Picco, D.J. and Platzeck, M.I.
Osaka J. Math.
11 (1974), 15–21

ON FREE ABELIAN EXTENSIONS

D.J. PICCO AND M.I. PLATZECK

(Received August 9, 1973)

Introduction. Let R be a commutative ring, and G a finite abelian group. In [2] (see also [5]) the set of isomorphism classes of Galois extensions of R with group G that have normal bases is described cohomologically by means of Harrison's complex of RG; to this end, Galois algebras are first classified and then Galois extensions with normal basis as a particular case. In this paper we use a different approach to classify Galois extensions which are free as R-modules; the restriction of this classification to extensions with normal basis yields the cohomological description of [2].

Free Abelian Extensions

Let R be a commutative ring, and G a finite abelian group. Recall that a faithful R-algebra A is said to be a Galois extension of R with respect to a representation of G by R-algebra automorphisms of A if the following equivalent conditions are satisfied:

1) $A^G=R$ and the map M_A from $A\otimes_R A$ to the ring of functions from G to A defined by $M_A\,(x\otimes y)\,(\sigma)=x\sigma(y)$ is an R-module isomorphism.

2) $A^G=R$, A is a finitely generated projective R-module and $L{:}AG\rightarrow\mathrm{End}_R(A)$ is an R-algebra isomorphism, where AG is the twisted group ring of G over A and L is defined by $L(a\sigma)(x)=a\sigma(x)$.

Let E denote the ring of functions from G to R; if we let G act on E by means of $(\sigma f)\,(\eta)=f(\sigma^{-1}\eta)$ then E is Galois over R with group G; we have $E=\oplus Re_\sigma$ with $\sum e_\sigma=1$, $e_\sigma e_\eta=\delta_{\sigma,\eta}\,e_\sigma$ and $\sigma(e_\eta)=e_{\sigma\eta}$. Clearly the condition 1) can be reformulated as follows:

3) $A^G=R$ and M_A: $A\otimes A\rightarrow E\otimes A$ defined by $M_A(x\otimes y)=\sum e_\sigma\otimes x\sigma(y)$ is an R-module isomorphism.

Note that for $M=M_E$: $E\otimes E\rightarrow E\otimes E$ we have $M(e_\alpha\otimes e_\beta)=e_{\alpha\beta^{-1}}\otimes e_\alpha$. Since $EG\simeq\mathrm{End}_R(E)$ we have $EG\otimes EG\simeq\mathrm{End}_R\,(E\otimes E)$; thus considering $E\otimes E$ as a left module over $EG\otimes EG$, the R-module automorphisms of $E\otimes E$ are produced by left multiplications by units of $EG\otimes EG$.

Suppose the Galois extension A is free as an R-module. Then there exists an R-module isomorphism j: $A\rightarrow E$ and $M^{-1}\cdot 1\otimes j\cdot M_A\cdot j^{-1}\otimes j^{-1}$: $E\otimes E\rightarrow E\otimes E$ is an isomorphism of R-modules. Therefore there exists a unique $u\in U(EG\otimes EG)$

such that $M^{-1}\cdot 1\otimes j\cdot M_A\cdot j^{-1}\otimes j^{-1}=L(u)$, that is, there is a unique u such that the diagram

$$\begin{tikzcd} A\otimes A \arrow[r, "M_A"] \arrow[dd, "j\otimes j"'] & E\otimes A \arrow[dr, "1\otimes j"] & \\ & & E\otimes E \\ E\otimes E \arrow[r, "L(u)"] & E\otimes E \arrow[ur, "M"'] & \end{tikzcd}$$

is commutative. In particular if $A=E$ and j is the identity then $u=1$.

Let A, A' be Galois extensions, $j\colon A\to E$, $j'\colon A'\to E$ R-module isomorphisms, and $f\colon A'\to A$ an isomorphism of Galois extensions. Since $jfj'^{-1}\colon E\to E$ is an R-isomorphism, there exists $v\in U(EG)$ auch that $jfj'^{-1}=L(v)$. If $u, u'\in U(EG\otimes EG)$ are obtained from j and j' respectively, if follows from $1\otimes f.M_{A'}=M_{A'}.f\otimes f$ that

$$(2)\quad L(u)\cdot L(v\otimes v) = M^{-1}\cdot 1\otimes L(v)\cdot M\cdot L(u')\,.$$

Let us now define R-algebra homomorphisms Δ_0, Δ_1, $\Delta_2\colon EG\to EG\otimes EG$ by means of $\Delta_0(x)=1\otimes x$, $\Delta_2(x)=x\otimes 1$ and $\Delta_1(\sum a_\sigma\sigma)=\sum a_\sigma\sigma\otimes\sigma$. Then $M^{-1}\cdot 1\otimes L(v)\cdot M=L(\Delta_1(v))$ and (2) becomes

$$(3)\quad u\cdot\Delta_0(v)\Delta_2(v) = \Delta_1(v)u'.$$

Note that if $u\in U(EG\otimes EG)$ is obtained from a Galois extension A by means of $j\colon A\to E$ and $v\in U(EG)$ then the element of $U(EG\otimes EG)$ obtained from $j'=L(v^{-1})j\colon A\to E$ is $u'=\Delta_1(v^{-1})\cdot u\cdot v\otimes v$.

Consider the following relation in $U(EG\otimes EG)$: if u, $u'\in U(EG\otimes EG)$ then $u\sim u'$ if there exists $v\in U(EG)$ such that $\Delta_1(v)\cdot u'=u\cdot v\otimes v$. It is easy to verify that this is an equivalence relation. Let $\boldsymbol{C}$ be the quotient set. (The previous remark shows that if $u\in U(EG\otimes EG)$ is obtained from a Galois extension, then every element in its equivalence class is obtained from the same extension.)

Let $E_G^f(R)$ be the set of Galois isomorphism classes $[A]$ of Galois extensions A of R with group G that are free as R-modules. The above construction defines a map $\psi\colon E_G^f(R)\to\boldsymbol{C}$ and we have:

Proposition 1. *$\psi\colon E_G^f(R)\to\boldsymbol{C}$ is injective.*

Proof. Let $[A]$, $[B]\in E_G^f(R)$, $j\colon A\to E$, $j'\colon B\to E$ and u, $u'\in U(EG\otimes EG)$ the units associated to A and B by means of j and j' respectively. Assume $u\sim u'$. As remarked before, we may suppose $u=u'$. We then have a commutative diagram

$$\begin{tikzcd} A\otimes A \arrow[r, "M_A"] \arrow[d, "j\otimes j"'] & E\otimes A \arrow[dr, "1\otimes j"] & \\ E\otimes E \arrow[r, "L(u)"] & E\otimes E \arrow[r, "M"] & E\otimes E \\ B\otimes B \arrow[u, "j'\otimes j'"] \arrow[r, "M_B"] & E\otimes B \arrow[ur, "1\otimes j'"'] & \end{tikzcd}$$

If $f=j'^{-1}\cdot j$ then the commutativity of the diagram

$$\begin{array}{ccccc} & A\otimes A & \xrightarrow{M_A} & E\otimes A & \\ f\otimes f & \downarrow & & \downarrow & 1\otimes f \\ & B\otimes B & \xrightarrow[]{M_B} & E\otimes B & \end{array}$$

shows that $f(x)\sigma f(y)=f(x\sigma(y))\ \forall\ x,\ y\in A,\ \sigma\in G$. Thus f is an R-algebra isomorphism. In particular, $f(1)=1$ and $f(\sigma(y))=\sigma f(y)$; then $f\colon A\to B$ is an isomorphism of Galois extensions.

Let us now determine the image of ψ. Let A be a free Galois extension of R with group G and $j\colon A\to E$ an R-module isomorphism. Then the diagram

$$(3)\quad \begin{array}{ccccc} A\otimes A\otimes A & \xrightarrow{1\otimes M_A} & A\otimes E\otimes A & \xrightarrow{1\otimes p} & A\otimes A\otimes E \\ {\scriptstyle M_A\otimes 1}\Big\downarrow & & & & \\ E\otimes A\otimes A & \xrightarrow[1\otimes M_A]{} & E\otimes E\otimes A & \xrightarrow[1\otimes 1\otimes j]{} & E\otimes E\otimes E \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{M_A\otimes 1} & E\otimes A\otimes E & \xrightarrow{1\otimes p} & E\otimes E\otimes A & \\ & & & \Big\downarrow & {\scriptstyle 1\otimes 1\otimes j} \\ & \xrightarrow[h]{\qquad\qquad\qquad} & & E\otimes E\otimes E & \end{array}$$

is commutative, where $p\colon X\otimes Y\to Y\otimes X$ is defined by $p(x\otimes y)=y\otimes x$ and $h\colon E\otimes E\otimes E\to E\otimes E\otimes E$ is defined by $h(e_\sigma\otimes e_\eta\otimes z)=e_\sigma\otimes e_{\sigma^{-1}\eta}\otimes z$. Let u be an element of $U(EG\otimes EG)$ making diagram (1) commutative. Then $M_A=1\otimes j^{-1}\cdot M\cdot L(u)\cdot j\otimes j$ and it follows from $h\cdot 1\otimes 1\otimes j\cdot 1\otimes M_A\cdot M_A\otimes 1=1\otimes 1\otimes j\cdot 1\otimes p\cdot M_A\otimes 1\cdot 1\otimes p\cdot 1\otimes M_A$ that

$$(4)\quad \begin{aligned}&[M^{-1}\otimes 1\cdot 1\otimes L(u)\cdot M\otimes 1]\cdot L(u\otimes 1)\\ &=[M^{-1}\otimes 1\cdot 1\otimes M^{-1}\cdot h^{-1}\cdot 1\otimes p\cdot M\otimes 1\cdot L(u\otimes 1)\cdot 1\otimes p\cdot 1\otimes M]\cdot L(1\otimes u)\end{aligned}$$

Consider now $\Delta_i\colon EG\otimes EG\to EG\otimes EG\otimes EG$ dened by $\Delta_0(x)=1\otimes x$, $\Delta_3(x)=x\otimes 1$, $\Delta_1(x\otimes y)=\Delta_1(x)\otimes y$, $\Delta_2(x\otimes y)=x\otimes\Delta_1(y)$. Then the Δ_i's are R-algebra homomorphisms and we can verify that

$$\begin{aligned}L(\Delta_1(u)) &= M^{-1}\otimes 1\cdot 1\otimes L(u)\cdot M\otimes 1\,,\\ L(\Delta_2(u)) &= M^{-1}\otimes 1\cdot 1\otimes M^{-1}\cdot h^{-1}\cdot 1\otimes p\cdot M1\cdot L(u\otimes 1)\cdot 1\otimes p\cdot 1\otimes M\,.\end{aligned}$$

Thus the relation (4) is equivalent to

$$(5)\quad \Delta_1(u)\ \Delta_3(u)=\Delta_2(u)\ \Delta_0(u).$$

On the other side, if $1_A\in A$ is the identity element of A, we have $M_A(x\otimes 1_A)=\sum_\sigma e_\sigma\otimes x=1\otimes x$ and therefore $L(u)\ (j(x)\otimes j(1_A))=M^{-1}(1\otimes j(x))=j(x)\otimes 1$ for every $x\in A$. Then $u(x\otimes j(1_A))=x\otimes 1$ for every $x\in E$; therefore $1\otimes j(1_A)=u^{-1}(1\otimes 1)$ and if $m\colon E\otimes E\to E$ is the product map, it follows that $j(1_A)=m(u^{-1}(1\otimes 1))$. Thus

$$(6)\qquad u(x\otimes m(u^{-1}(1\otimes 1))) = x\otimes 1 \quad \text{for every } x\in E\,.$$

Suppose now $u\in U(EG\otimes EG)$ verifies relations (5) and (6). We shall construct a Galois extension A with $[A]\in E_G^f(R)$ such that $\psi([A])$ is the class of u. Consider $ML(u)\colon E\otimes E\to E\otimes E$; if $t\in E\otimes E$ we have $ML(u)(t)=\sum_\sigma e_\sigma\otimes t_\sigma$ with $t_\sigma\in E$ uniquely determined. Given $x, y\in E$, we set

$$x * y=(x\otimes y)_1$$

(that is, $x*y$ is the coefficient of $e_1\otimes 1$ in $ML(u)(x\otimes y)$). Since $ML(u)$ is additive, it follows that $(x, y)\to x * y$ is a distributive product.

Let $1_A = m(u^{-1}(1\otimes 1))\in E$. Then $ML(u)(x\otimes 1_A)=M(x\otimes 1)=1\otimes x=\sum_\sigma e_\sigma\otimes x$, that is

$$(7)\qquad x * 1'=x \qquad \text{for every } x\in E.$$

For every $\sigma\in G$, let $\tilde\sigma\colon E\to E$ be defined by $\tilde\sigma(x)=(1'\otimes x)_\sigma$. Then $ML(u)(1'\otimes x)=\sum_\sigma e_\sigma\otimes\tilde\sigma(x)$. Since $ML(u)(1'\otimes 1')=1\otimes 1'=\sum_\sigma e_\sigma\otimes 1'$, we have $\tilde\sigma(1')=1'$ for all $\sigma\in G$.

Let us now show that

$$(8)\qquad ML(u)(x\otimes y)=\sum_\sigma e_\sigma\otimes(x * \tilde\sigma(y)).$$

Since u verifies (5) and therefore (4), we have

$h\cdot 1\otimes M\cdot 1\otimes L(u)\cdot M\otimes 1\cdot L(u\otimes 1)=1\otimes p\cdot M\otimes 1\cdot L(u)\otimes 1\cdot 1\otimes p\cdot 1\otimes M\cdot 1\otimes L(u)$. Applying this functions to $x\otimes 1'\otimes y$, $x, y\in E$, we obtain

$$\sum_{\sigma,\eta} e_\sigma\otimes e_{\sigma^{-1}\eta}\otimes(x\otimes y)_\eta=\sum_{\sigma,\eta} e_\sigma\otimes e_\eta\otimes(x\otimes\tilde\eta(y))_\sigma$$

and therefore $(x\otimes y)_\eta=(x\otimes\tilde\eta(y))_1=x*\tilde\eta(y)$.

If we apply the same relation to $x\otimes y\otimes z$ and use (8), then we obtain

$$\sum_{\sigma,\eta} e_\sigma\otimes e_{\sigma^{-1}\eta}(x*\tilde\sigma(y))*\tilde\eta(z)=\sum_{\sigma,\eta} e_\sigma\otimes e_\eta\otimes x*\tilde\sigma(y*\tilde\eta(z))$$

and it follows that

$$(9)\qquad x*\tilde\sigma(y*\tilde\tau(z)) = (x*\tilde\sigma(y))*\widetilde{\sigma\tau}(z)\,.$$

In particular, $x*\tilde\sigma(1'*\tilde 1(z))=(x*\tilde\sigma(1'))*\tilde\sigma(z)$. But (8) implies $1'\cdot z=1'\cdot\tilde 1(z)$ and $x*\tilde\sigma(1')=x*1'=x$ by (7); then $1'*\tilde\sigma(1'*z)=1'*\tilde\sigma(z)$. Thus $ML(u)(1'\otimes z)=$

$\sum_\sigma e_\sigma \otimes (1' * \tilde{\sigma}(z)) = \sum_\sigma e_\sigma \otimes (1' * \tilde{\sigma}(1' * z)) = ML(u)(1' \otimes (1' * z))$; therefore $1' \otimes z = 1' \otimes (1' * z)$. Thus $z \otimes 1' = (1' * z) \otimes 1'$ and from (7) we obtain $1' * z = z$ for all $z \in E$; therefore $1'$ is the identity element for the product $(x, y) \to x * y$. Moreover $z = 1' * z = \tilde{1}' * 1(z)$ and then $\tilde{1}(z) = z$ for every $z \in E$. Now (9) shows the associativity of the product, and we have $\sigma(x * y) = \tilde{\sigma}(x) * \tilde{\sigma}(y)$ and $\tilde{\sigma}\tilde{\tau}(z) = \widetilde{\sigma\tau}(z)$. If $r, s \in R$ then $ML(u)\ (r1' \otimes s1') = ML(u)\ (1' \otimes rs1') = \sum_\sigma e_\sigma \otimes (1' * \tilde{\sigma}(rs1')) = \sum e_\sigma \otimes \tilde{\sigma}(rs1')$, and so we have $r1' * s1' = \tilde{1}(rs1') = rs1'$. Since $r1' * x = x * r1' = rx$, the map $R \to E$ $(r \to r1')$ defines on E with product $*$ *a* structure of R-algebra, which we shall denote by A. Also $\sum_\sigma e_\sigma \otimes \tilde{\sigma}(rs1') = ML(u)\ (r1' \otimes s1') = ML(u)\ (rs1' \otimes 1') = \sum_\sigma e_\sigma \otimes rs1' * \tilde{\sigma}(1') = \sum_\sigma e_\sigma \otimes rs1'$ and therefore $\tilde{\sigma}(r1') = r1'$, $\forall \sigma \in \mathrm{G}$, $r \in R$. Thus G acts on A as a group of R-algebra automorphisms. Note that $r1' * x = x * r1' = rx$ implies that A is free over $R1'$. If $x \in A^G$ we have $ML(u)\ (1' \otimes x) = \sum_\sigma e_\sigma \otimes x = 1 \otimes x = ML(u)\ (x \otimes 1')$. Therefore $1' \otimes x = x \otimes 1'$, and we must have $x \in R1'$, thus $A^G = R1'$.

Since M_A: $A \otimes A \to E \otimes A$ is $ML(u)$ by (8), we conclude that A is Galois extension of R with group G. Clearly the diagram (1) with $j =$ identity shows that ψ (class of A) is the class of u.

We have remarked that if $u \in U(EG \otimes EG)$ is obtained from a Galois extension by means of the diagram (1) then every $v \in U(EG \otimes EG)$ equivalent to u is obtained in that way. It follows therefore that if u verifies (5) and (6) and v is equivalent to u then v also satisfies (5) and (6). Let $\boldsymbol{H}$ be the subset of $\boldsymbol{C}$ of classes whose elements satisfy (5) and (6). Then we have

Theorem. ψ: $E_G^f(R) \to \boldsymbol{H}$ *is bijective.*

REMARKS.

1) Since ψ $([E])$=class of 1, we have ψ $([E]) = \{\Delta_1(v^{-1}).\ v \otimes v\colon v \in U(EG)\}$

2) The group structure of $U(EG \otimes EG)$ does not induce a group structure on $\boldsymbol{H}$; in fact, the inverse of an element verifying (5) and (6) does not necessarily verify (5) and (6), nor does the product of two such units. For example, if K is a field and $G = Z_2$, and $x = -e_\sigma + \sigma$, $x^{-1} = e_1 + \sigma$ then $y = \Delta_1(x) \cdot x^{-1} \otimes x^{-1}$ verifies (5) and (6) but y^{-1} does not; if $K = Z_2$ then $z = \Delta_1(x^{-1}) \cdot x \otimes x$ verifies (5) and (6) but z^2 does not.

3) If α is an automorphism of G and A is a Galois extension of R with group G, we can obtain a new structure of Galois extension A^α on A by defining $\tilde{\sigma}(a) = \alpha(\sigma)(a)$. If α: $E \to E$ is defined by $\alpha(e_\sigma) = e_{\alpha(\sigma)}$, and α: $EG \otimes EG \to EG \otimes EG$ by $\alpha(a\sigma \otimes b\eta) = \alpha(a)\alpha(\sigma) \otimes \alpha(\mathrm{b})\alpha(\eta)$, then $\psi(A^\alpha) = \alpha^{-1}\psi(A)$; more precisely, if u is obtained from $j: A \to E$, then $\alpha^{-1}(u)$ is obtained from $\alpha^{-1} j$: $A \to E$.

Abelian Extensions with normal basis.

If A is Galois over R with group G, we have a commutative diagram

$$\begin{array}{ccc} A\otimes A & \xrightarrow{M_A} & E\otimes A \\ \sigma\otimes\eta\big\downarrow & & \big\downarrow\sigma\eta^{-1}\otimes\sigma \\ A\otimes A & \xrightarrow{M_A} & E\otimes A \end{array}$$

for every $\sigma,\ \eta\in G$.

Let $j\colon A\to E$ be an R-module isomorphism, and $u\in U(EG\otimes EG)$ the corresponding element in diagram (1). Then it follows from $M_A\cdot\sigma\otimes\eta=\sigma\eta^{-1}\otimes\sigma\cdot M_A$ (and $M\sigma\otimes\eta=\sigma\eta^{-1}\otimes\sigma M$) that j is an RG-isomorphism if and only if $L(u)$ is an $RG\otimes RG$-isomorphism, *i.e.* if $u\in U(RG\otimes RG)$. Note that there exists an RG-isomorphism $A\to E$ if and only if A has a normal basis.

Let us recall the definition of Harrison's complex; if $(RG)^n=RG\otimes\cdots\otimes RG$ (n-times) we define $\Delta_i\colon (RG)^n\to(RG)^{n+1}$, $i=0,\ 1,\ \cdots,\ n+1$ by $\Delta_0(x)=1\otimes x$, $\Delta_{n+1}(x)=x\otimes 1$ and for $i=1,\ \cdots,\ n$, Δ_i is the map induced by $\Delta_i(\sigma_1\otimes\cdots\otimes\sigma_n)=\sigma_1\otimes\cdots\otimes\sigma_i\otimes\sigma_i\otimes\sigma_{i+1}\cdots\sigma_n$. Then the Δ_j's are algebra homomorphisms and therefore $\Delta_j\colon U(RG^n)\to U(RG^{n+1})$. Since RG is commutative, setting $\Delta(x)=\prod_{i=0}^{n+1}\Delta_i(x)^{(-1)^i}$ for $x\in U(RG)$ we obtain group homomorphisms $\Delta\colon U(RG^n)\to U(RG^{n+1})$ such that $\Delta\Delta=0$; we thus have a complex whose cohomology groups are denoted by $H^n(R,\ G)$, ([3]).

Note that the Δ_i's defined on RG and on $RG\otimes RG$ are the restrictions of the Δ_i's considered before; thus for $u\in(RG\otimes RG)$ the condition (5) is equivalent to $\Delta(u)=1$; also if $u,\ u'\in U(RG\otimes RG)$ then $u\sim u'$ is equivalent to $u'=u\Delta(v)$ for some $v\in U(RG)$. If it is known that for $u\in U(RG\otimes RG)$ (5) implies (6), it follows that there is a one to one correspondence from the set of Galois isomorphism classes of Galois extensions of R with group G that have normal basis onto $H^2(R,\ G)$. Now, let us show that (5) implies (6) for $u\in\ U(RG\otimes RG)$: If $\pi\colon RG\otimes RG\otimes RG\to RG\otimes RG$ is the map induced by $\pi(\sigma\otimes\eta\otimes\tau)=\sigma\otimes\tau$ then π is an R-algebra homomorphism and from $\Delta_2(u)\Delta_0\ (u)=\Delta_3(u)\ \Delta_1(u)$ we obtain $u\cdot\pi\Delta_0(u)=\pi\Delta_3(u)\cdot u$, and then $\pi\Delta_0(u^{-1})=\pi\Delta_3(u^{-1})$. If $u^{-1}=\sum_{\sigma,\eta}r_{\sigma,\eta}\sigma\otimes\eta$ we have $\sum_{\sigma,\eta}r_{\sigma,\eta}1\otimes\eta=\sum_{\sigma,\eta}r_{\sigma,\eta}\sigma\otimes 1$ and therefore for $x\in E$, $L(u^{-1})\ (x\otimes 1)=\sum_{\sigma,\eta}r_{\sigma,\eta}\sigma(x)\otimes 1=\sum_{\sigma,\eta}r_{\sigma,\eta}x\otimes 1=x\otimes(\sum_{\sigma,\eta}r_{\sigma,\eta})$. Since $\sum_{\sigma,\eta}r_{\sigma,\eta}=m\cdot u^{-1}(1\otimes 1)$, we have (6).

Recall now that the set $E_G(R)$ of Galois isomorphism classes of all Galois extensions of R with group G is a group, whose product is defined as follows: Let $g=\{\sigma,\ \sigma^{-1}\}\in G\times G$. If $A,\ B\in E_G(R)$ then $A.B=(A\otimes B)^g$ with $\sigma\in G$ acting on $A.B$ as the restriction of $\sigma\otimes 1$. (see [1], [4]). The subset $E_G^n(R)$ of extensions with normal basis is a subgroup of $E_G(R)$; indeed, if A and B have normal bases there exist RG-isomorphisms $j_A\colon A\to E,\ j_B\colon B\to E$, then $j_A\otimes j_B\colon (A\otimes B)^g\to(E\otimes E)^g$ and it is an RG-isomorphism. Since $t\colon E\to(E\otimes E)^g$ given by $t(e_\sigma)=\sum_\alpha e_{\sigma\alpha}\otimes e_{\alpha^{-1}}$ is a Galois isomorphism, we obtain that $j_{AB}=t^{-1}\cdot j_A\otimes i_B|_{A.B}$:

$A \cdot B \to E$ is an RG-isomorphism, *i.e.* $A \cdot B$ has a normal basis. On the other side, if $u, v \in U(RG \otimes RG)$ are the cocycles associated to j_A and j_B respectively, it is easy to verify that the cocylce associated to j_{AB} is $u.v$. Thus we have ([2], [4])

Proposition. *Let R be a commutative ring, and G a finite abelian group. Then there is an isomorphism* $\psi: E_G^n(R) \to H^2(R, G)$; *if A has a normal basis and* j: $A \to E$ *is an RG-isomorophism, then* $\psi([A])$ *is the cohomology class of the cocylce u defined by the diagram* (1).

UNIVERSIDAD NACIONAL DEL SUR, BAHIA BLANCA

References

[1] S. Chase, D. Harrison and A. Rosenberg: *Galois theory and cohomology of commutative rings,* Mem. Amer. Math. Soc. No. 52 (1965).
[2] S. Chase and A. Rosenberg: *A theorem of Harrison, Kummer theory and Galois algebras,* Nagoya Math. J. **27** (1966), 663–685.
[3] D. Harrison: *Abelian extensions of arbitrary fields,* Trans. Amer. Math. Soc. **106** (1963), 230–235.
[4] M. Orzech: *Cohomological description of Abelian Galois extensions,* Trans. Amer. Math. Soc. **137**(1969), 481–500.
[5] P. Wolf: Algebraische Theorie der Galoischen Algebren, Deutscher Verlag der Wissenschaften, Berlin, 1956.