} \phi_{jk} \cap \operatorname{supp} \phi_{lm} \neq \phi\}.$$

It is obvious that there is a constant $C>0$ such that

$$C^{-1}\leqq\frac{\delta_m}{\delta_k}\leqq C. \qquad \text{if} \quad (l, m)\in I(j, k).$$

The number of indices (l, m) in $I(j, k)$ is bounded.

Proposition 6.5. *We have the following estimates for commutators: If $(l, m)\in I(j, k)$, then*

(6.22) $\|[P_{jk}, \phi_{lm}]\|\leqq C\delta_k^{-2}$,

(6.23) $\|[[P_{jk}, \phi_{lm}], P_{jk}]\|\leqq C\delta_k^{-4}$,

(6.24) $\|[[P_{jk}, \phi_{lm}], \phi_{lm}]\|\leqq C\delta_k^{-2}$,

(6.25) $\|[[P_{jk}, \phi_{lm}{}^*], \phi_{lm}]\|\leqq C\delta_k^{-2}$,

(6.26) $\|[P_{jk}^{\pm}, \phi_{lm}]\|\leqq C\delta_k^{-2}$,

(6.27) $\|[P_{jk}, [E_{jk}^{\pm}, \phi_{lm}]]\|\leqq C\delta_k^{-2}$.

Proof.
$$\begin{aligned}[P_{jk}, \phi_{lm}] &= [P_{jk}, \varphi_{lm}(x)\psi_m(D)]\\ &= [P_{jk}, \varphi_{lm}]\varphi_m(D)+\varphi_{lm}[P_{jk}, \psi_m(D)]\\ &= \delta_k\sum_\nu P_0^{(\nu)}(x^{jk}, \xi^k)D_\nu\varphi_{lm}(x)\psi_m(D)\\ &\quad -\varphi_{lm}\delta_k^{-2}\sum_\nu D_\nu\psi_m(D)P_{0(\nu)}(x^{jk}, \xi^k).\end{aligned}$$

This proves that

$$\|[P_{jk}, \phi_{lm}]\|\leqq C\delta_k^{-2}.$$

More precisely, $\{\delta_k^2[P_{jk}, \phi_{lm}]\}_{jk}$ is bounded sequence of operators in $L^0_{2/3,1/3}$ of Hörmander. By just the same argument we can prove (6.23). (6.24) and (6.25) are consequences of the fact that

$$\{\delta_k[P_{jk}, \phi_{lm}]\}_{jk}$$

is a bounded set in $L^0_{2/3,1/3}$.

We set $A=\delta_k^2P_{jk}$, $B=\delta_k^{-2}\phi_{lm}$ and apply Lemma 5.2.

Then we have

$$||[P^{\pm}_{jk}, \phi_{lm}]|| \leqq C\delta_k^{-2} .$$

Since

$$(6.29) \qquad P_{jk}[E^{\pm}_{jk}, \phi_{lm}] = [P^{\pm}_{jk}, \phi_{lm}] - E^{\pm}_{jk}[P_{jk}, \phi_{lm}] ,$$

(6.27) is a consequence of (6.26).
Now we are ready for proving our Theorem I.

Proof of (iii). Let (j, k) and (j', k') be two pairs of indices. Then we put

$$I(jk, j'k') = \{(l, m) \quad | \operatorname{supp} \phi_{lm} \cap \operatorname{supp} \phi_{jk} \neq \phi \quad \operatorname{supp} \phi_{lm} \cap \operatorname{supp} \phi_{j'k'} \neq \phi\} .$$

By definition of P^+, F^+ and F^-, we have

$$(6.30) \qquad (F^-P^-F^+u, v) = \sum_{jk} \sum_{j'k'} (P^-\phi^*_{jk}E^+_{jk}\phi_{jk}u, \phi^*_{j'k'}E^-_{j'k'}\phi_{j'k'}v) .$$

If $\operatorname{supp} \phi_{lm} \cap \operatorname{supp} \phi_{jk} = \phi$ and $\operatorname{supp} \varphi_k \cap \operatorname{supp} \psi_m \neq 0$, then

$$(6.31) \qquad ||\phi_{lm}(x, D)\phi_{jk}(x, D)^*w|| \leqq C\delta_k^{-N}||w||$$

for any $N>0$. If $\operatorname{supp} \psi_k \cap \operatorname{supp} \psi_m = \phi$, then $\phi_{lm}(x, D)\phi_{jk}(x, D)^*u = 0$.

Thus we have

$$\begin{aligned}(6.32) \qquad & \sum_{(l,m)\notin I(j,k)} ||\phi^*_{lm}P^-_{lm}\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u|| \\ & \leqq C|\Omega|\delta_k^n\delta_k^{-N}||E^+_{jk}\phi_{jk}u|| \\ & \leqq C|\Omega|\delta_k^{n-N}||\phi_{jk}u|| ,\end{aligned}$$

where $|\Omega|$ is the volume of the domain Ω.
Similarly

$$(6.33) \qquad \sum_{(l,m)\notin I(j',k')} ||\phi^*_{lm}P^-_{lm}\phi_{lm}\phi^*_{j'k'}E^*_{j'k'}\phi_{j'k'}v||^2 \leqq C|\Omega|\delta_k^{n-N}||\phi_{j'k'}v||^2.$$

(6.32) and (6.33) imply that

$$\begin{aligned}(6.34) \qquad & (F^-P^-F^+u, v) - \sum_{jk}\sum_{j'} \sum_{(l,m)I(jk,j'k')} (\phi^*_{lm}P^-\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u\phi_{j'k'}E'_{jk}\phi_{jk'}v) \\ & \leqq C|\Omega|(\sum_{jk,j'k'} \delta_k^{n-N}|||\phi_{jk}u||\,||\phi_{j'k'}v||) \\ & \leqq C|\Omega|\,||u||_{-1/3}||v||_{-1/3} .\end{aligned}$$

We have

$$(6.35)\qquad \sum_{(lm)\in I(jk,j'k')} \phi^*_{lm}P^-_{lm}\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u = \sum_{(lm)\in I(jk,j'k')} \phi^*_{lm}P^-_{jk}\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u + \sum_{(lm)\in I(jk,j'k')} \phi^*_{lm}(P^-_{lm}-P^-_{jk})\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u\,.$$

We apply Proposition 6.3 and have

$$(6.36)\qquad \|\sum_{lm} \phi^*_{lm}(P^-_{lm}-P^-_{jk})\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u\| \leqq C\delta_k^{-2}\|\phi_{jk}u\|\,.$$

On the other hand,

$$(6.37)\qquad \sum_{(l,m)\in I(jk,j'k')} \phi^*_{lm}P^-_{jk}\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u = \sum_{lm} \{\phi^*_{lm}[P^-_{jk}\phi_{lm}]\phi^*_{jk}E^+_{jk}\phi_{jk}u + \phi^*_{lm}\phi_{lm}[P^-_{jk},\ \phi^*_{jk}]E^+_{jk}\phi_{jk}u\}\,.$$

By proposition 6.5, we have

$$(6.38)\qquad \|\phi^*_{lm}[P^-_{jk},\ \phi^*_{lm}]\phi^*_{jk}E^+_{jk}\phi_{jk}\|u \leqq C\delta_k^{-2}\|\phi_{jk}u\|$$

and

$$(6.39)\qquad \|\phi^*_{lm}\phi_{lm}[P^-_{jk},\ \phi^*_{jk}]E^+_{jk}\phi_{jk}u\| \leqq C\delta_k^{-2}\|\phi_{jk}u\|\,.$$

(6.37), (6.38) and (6.39) imply that

$$(6.40)\qquad \|\sum_{(lm)\in I(jk,j'k')} \phi^*_{lm}P^-_{jk}\phi_{lm}\phi^*_{jk}E^+_{jk}\phi_{jk}u\| \leqq C\delta_k^{-2}\|\phi_{jk}u\|\,.$$

As a consequence of (6.34) and (6.40), we have

$$(6.41)\qquad |(F^-P^-F^+u,\ v)| \leqq \sum_{(jk)(j'k')} C\delta_k^{-2}\|\phi_{jk}u\|\,\|\phi_{j'k'}v\| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}\,,$$

where the summation ranges over those (jk) and $(j',\ k')$ that $I(jk,\ j'k')\neq\phi$. This proved (iii). Proof of remaining part of Theorem I is the same.

7. The role of characteristics

So far the choice of sequence $\{(x^{jk},\ \xi^k)\}$ is not specified. In the following we shall make use of special choice of it in order to simplify operators $P^{\pm}_{jk}$ and $E^{\pm}_{jk}$.

The set

$$(7.1)\qquad \Sigma^0(P) = \{(x,\ \xi)\in \boldsymbol{R}^{2n}\,|\,\xi\neq 0,\ P_0(x,\ \xi)=0\}$$

is called the characteristics of the operator P. We also use the following notations;

$$\Sigma^+(P) = \{(x, \xi) \in \boldsymbol{R}^{2n} \mid \xi \neq 0,\ P_0(x, \xi) > 0\}\ , \tag{7.2}$$

$$\Sigma^-(P) = \{(x, \xi) \in \boldsymbol{R}^{2n} \mid \xi \neq 0,\ P_0(x, \xi) < 0\}\ . \tag{7.3}$$

Proposition 7.1. *Assume that* $(x^{jk}, \xi^k) \in \Sigma^+(P) \cup \Sigma^0(P)$ *and that* $P(x, \xi) \geqq 0$ *for any* $x \in \mathrm{supp}\ \phi_{jk}$ *and* ξ *with* $|\xi - \xi^k| < \alpha\delta_k^2$, *where* α *is the constant appeared in Proposition* 6.1. *Then we can replace* E_{jk}^+ *by the identity operator without altering results in Theorem* I.

Proof of Proposition 7.1.

We put $L_k = \{j \mid (x^{jk}, \xi^k)$ satisfies the assumption of Proposition 7.1$\}$

$$Q_k = \sum_{j \in L_k} \phi_{jk}^*(x, D) P_{jk}^- \phi_{jk}(x, D) \tag{7.4}$$

and

$$G_k = \sum_{j \in L_k} \phi_{jk}^*(x, D) E_{jk}^- \phi_{jk}(x, D)\ . \tag{7.5}$$

We claim that there exists a constant $C > 0$ such that

$$\|Q_k u\| \leqq C\delta_k^{-2} \|\psi_k(D) u\|\ . \tag{7.6}$$

We admit this for a moment. Replacing E_j^+ $(j \in L_k,\ k = 0, 1, 2, \cdots)$ in (4.1)~(4.4) with the identity, we obtain operators $Q^\pm$ and $G^\pm$.
Differences between old and new operators are

$$Q^\pm - P^\pm = \sum_k Q_k\ , \tag{7.7}$$

$$G^\pm - F^\pm = \sum_k G_k\ . \tag{7.8}$$

These relations imply that

$$\begin{aligned}(G^- Q^+ G^+ u, v) = \sum_k (G^- Q_k G^+ u, v) &- \sum_k (G_k P^+ F^+ u, v) \\ &+ \sum_k (F^- P^+ G_k u, v) - \sum_{k,l} (G_k P^+ G_l u, v) \\ &+ (F^- P^+ F^+ u, v)\ .\end{aligned} \tag{7.9}$$

We know by Theorem I that

$$|(F^- P^+ F^+ u, v)| \leqq C\|u\|_{-1/3} \|v\|_{-1/3}\ . \tag{7.10}$$

On the other hand we can use (7.6) and prove the following inequalities in the same way as the proof of (6.34):

$$
(7.11)\qquad
\begin{aligned}
&\sum_k |(G_kP^+F^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}\,,\\
&\sum_k |(F^-P^+G_ku, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}\,,\\
&\sum_{k,l} |(G_kP^+G_lu, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}\,,\\
&\sum_k |(G^-Q_kG^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}\,.
\end{aligned}
$$

These prove

$$
(7.12)\qquad |(G^-Q^+G^+u, v)| \leqq C\|u\|_{-1/3}\|v\|_{-1/3}
$$

which corresponds to (4.7). Other inequalities can be proved in the same manner.

Now we must prove our claim (7.6). We choose $\bar{x}$ as in Proposition 6.1. Let

$$
(7.13)\qquad
\begin{aligned}
Q_{jk}(x, D) = p_0(\bar{x}, \xi^k) &+ \sum_{\nu=1}^{n} p_{0(\nu)}(\bar{x}, \xi^k)(x-\bar{x})_\nu\\
&+ \sum_{\nu=1}^{n} p_0^{(\nu)}(\bar{x}, \xi^k)(D-\xi^k)_\nu\,.
\end{aligned}
$$

Then

$$
(7.14)\qquad
\begin{aligned}
&(Q_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\\
&\quad = ((p_0(\bar{x}, \xi^k)+\sum_\nu p_0^{(\nu)}(\bar{x}, \xi^k)(D-\xi^k)_\nu)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\\
&\quad = p_0(\bar{x}, \xi^k)((1-\dot{\psi}(D))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\\
&\qquad +((p_0(\bar{x}, \xi^k)+\sum_\nu p_0^{(\nu)}(\bar{x}, \xi^k)(D-\xi^k)_\nu)\dot{\psi}_k(D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\\
&\qquad +\sum_\nu p_0^{(\nu)}(\bar{x}, \xi^k)((D-\xi^k)_\nu(1-\dot{\psi}_k(D))\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)
\end{aligned}
$$

because of (6.2).

Since α is large, we may assume that $p_0(\bar{x}, \xi)\geqq 0$ if $\xi\in \operatorname{Supp}\dot{\psi}_{k0}$. Taylor's expansion of $p_0(\bar{x}, \xi)$ at $\xi=\xi^k$ imply that there exists a constant $C>0$ such that

$$
\begin{aligned}
&((p_0(\bar{x}, \xi^k)+\sum_\nu p_0^{(\nu)}(\bar{x}, \xi^k)(D-\xi^k)_\nu)\dot{\psi}_k(D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)\\
&\quad \geqq -C\delta_k^{-2}\|\phi_{jk}(x, D)u\|^2\,.
\end{aligned}
$$

We know that

$$
(D-\xi^k)_\nu(1-\dot{\psi}_k(D))\phi_{jk}(x, D)u = (D-\xi^k)_\nu(1-\dot{\psi}_k(D))\varphi_{jk}(x)\psi_k(D)\dot{\psi}_k(D)u
$$

and that the sequence of double symbols $\{(\xi-\xi^k)(1-\dot{\psi}_k(\xi))\varphi_{jk}(x)\psi_k(\eta)\}_{j,k}$ is bounded in $S^{-\infty}$. Therefore we have estimate for any $N>0$,

$$
\|(D-\xi^k)(1-\dot{\psi}_k(D))\phi_{jk}(x, D)u\|^2 \leqq C\delta_k^{-N}\|\psi_k(D)u\|^2\,.
$$

This implies that

$$(Q_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)+C\delta_k^{-2}\|\phi_{jk}(x, D)u\|^2+C\delta_k^{-N}\|\psi_k(D)u\|^2\geqq 0 .$$

This and Proposition 6.3 prove that

$$(P_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)+C(\delta_k^{-2}\|\phi'_{jk}(x, D)u\|^2+\delta_k^{-N}\|\psi_k(D)u\|^2)\geqq 0 ,$$

where $\{\phi'_{jk}(x, \xi)\}$ is a bounded sequence in $S^0_{2/3,1/3}$ as of Proposition 6.3. Taking sum of these with respect to $j\in L_k$, we have

$$\sum_{j\in L_k}(P_{jk}(x, D)\phi_{jk}(x, D)u, \phi_{jk}(x, D)u)+C\delta_k^{-2}\|\psi_k(D)u\|^2\geqq 0 .$$

Our claim is an immediate consequence of this inequality.

REMARK. Result similar to Proposition 7.1 holds for E^-_{jk}.

Next we discuss the case that $P_0(x, \xi)$ changes sign in the neighbourhood of supp ϕ_{jk}. In this case we compare $P_{jk}(x, D)$ with the operator $\dot{P}_{jk}(x, D)$ which is determined at a characteristic point.

Proposition 7.2. *Assume that $P_0(x, \xi)$ changes sign at some point $(\dot{x}, \dot{\xi})$ with*

$$|X^{jk}-\dot{x}|<\alpha\delta_k^{-1}, \quad |\xi^k-\dot{\xi}|<\alpha\delta_k^2 . \tag{7.6}$$

Then we can replace $P_{jk}(x, D)$ by

$$\bar{P}_{jk}(x, D) = \sum_\nu P_{0(\nu)}(\dot{x}, \dot{\xi})(x-\dot{x})_\nu + \sum_\nu P_0^{(\nu)}(\dot{x}, \dot{\xi})(D-\dot{\xi})_\nu \tag{7.7}$$

without altering results in Theorem I.

Proof. This proposition is contained in Proposition 6.3.
Finally we discuss the case where the operator $E^\pm_{jk}$ can be arbitrarily chosen.

Proposition 7.3. *Assume that we have*

$$P_0(\dot{x}, \dot{\xi}) = 0 \quad \mathrm{grad}_{x,\xi}P_0(\dot{x}, \dot{\xi}) = 0$$

at some point $(\dot{x}, \dot{\xi})$ with $|\dot{x}-x^{jk}|<\alpha\delta_k^{-1}$, $|\dot{\xi}-\xi^k|<\alpha\delta_k^2$. Then we can replace $P_{jk}(x, D)$ by zero operator 0 *without altering Theorem* I.

Proof. This is because of Proposition 6.3.

REMARK 7.4. In this case, the operator $E^\pm_{jk}$ does not matter. We can put $E^+_{jk}=Id$ or 0 at our disposal. From Proposition 7.1, 7.2 and 7.3, we can see F^+ and F^- depend only on location of sets $\sum^+(P)$, $\sum^-(P)$ and $\sum^0(P)$. An interesting consequence comes out when one compare two pseudo-differential operators whose characteristics are the same. Let Q be another self-adjoint pseudo-differential operator of class $L^0_{1,0}$. We assume Q has homogeneous principal symbol $q_0(x, \xi)$ and $Q-q_0(x, D)\in L^{-1}_{1,0}$. Just as we did for the operator $P(x, D)$ we can consider operators Q^+, Q^-, F^+_Q, F^-_Q and sets $\sum^0(Q)$, $\sum^+(Q)$, $\sum^-(Q)$.

Theorem II. If $\sum^+(Q)\cup\sum^0(Q)\supset\sum^+(P)\cup\sum^0(P)$ and $\sum^-(Q)\cup\sum^0(Q)\supset\sum^-(P)\cup\sum^0(P)$, then we can take $F^+=F^+_Q$ and $F^-=F^-_Q$.

Proof. If Proposition 7.1 applies to (x^{jk}, ξ^k) and operator P, then the same applies to the operator Q. If Proposition 7.2 applies to (x^{jk}, ξ^k) and P, then we have $(\dot{x}, \xi)\in\sum^0(P)\subset\sum^0(Q)$. If Proposition 7.2 does not apply to (x^{jk}, ξ^k) and Q, then $(\dot{x}, \xi)$ satisfies $q_0(\dot{x}, \xi)=0$, $\text{grad}_{x,\xi}q_0(\dot{x}, \xi)=0$. Proposition 7.3 can be applied to this case and we come to the conclusion that we may take $Q_{jk}=0$ and the operator $E^{\pm}_{jk}$ does not matter so far as Q is concerned.

Acknowledgement:

The original manuscript of the author fallaciously asserted that operators $\phi'_{jk}(x, D)$, $\phi^{(l)}_{jk}(x, D)$, $l=1, 2, 3$, in Propositions 6.2 and 6.3 could be replaced by $\phi_{jk}(x, D)$ itself. This error was pointed out by the editors. The author expresses his hearty thanks to the editors.

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