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DISCONTINUOUS SUBGROUPS OF EXTENSIONS OF SEMI-SIMPLE LIE GROUPS

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Introduction

Let G be a semi-simple Lie group acting as a group of linear transformations on the vector space V, K a maximal compact subgroup of G, Γ a discrete subgroup of normalizing a lattice L in V and such that $\Gamma\backslash G$ has finite invariant measure. In this paper we investigate deformations of and cohomology groups attached to $\Gamma\cdot L\subset G\cdot V$, where $\cdot$ donotes semi-direct product. In fact, we give a local description of the space of homomorphisms of $\Gamma\cdot L$ into $G\cdot V$ topologized by compact-open topology, and compute $H^1(\Gamma.L, Ad)$ in certain cases. We also introduce the notion of $\Gamma\cdot L$-invariant form à la Matsushima-Murakami and prove a type decomposition theorem for harmonic forms on $G\cdot V/K$. A special case of this theorem was first proved by Kuga [6].

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1. Deformations of $\Gamma\cdot L$

Let H be a Lie group and Λ discrete subgroup of H. By a *deformation* of Λ in H we mean a family r_t (t ranging over an open interval containing 0) of homomorphisms of Λ into H depending in a C^∞ fashion on t and such that r_0 is the canonical injection of Λ into H. Let $R(\lambda)$ be the tangent vector to $r_t(\lambda)$ at $t=0$. Then $Z: \lambda\to R(\lambda)\lambda^{-1}$ is a crossed homomorphism of Λ into the Lie algebra $\mathfrak{H}$ of H for the adjoint action of Λ on $\mathfrak{H}$. ($\mathfrak{H}$ is identified with the tangent space to H at e.) In fact, differentiating the relation $r_t(\lambda\cdot\lambda')=r_t(\lambda)r_t(\lambda')$ we obtain $R(\lambda\cdot\lambda')=R(\lambda)\lambda'+\lambda R(\lambda')$. Upon right multiplication by $\lambda'^{-1}\lambda^{-1}$ we get:

$$R(\lambda\cdot\lambda')\lambda'^{-1}\lambda^{-1} = R(\lambda)\lambda^{-1}+\lambda(R(\lambda')\lambda'^{-1})\lambda^{-1}$$

Since $R(\lambda)\lambda^{-1}\in\mathfrak{H}$ we have proved the assertion. The mapping $Z: \lambda\to R(\lambda)\lambda^{-1}$ is called the *crossed homomorphism tangent to* the deformation r_t of Λ.

Two deformations r_t and s_t of Λ are *equivalent* if there is a smooth curve h_t in H such that $r_t(\lambda)=h_t s_t(\lambda)h_t^{-1}$ for all $\lambda\in\Lambda$ and sufficiently small t.

Lemma 1. *The crossed homomorphisms tangent to equivalent deformations differ by principal homomorphisms. Hence there is a natural map from equivalence classes of deformations of* Λ *into* $H^1(\Lambda, Ad)$.

We call $H^1_0(\Lambda, Ad)$ the image of this map.

Proof. Suppose $r_t(\lambda)=h_t s_t(\lambda)h_t^{-1}$. Then upon differentiating we obtain

$$R(\lambda) = P\cdot\lambda+S(\lambda)-\lambda\cdot p$$

hence $R(\lambda)\lambda^{-1}=P-Ad(\lambda)P+S(\lambda)\lambda^{-1}$ where P is the tangent vector to h_t at e.

q.e.d.

Let G be a connected semi-simple real Lie group without compact factors, ρ a representation of G on the finite dimensional real vector space V, Γ a finitely generated discrete subgroup of G such that G/Γ has finite invariant measure, L a lattice in V normalized by Γ, $\mathfrak{G}$ and $\mathfrak{V}$ Lie algebras of G and V, respectively. Clearly $\Gamma\cdot L$ is a discrete subgroup of the semi-direct product $G\cdot V$ of G and V defined via ρ.

Theorem 1. *Under the assumption* $H^1(\Gamma, V)=H^1(\Gamma, G)=0$, *every equivalence class of deformations of* $\Gamma\cdot L$ *in* $G\cdot V$ *is represented by a deformation of the form* $s_t(\gamma, l)=(\gamma, l+S_t(l))$ *where* S_t *is an element of* $Z(\rho(G))$ *depending differentially on* $t\in(-\alpha, \alpha)$ *and* $S_0=0$, *where* $Z(\rho(G))$ *is the centralizer of* $\rho(G)$ *in the full matrix algebra over* V. *In this way, one has a bijection between germs around* 0 *of* C^∞ *maps* S: $(-\alpha, \alpha)\to Z(\rho(G))$ *such that* $S_0=0$ *and equivalence classes of deformations of* $\Gamma\cdot L$ *in* $G\cdot V$.

Proof. We begin with a lemma

Lemma 2. *Let* r_t *be a deformation of* $\Gamma\cdot L$ *in* $G\cdot V$. *Then* $r_t(L)\subset V$ *for* t *sufficiently small.*

Proof of Lemma 2. Since $H^1(\Gamma, \mathfrak{G})=H^1(\Gamma, V)=0$ there is a curve (g_t, v_t) in $G\cdot V$ such that $r_t(\gamma)=(g_t, v_t)\gamma(g_t, v_t)^{-1}$ for every $\gamma\in\Gamma$ and sufficiently small t [12]. Put $s_t(\gamma, l)=(g_t, v_t)^{-1}r_t(\gamma, l)(g_t, v_t)$. Then one has $s_t(\gamma)=\gamma$. We can write

$$s_t(l) = (s^1_s(l), l+s^2_t(l))$$

It is enough to show that $s^1_t(l)=e$ for all $l\in L$ and small t. Since s_t is a homomorphism, $s_t(\gamma)s_t(l)s_t(\gamma)^{-1}=s_t(\rho(\gamma)l)$. Substituting $s_t(\gamma)=\gamma$ we obtain

$$\gamma s^1_t(l)\gamma^{-1} = s^1_t(\rho(\gamma)l) \qquad \gamma, l\in\Gamma, L$$

Furthermore, from $s_t(l+l^1)=s_t(l)s_t(l^1)$ we deduce $s^1_t(l+l^1)=s^1_t(l)s^1_t(l^1)$. The map ϕ: $L\to G$ defined by $\phi(l)=s^1_t(l)$ is thus a homomorphism. Consider the algebraic hull M of $\phi(L)$ in G. Since $\phi(L)$ is invariant under conjugations by elements of Γ, M is invariant under inner automorphisms by elements of Γ, and therefore by a theorem of Borel and Selberg Lie algebra of M is an ideal in G. Hence the connected component M_0 of M is a normal subgroup of G. Since L is abelian, M_0 is commutative contradicting semi-simplicity of G unless $M_0=\{e\}$. Hence M is discrete and $s_t^1(l)=e$ for t sufficiently small and for all $l\in L$ thereby proving the lemma.

It follows from this lemma that every deformation r_t of $\Gamma\cdot L$ is equivalent to a deformation s_t of $\Gamma\cdot L$ of the form $s_t(\gamma, l)=(\gamma, l+s_t^2(l))$. Then we check easily that $s_0^2=0$ and $s_t^2(l+l^1)=s_t^2(l)+s_t^2(l^1)$. Hence s_t^2 can be extended to a linear endomorphism S_t of V. Denoting the full matrix algebra over V by $M(V)$, we see that S_t can be regarded as a C^∞ map from $(-\alpha, \alpha)$ into $M(V)$ with $S_0=0$. It is then trivial to verify that s_t is a homomorphism of $\Gamma\cdot L$ if and only if

$$\rho(\gamma)S_t=S_t\rho(\gamma) \qquad \text{for all } \gamma\in\Gamma$$

i.e., $S_t\in Z(\rho(\Gamma))=Z(\rho(G))$. [2]

It is readily checked that if q_t is another deformation of the same type, viz., $q_t(\gamma, l)=(\gamma, l+Q_t(l))$, then s_t and q_t are equivalent if and only if $S_t=Q_t$ for t sufficiently small. Hence there is a natural injection from equivalence classes of $\Gamma\cdot L$ into germs around 0 of S_t. Conversly, for every C^∞ map S: $(-\alpha, \alpha)\to Z(\rho(G))$ with $S_0=0$, set $s_t(\gamma, l)=(\gamma, l+S_t(l))$. Then it is readily verified that s_t is a homomorphism of $\Gamma\cdot L$ into $G\cdot V$ for $t\in(-\alpha, \alpha)$, thus showing that the above map is actually a bijection.

Corollary 1. $H_0^1(\Gamma\cdot L, Ad)\cong Z(\rho(G))$.

Let $\mathcal{R}(\Gamma\cdot L)$ denote the space of homomorphisms of $\Gamma\cdot L$ into $G\cdot V$ topologized by compact-open topology, and $i\in\mathcal{R}(\Gamma\cdot L)$ the canonical injection of $\Gamma\cdot L$ into $G\cdot V$. Two homomorphisms of $\Gamma\cdot L$ into $G\cdot V$ are called *equivalent* if they differ by an inner automorphism of $G\cdot V$. Then by a similar argument as above we can actually prove the following result:(1)

Theorem 1′. *Under the assumption $H^1(\Gamma, G)=H^1(\Gamma, V)=0$, there exists an open neighborhood U of i in $\mathfrak{R}(\Gamma\cdot L)$ such that every homomorphism $r\in U$ is equivalent to a homomorphism s of the form*

$$s(\gamma, l)=(\gamma, l+S(l))$$

where S is in an open neighborhood of zero in $Z(\rho(G))$.

This implies that i has an arc-wise connected neighborhood.

(1) We first prove an analogue of lemma 2 by replacing "r_t" with "r" and "t sufficiently small" with "r sufficiently close to i."

Corollary 2. *Under the condition $H^1(\Gamma, G)=H^1(\Gamma, V)=0$, i has a neighborhood in $\mathfrak{R}(\Gamma\cdot L)$ homeomorphic to R^{k+n+m} where $k=\dim G$, $n=\dim V-\dim V_0$, $m=\dim Z(\rho(G))$ and $V_0=\{v\in V \mid \rho(g)v=v$ for all $g\in G\}$.*

Proof. Let $Z^x(\rho(G))$ denote the centralizer of $\rho(G)$ in $Gl(V)$, and $\tilde{G}$ be the semi-direct product of $Z^x(\rho(G))$ and $G\cdot V$, where the action of $Z(\rho(G))^x$ on $G\cdot V$ is given by $R(g, v)=(g, R(v))$ for $R\in Z^x(\rho(G))$. Then $\tilde{G}=Z^x(\rho(G))\cdot G\cdot V$ acts on $G\cdot V$ as a group of automorphisms of $G\cdot V$ as follows:

$$(1)\qquad \begin{aligned}(R, g, v)[(h, u)] &= (g, v)[R(h, u)](g, v)^{-1}\\ &= (ghg^{-1}, v+\rho(g)R(u)-\rho(ghg^{-1})v)\end{aligned}$$

for $(h, u)\in G\cdot V$. Denote this automorphism of $G\cdot V$ by $\sigma_{(R,g,v)}$.(2)

By choosing the neighborhood U of i provided by theorem $1'$ sufficiently small we can assume that $(I+S)\in Z^x(\rho(G))$. In fact, it suffices to show $(I+S)(L)$ or equivalently $r(L)$ is a lattice in V. In view of footnote (1) the latter condition is satisfied for all r sufficiently close to i. Since, in the notation of theorem $1'$, r and s differ by an inner automorphism of $G\cdot V$, there is (g_1, v_1) in $G\cdot V$ such that

$$(2)\qquad \begin{aligned}r(\gamma, l) &= (g_1, v_1)s(\gamma, l)(g_1, v_1)^{-1}\\ &= (g_1\gamma g_1^{-1}, v_1+\rho(g_1)(I+S)(l)-\rho(g_1\gamma g_1^{-1})(v_1)).\end{aligned}$$

The group $\tilde{G}$ acts on $\mathfrak{R}(\Gamma\cdot L)$ by $r\to\sigma_{(R,g,v)}\cdot r$. By virtue of (1), (2) and the fact that $I+S\in Z^x(\rho(G))$ if U is sufficiently small, the orbit $\tilde{G}(i)$ of i contains neighborhood of i in $\mathfrak{R}(\Gamma\cdot L)$. It follows that $\tilde{G}(i)$ is an open subset of $\mathfrak{R}(\Gamma\cdot L)$.

Let $C(G)$ denote the center of G which is a discrete subgroup of G. Then we have

Lemma 3. *The isotropy subgroup $\tilde{G}_i$ of $\tilde{G}$ at i is isomorphic to $C(G)\times V_0$.*

Proof of Lemma 3. Setting $h=\gamma$ and $u=l$ in (1), one sees that if $(R, g, v)\in\tilde{G}_i$ then $g\gamma g^{-1}=\gamma$ for all $\gamma\in\Gamma$ and therefore $g\in C(G)$. Then, setting $l=0$, we have $v-\rho(\gamma)v=0$ for all $\gamma\in\Gamma$ and hence $v=\rho(g)v$ for all $g\in G$, that is, $v\in V_0$. Finally, for $g\in C(G)$, $v\in V_0$, the condition $\sigma_{(R,g,v)}(\gamma, l)=(\gamma, l)$ implies $\rho(g)R(l)=l$ for all $l\in L$, i.e., $R=\rho(g)^{-1}$. Conversely, it is trivial to check that $(\rho(g)^{-1}, g, v)\in\tilde{G}_i$ if $g\in C(G)$ and $v\in V_0$. q.e.d.

REMARK. $\tilde{G}_i$ coincides with the center of $\tilde{G}$.

We have thus constructed an injective continuous map

(2) One can actually prove that the connected component of the unit element of the full automorphism group of $G\cdot V$ is given by

$$\{\sigma_{(R,g,v)} \mid (R, g, v)\in\tilde{G}\}$$

$$\psi\colon \tilde{G}/\tilde{G}_i \to \tilde{G}(i) \subset \mathfrak{R}(\Gamma\cdot L)$$

Suppose $\Gamma\cdot L$ has a set of p generators. Then $\mathcal{R}(\Gamma\cdot L)$ can be identified with the real subvariety of $(G\cdot V)^p$ defined by the set of relations between generators of $\Gamma\cdot L$. Hence $\mathcal{R}(\Gamma\cdot L)$ and therefore $\tilde{G}(i)$ is locally compact. This ensures that ψ is actually a homeomorphism of $\tilde{G}/\tilde{G}_i$ onto $\tilde{G}(i)$ and completes the proof of corollary 2.

2. The cohomology group $H^1(\Gamma\cdot L, Ad)$

Theorem 2. *Under the assumption $H^1(\Gamma, \mathfrak{G})=H^1(\Gamma, \mathfrak{V})=0$, one has*

$$H^1(\Gamma\cdot L, Ad) \simeq Z(\rho(G))$$

We first introduce some notation. Let Z be a crossed homomorphism of L into $\mathfrak{G}+\mathfrak{V}$. We say $Z\in Z^1_\Gamma$ if there exists a crossed homomorphism X of Γ into $\mathfrak{G}+\mathfrak{V}$ such that

$$Z(\rho(\gamma)l)-Ad(\gamma)Z(l) = X(\gamma)-Ad(\rho(\gamma)l)X(\gamma) \tag{3}$$

for all $\gamma, l\in\Gamma, L$. Let $Z\colon L\to\mathfrak{G}+\mathfrak{V}$ be the principal homomorphism, i.e., $Z(l)=R-Ad(l)R$. Then it is readily checked that Z satisfies (3) with $X(\gamma)=R-Ad(\gamma)R$. Therefore the group of principal homomorphisms B^1 is contained in Z^1_Γ and we set

$$H^1_\Gamma(L, \mathfrak{G}+\mathfrak{V}) = Z^1_\Gamma/B^1$$

Lemma 4. *Let $X\in\mathfrak{G}+\mathfrak{V}$. Then $X-Ad(\gamma)X-Ad(l)X+Ad(l\cdot\gamma)X=0$ for all γ, $l\in\Gamma$, L implies $X\in\mathfrak{G}_\rho+\mathfrak{V}$, where $\mathfrak{G}_\rho$ denotes the Lie algebra of kernel of ρ.*

Proof of Lemma 4. Let $l=\exp Y$, $Y\in\mathfrak{V}$, and $R\in\mathfrak{G}+\mathfrak{V}$. Then

$$Ad(l)R = R+[Y, R]+\cdots \text{ higher terms.}$$

Since $\mathfrak{V}$ is commutative and $[Y, R]\in\mathfrak{V}$, all higher terms vanish and we have

$$Ad(l)R = R+[Y, R] \tag{4}$$

In particular, let $R=X-Ad(\gamma)X$. Then the hypothesis implies $Ad(l)R=R$, hence $[Y, R]=0$ for all $Y\in\mathfrak{V}$. Let $X=X_1+X_2$ with $X_1\in\mathfrak{G}$ and $X_2\in\mathfrak{V}$. Since $\mathfrak{G}$ and $\mathfrak{V}$ are invariant under the adjoint action of Γ, $[Y, R]=0$ implies $[X_1-Ad(\gamma)X_1, Y]=0$ for all $Y\in\mathfrak{V}$. Since $[X_1, Y]=\rho(X_1)Y$, one has $\rho(X_1)=\rho(Ad(\gamma)X_1)=\rho(\gamma)\rho(X_1)\rho(\gamma)^{-1}$ for all $\gamma\in\Gamma$. By virtue of a theorem of Borel and Selberg [2] $\rho(X_1)$ belongs to the center of $\rho(G)$, that is, $\rho(X_1)=0$ or $X_1\in G_\rho$. Hence $X\in\mathfrak{G}_\rho+\mathfrak{V}$.

Lemma 5. *Every $Z\in Z^1_\Gamma$ is cohomologous to a unique $Y\in Z^1_\Gamma$ satisfying*

$$(5) \qquad Y(\rho(\gamma)l) = Ad(\gamma)Y(l)$$

for all $\gamma, l \in \Gamma, L$.

Proof of Lemma 5. Since $Z \in Z^1_\Gamma$ there is a crossed homomorphism X of Γ into $\mathfrak{G}+\mathfrak{V}$ such that (3) is satisfied. In view of vanishing of $H^1(\Gamma, \mathfrak{G}+\mathfrak{V})$ one may write $X(\gamma)=R-Ad(\gamma)R$. Define $Y\colon L \to \mathfrak{G}+\mathfrak{V}$ by $Y(l)=Z(l)-R+Ad(l)R$ and substitute in (3) to obtain $Y(\rho(\gamma)l)=Ad(\gamma)Y(l)$.

To prove uniqueness let Y' be another candidate. Then $Y-Y'$ is a coboundary, so $(Y-Y')(l)=R-Ad(l)R$ for some $R \in G+V$. Since Y and Y' satisfy (5) one has $(Y-Y')(l)=Ad(\gamma)(Y-Y')(\rho(\gamma)^{-1}l)$ and so

$$R-Ad(l)R = Ad(\gamma)R-Ad(l\cdot\gamma)R\,.$$

By Lemma 4, $R \in \mathfrak{G}_\rho+\mathfrak{V}$ and hence $(Y-Y')(l)=R-Ad(l)R=0$. In fact, if $l=\exp T$, $T \in V$, then by (4) $R-Ad(l)R=[R, T]=0$ since $R \in G_\rho+V$. q.e.d.

Lemma 6. $H^1_\Gamma(L, \mathfrak{G}+\mathfrak{V}) \simeq H^1(\Gamma\cdot L, \mathfrak{G}+\mathfrak{V})$.

Proof of Lemma 6. Let Z be a representative of an element of $H^1(\Gamma\cdot L, \mathfrak{G}+\mathfrak{V})$, then $Z|_L \in Z^1_\Gamma$. In fact, $Z(\gamma)+Ad(\gamma)Z(l)=Z(\gamma\cdot l)=Z(\rho(\gamma)l\cdot\gamma)$ $=Z(\rho(\gamma)l)+Ad(\rho(\gamma)l)Z(\gamma)$ so that (3) is satisfied with $X=Z|_L$. It is clear that $Z\sim 0$ implies $Z|_L\sim 0$. Therefore $Z\to Z|_L$ induces a homomorphism $\beta\colon H^1(\Gamma\cdot L, \mathfrak{G}+\mathfrak{V}) \to H^1_\Gamma(L, \mathfrak{G}+\mathfrak{V})$.

To prove injectivity of β, let $Z|_L\sim 0$. Since $H^1(\Gamma, \mathfrak{G}+\mathfrak{V})=0$, $Z|_\Gamma$ is also a principal homomorphism. Hence suppose $Z|_\Gamma(\gamma)=R-Ad(\gamma)R$ and $Z|_L(l)=R'-Ad(\gamma)R'$. Then

$$Z(l\cdot\gamma) = R'-Ad(l)R'+Ad(l)(R-Ad(\gamma)R)$$

We saw earlier that (3) is satisfied by every crossed homomorphism Z of $\Gamma\cdot L$ into $\mathfrak{G}+\mathfrak{V}$ with $X=Z|_\Gamma$. In our case, it reads as follows:

$$\begin{aligned} &R'-Ad(\rho(\gamma)l)R'-Ad(\gamma)R'+Ad(\gamma)Ad(l)R' \\ &\quad = R-Ad(\gamma)R-Ad(\rho(\gamma)l)R+Ad(\rho(\gamma)l)R \end{aligned}$$

for all $\gamma, l \in \Gamma, L$. Replacing l by $\rho(\gamma)^{-1}l$ and setting $R''=R-R'$ we obtain

$$R''-Ad(\gamma)R''-Ad(l)R''+Ad(l\cdot\gamma)R'' = 0$$

In view of Lemma 4, $R'' \in G_\rho+V$. Since L acts trivially on $\mathfrak{G}_\rho+\mathfrak{V}$ by adjoint representation, one has $R''-Ad(l)R''=0$ and hence

$$R'-Ad(l)R' = R-Ad(l)R$$

Therefore one has $Z(l\cdot\gamma)=R-Ad(l\cdot\gamma)R$ proving $Z\sim 0$. Hence β is an injective homomorphism.

To prove surjectivity of β, let $Z: L \to \mathfrak{G}+\mathfrak{V}$ be representative of a cohomology class in $H^1_\Gamma(L, \mathfrak{G}+\mathfrak{V})$. Then by lemma 5, there is a unique $Y \sim Z$ such that $Y(\rho(\gamma)l) = Ad(\gamma)Y(l)$ for all $\gamma, l \in \Gamma, L$. Now for $l\cdot\gamma \in \Gamma\cdot L$ set $\tilde{Y}(l\cdot\gamma) = Y(l)$. Then

$$\begin{aligned}\tilde{Y}((l\cdot\gamma)(l'\cdot\gamma')) &= \tilde{Y}((l+\rho(\gamma)l')(\gamma\gamma')) \\ &= Y(l+\rho(\gamma)l') \\ &= Y(l)+Ad(l)Y(\rho(\gamma)l') \\ &= Y(l)+Ad(l\cdot\gamma)Y(l') \\ &= \tilde{Y}(l\cdot\gamma)+Ad(l\cdot\gamma)\tilde{Y}(l'\cdot\gamma')\end{aligned}$$

Hence $\tilde{Y}$ is a crossed homomorphism. Clearly $\beta(\{\tilde{Y}\}) = \{Y\}$ where by $\{\tilde{Y}\}\{(Y)\}$ we mean cohomology class of $\tilde{Y}(Y)$ respectively, therefore β is surjective and proof of lemma 6 is complete.

Lemma 7. $H^1_\Gamma(L, \mathfrak{G}+\mathfrak{V}) \simeq Z(\rho(G))$.

Proof of Lemma 7. In view of lemma 5

$$H^1_\Gamma(L, \mathfrak{G}+\mathfrak{V}) = \{\text{crossed homomorphisms } Y: L \to \mathfrak{G}+\mathfrak{V} \text{ satisfying (5)}\}$$

Write $Y = Y_1 + Y_2$ where $Y_i = \pi_i \cdot Y$, π_i being projection of $\mathfrak{G}+\mathfrak{V}$ onto its i^{th} $(i=1,2)$ summand. Identifying $\mathfrak{V}$ and V via the exponential map, we can write using (4)

$$\begin{aligned}Ad(l)Y(l') &= Y(l')-[Y(l'), l] \\ &= Y(l')-[Y_1(l'), l] \\ &= Y(l')-\rho(Y_1(l'))l.\end{aligned}$$

Therefore we have

$$\begin{aligned}Y_1(l+l')+Y_2(l+l') &= Y(l+l') \\ &= Y(l)+Ad(l)Y(l') \\ &= Y_1(l)+Y_1(l')+Y_2(l)+Y_2(l')-\rho(Y_1(l'))l\end{aligned}$$

Hence

$$\begin{cases} Y_1(l+l') = Y_1(l)+Y_1(l'), \\ Y_2(l+l') = Y_2(l)+Y_2(l')-\rho(Y_1(l'))l. \end{cases}$$

Therefore Y_1 can be extended to a linear map of V into $\mathfrak{G}$ such that $Y_1(\rho(\gamma)v) = Ad(\gamma)Y_1(v)$ for all $\gamma, v \in \Gamma, V$. By a theorem of Borel and Selberg $Y(\rho_1(g)v) = Ad(g)Y_1(v)$ and consequently $Y_1(\rho(X)v) = ad(X)Y_1(v)$ for all $X \in G$. Therefore $Y_1 \in \mathrm{Hom}_G(V, \mathfrak{G})$ and im $Y_1 = \mathfrak{G}_1$ is an ideal in $\mathfrak{G}$. Now we have

$$\begin{aligned}Y_1(\rho(Y_1(l'))l) &= ad(Y_1)l'))Y_1(l) \\ &= [Y_1(l'), Y_1(l)]\end{aligned}$$

which is anti-symmetric in l and l'. But $\rho(Y_1(l'))l=-Y_2(l+l')+Y_2(l)+Y_2(l')$ is symmetric in l and l', therefore $Y_1(\rho(Y_1(l'))l)=[Y_1(l'), Y_1(l)]=0$, that is, $[\mathfrak{G}_1, \mathfrak{G}_1]=0$. Since $\mathfrak{G}_1$ is an ideal of semi-simple Lie algebra, commutativity of $\mathfrak{G}_1$ implies $\mathfrak{G}_1=0$. Hence $Y_1=0$ and

$$Y_2 = Y \in \mathrm{Hom}_\Gamma(V, V) = \mathrm{Hom}_G(V, V) = Z(\rho(G))$$

q.e.d.

Lemmas 6 and 7 complete the proof of Theorem 2.

In view of Theorems 1 and 2 every crossed homomorphism of $\Gamma\cdot L$ into $\mathfrak{G}+\mathfrak{V}$ is tangent to a deformation of $\Gamma\cdot L$ in $G\cdot V$ and we have

Corollary. $H_0^1(\Gamma\cdot L, Ad)=H^1(\Gamma\cdot L, Ad)$.

3. Cohomology of $\Gamma\cdot L$-invariant forms

Let be Γ a uniform discrete subgroup of a connected semi-simple real Lie group G, K a maximal compact subgroup of G, ρ and ψ finite dimensional real representations of G on V and W, respectively, and L a lattice in V normalized by Γ. It is convenient to assume Γ has no elements of finte order, that is, $\Gamma\cap gKg^{-1}=\{e\}$ for all $g\in G$. Since Γ and therefore $\Gamma\cdot L$ has no elements of finite order, $\Gamma\cdot L$ acts freely on $\tilde{X}=G\cdot V/K$, that is, $(\gamma, l)(\tilde{x})=(\gamma', l')(\tilde{x})$ for some $\tilde{x}\in\tilde{X}$ implies $(\gamma, l)=(\gamma', l')$, and the quotient space becomes a real analytic variety. $\Gamma\cdot L\backslash\tilde{X}$ has a natural structure of a fibre bundle over $\Gamma\backslash X$ $(X=G/K)$ with fibre isomorphic to $L\backslash V$. We have canonical isomorphisms $L_{l\cdot\gamma}$: $T^*_{\tilde{x}}(\tilde{X})\to T^*_{(l\cdot\gamma)(\tilde{x})}(\tilde{X})$ induced by the free action of $\Gamma\cdot L$ on $G\cdot V/K$.

We can choose an inner product $(,)$ on W such that $\psi(K)$ is contained in the orthogonal group on W relative to $(,)$. Now for each $x\in X$ define an inner product $(,)_x$ on W by $(u, v)_x=(\psi(g)u, \psi(g)v)$, where $x=gK$. It is easily verified that $(,)_x$ is independent of the choice of coset representative g, and its dependence on x is smooth. We similarly define an inner product $[,]_x$ on V for each $x\in X$. Then $[,]_x$ induces a flat Riemannian metric $B(x)$ on V (and therefore on $L\backslash V$) which depends smoothly on x. Finally, we define an invariant Riemannian metric on the vector bundle $T(X)\times T(V)\to\tilde{X}\approx X\times V$ by

$$ds^2 = ds_0^2+B(x)$$

where ds_0^2 is an invariant metric on X. Note that one has $\tilde{X}\approx X\times V$ by the correspondence $\tilde{x}=(g, v)\leftrightarrow(gK, \rho(g)v)$.

By a W-valued p-form on $\tilde{X}$ we mean a C^∞ section ω of the vector bundle $W\otimes\wedge^p T^*(\tilde{X})\to\tilde{X}$. We say ω is $\Gamma\cdot L$-invariant if

$$\omega\cdot(\gamma, l) = (\psi(\gamma)\otimes(\wedge^p\cdot L_{(\gamma,l)}))\omega$$

We denote the space of W-valued $\Gamma\cdot L$-invariant p-forms on $\tilde{X}$ by $\Omega^p(\tilde{X}, \Gamma\cdot L, W, \psi)$ or simply $\Omega^p(W)$ if no confusion arises.

For any vector spaces E and F one has canonical isomorphism

$$\sum_{i+j=r} \wedge^i E \otimes \wedge^j F \simeq \wedge^r(E\oplus F)\,.$$

Since $T^*(\tilde{X}) = T^*(X)\oplus T^*(V)$, it follows that we have a decomposition

$$\Omega^p(W) = \sum_{a+b=p} \Omega^{a,b}(W)$$

where $\Omega^{a,b}(W)$ is the subspace of $\Omega^p(W)$ consisting of C^∞ sections of $W\otimes \wedge^a T^*(X)\otimes \wedge^b T^*(V) \to \tilde{X}$ and $\wedge^a T^*(X)\otimes \wedge^b T^*(V)$ is identified with a subspace of $\wedge^p T^*(\tilde{X})$ by the isomorphism mentioned above.

The operator d (exterior differentiation) makes the diagram

$$\cdots \to \Omega^p(W) \overset{d}{\to} \Omega^{p+1}(W) \to \cdots$$

into a cochain complex. The cohomology groups of this cochain complex will be denoted by $H^p(\tilde{X}, \Gamma\cdot L, W, \psi)$. Since $\tilde{X}$ is contractible, it is known that $H^p(\tilde{X}, \Gamma\cdot L, W, \psi)$ is isomorphic to the cohomology group $H^p(\Gamma\cdot L, W)$ defined algebraically [4].

Let U be an open set in $X=G/K$ on which one has a positively oriented orthonormal basis $\{L_1, \cdots, L_k\}$ for $T_x(X)$, $x\in U$, depending smoothly on x. Then on $\tilde{U}=\pi^{-1}(U)$ (π denoting the canonical projection $\tilde{X}\to X$), one can obtain a positively oriented orthonormal basis $\{L_1, \cdots, L_k, L_{k+1}, \cdots, L_{k+n}\}$, or more precisely $\{L_1(\tilde{x}), \cdots, L_{k+n}(\tilde{x})\}$, for $T_{\tilde{x}}(\tilde{X})$, $\tilde{x}=(x, v)$, relative to metric $ds_0^2+B(x)$, depending smoothly on $\tilde{x}$ such that $\{L_1, \cdots, L_k\}$ and $\{L_{k+1}, \cdots, L_{k+n}\}$ span $T_x(X)$ and $T_v(V)$, respectively. Denote the basis dual to $\{L_1, \cdots, L_{k+n}\}$ by $\{w^1(x), \cdots, w^{k+n}(x)\}$ or simply $\{w^1, \cdots, w^{k+n}\}$. When one identifies $T_v(V)$ with V, one may assume that $L_1, \cdots, L_{k+n}$ are independent of v and depend smoothly on x. On $\tilde{U}$, an element $\omega\in\Omega^{a,b}(W)$ has an expression of the form

$$(6)\qquad \omega(\tilde{x}) = \sum_{i_1,\cdots,i_a} \sum_{j_1,\cdots,j_b} u_{i_1\cdots i_a j_1\cdots j_b}(\tilde{x})\otimes w^{i_1}\wedge\cdots\wedge w^{i_a}\otimes w^{j_1}\wedge\cdots\wedge w^{j_b}$$

where $u_{i_1\cdots i_a j_1\cdots j_b}(\tilde{x})\in W$, i's and j's range over $\{1, \cdots, k\}$ and $\{k+1, \cdots, k+n\}$, respectively. The operator $*$ is defined by

$$(7)\qquad (*\omega)(\tilde{x}) = \sum_{i_1,\cdots,i_a} \sum_{j_1,\cdots,j_b} \delta^{1\cdots k+n}_{i_1\cdots i_a j_1\cdots j_b i'_1\cdots i'_{k-a} j'_1\cdots j'_{n-b}} u_{i_1\cdots i_a j_1\cdots j_b}(\tilde{x}) \otimes w^{i'_1}\wedge\cdots\wedge w^{i'_{k-a}}\otimes w^{j'_1}\wedge\cdots\wedge w^{j'_{n-b}}$$

where $\{i_1, \cdots, i_a, i'_1, \cdots, i'_{k-a}\}$ (resp. $\{j_1, \cdots, j_b, j'_1, \cdots, j'_{n-b})\}$) coincide set-theoretically with $\{1, \cdots, k\}$ (resp. $\{k+1, \cdots, k+n\}$). One can show that the operator $*$ is independent of the choice of orthonormal basis $\{L_1, \cdots, L_{k+n}\}$.

The inner product $(\ ,\ )_x$ on W defines an isomorphism $W \to W^*$ which induces an isomorphism $\#: \Omega^{a,b}(W) \to \Omega^{a,b}(W^*) = \Omega^{b,a}(\tilde{X}, \Gamma \cdot L, W^*, {}^t\psi^{-1})$. Now let ∂ be the unique map which makes the following diagram commutative:

$$\begin{array}{ccc} \Omega^p(W) & \xrightarrow{\#} & \Omega^p(W^*) \\ \partial \big\downarrow & & \big\downarrow d \\ \Omega^{p+1}(W) & \xrightarrow{\#} & \Omega^{p+1}(W^*) \end{array}$$

We set for $\omega \in \Omega^p(W)$

$$\delta\omega = (-1)^{(k+n)(p+1)+1} * \partial * \omega$$

and

$$\Delta = d\delta + \delta d$$

A p-form ω is *harmonic* if $\Delta\omega = 0$. It can be shown that harmonicity of ω is equivalent to vanishing of $d\omega$ and $\delta\omega$. By Baily's generalization of Hodge's theorem

$$H^p(\tilde{X}, \Gamma \cdot L, W, \psi) \simeq \mathcal{H}^p(\tilde{X}, \Gamma \cdot L, W, \psi)$$

where $\mathcal{H}^p(\tilde{X}, \Gamma \cdot L, W, \psi)(\mathcal{H}^{a,b}(\tilde{X}, \Gamma \cdot L, W\psi))$ denotes the subspace of harmonic forms in $\Omega^p(\tilde{X}, \Gamma \cdot L, W, \psi)(\Omega^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi))$ [1].

4. Decomposition of space of harmonic forms

Theorem 3. We have a direct sum decomposition

$$\mathcal{H}^p(\tilde{X}, \Gamma \cdot L, W, \psi) = \sum_{a+b=p} \mathcal{H}^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi)$$

Proof. Let $K^{a,b}$ denote the subspace of $\Omega^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi)$ consisting of forms constant along the fibres of $\tilde{X} \to X$, and $\Omega^a(b)$ or more precisely $\Omega^a(X, \Gamma, W \otimes \wedge^b V^*, \psi \otimes (\wedge^b \cdot \rho)^*)$ the space of Γ-invariant a-forms on X with values in $W \otimes \wedge^b V^*$ for the representation $\psi \otimes (\wedge^b \cdot \rho)^*$. An element $\theta \in \Omega^a(b)$ has locally (i.e., in U) an expression of the form

$$\theta(x) = \sum_{i_1, \cdots, i_a} v_{i_1 \cdots i_a}(x) \otimes w^{i_1} \wedge \cdots \wedge w^{i_a}$$

where $v_{i_1 \cdots i_a}(x) \in W \otimes \wedge^b V^*$, i.e., $v_{i_1 \cdots i_a}(x) = \sum_{i_1, \cdots, j_b} a_{i_1, \cdots, i_a, j_1, \cdots j_b}(x) \otimes w^{j_1} \wedge \cdots \wedge w^{j_b}$ where $a_{i_1 \cdots i_a j_1 \cdots j_b}: X \to W$ is a C^∞ function and $\{w^{k+1}, \cdots, w^{k+n}\}$ is the orthonormal basis of V^* introduced above. Hence

$$\theta(x) = \sum_{i_1 \cdots i_a} \sum_{j_1 \cdots j_b} a_{i_1 \cdots i_a j_1 \cdots j_b}(x) \otimes w^{j_1} \wedge \cdots \wedge w^{j_b} \otimes w^{i_1} \wedge \cdots \wedge w^{i_a} \tag{8}$$

An (a, b) form $\omega \in \Omega^{a,b}(W)$ is constant along the fibres of $\tilde{X} \to X$ if and only if the functions $u_{i_1\cdots i_a j_1\cdots j_b}$ in (6) depend only on x. Furthermore, an expression of the form (6) with $u_{i_1\cdots i_a j_1\cdots j_b}$ depending only on x, is an element of $\Omega^{a,b}(W)$ if and only if

$$\omega(\gamma\tilde{x}) = \sum_{i_1,\cdots,i_a} \sum_{j_1,\cdots,j_b} \psi(\gamma)u_{i_1,\cdots i_a j_1\cdots j_b}(x)\otimes$$
$$(\wedge^a \cdot L_\gamma)w^{i_1}\wedge\cdots\wedge^{i_a}\otimes\wedge^b\cdot\rho)^* w^{j_1}\wedge\cdots\wedge w^{j_b}$$

Similarly, an expression of the form (8) is an element of $\Omega^a(b)$ is and only if

$$\theta(\gamma x) = \sum_{i_1,\cdots,i_a} \sum_{j_1,\cdots,j_b} \psi(\gamma)a_{i_1\cdots i_a j_1\cdots j_b}(x)\otimes$$
$$(\wedge^b\cdot\rho)^*(\gamma)w^{j_1}\wedge\cdots\wedge w^{j_b}\otimes(\wedge^a\cdot L_\gamma)w^{i_1}\wedge\cdots\wedge w^{i_a}\,.$$

We have therefore proved

Lemma 8. *There is a natural isomorphism*

$$\wedge:\ K^{a,b}\to\Omega^a(b)$$

(In the above notation, one defines

$$\hat{\omega}(x) = \sum_{i_1,\cdots,i_a} \sum_{j_1,\cdots,j_b} u_{i_1\cdots i_a j_1\cdots j_b}(x)\otimes w^{j_1}\wedge\cdots\wedge w^{j_b}\otimes w^{i_1}\wedge\cdots\wedge w^{i_a})\,.$$

The mapping $w^{j'_2}\wedge\cdots\wedge w^{j'_{n-b}}\mapsto\delta^{k+1\cdots k+n}_{j_1\cdots j_b j'_1\cdots j'_{n-b}}w^{j_1}\wedge\cdots\wedge w^{j_b}$ extends by linearity to an isomorphism $\wedge^{n-b}V^*\simeq\wedge^b V^*$ and thus induces isomorphism

$$\alpha:\ \Omega^a(n-b)\to\Omega^a(b)$$

Lemma 9. The following diagram commutes up to sign

$$\begin{array}{ccccc} K^{a,b} & & \xrightarrow{\quad *\quad} & & K^{k-a,n-b} \\ \downarrow\wedge & & & & \downarrow\wedge \\ \Omega^a(b) & \xrightarrow{*} & \Omega^{k-a}(b) & \xrightarrow{\alpha} & \Omega^{k-a}(n-b) \end{array}$$

Proof of Lemma 9. The operator $*$ on $K^{a,b}$ is given by (7) and on $\Omega^a(b)$ by

(9) $$(*\theta)(x) = \sum_{i_1,\cdots,i_a} \delta^{1\cdots k}_{i_1\cdots i_a j'_1\cdots j'_{k-a}} \sum_{j_1,\cdots,j_b} a_{i_1\cdots i_a j_1\cdots j_b}(x)\otimes$$
$$w^{j_1}\wedge\cdots\wedge w^{j_b}\otimes w^{i'_1}\wedge\cdots\wedge w^{i'_{k-a}}\,.$$

Since

$$\delta^{1\cdots k+n}_{i_1\cdots i_a j_1\cdots j_b i'_1\cdots i'_{k-a} j'_1\cdots j'_{n-b}} = (-1)^{b(k-a)}\delta^{1\cdots k}_{i_1\cdots i_a i_1\cdots i'_{k-a}}\delta^{k+1\cdots k+n}_{j_1\cdots j_b j'_1\cdots j'_{n-b}}$$

the assertion of the lemma follows from (7), (9) and definition of α.

It is clear that restrictions of d and δ to $\sum_{a,b} K^{a,b}$ are operators of bidegrees (1, 0) and (−1, 0), respectively. Similarly, d and δ are operators of degrees

$+1$ and -1 on $\sum_a \Omega^a(b)$, respectively.

Let $C(x)$ be the matrix representation for $(\, , \,)_x$ with respect to some basis, i.e., for $w_1, w_2 \in W$, $(w_1, w_2)_x = {}^t w_1 C(x) w_2$. Then the isomorphisms $\sharp$ can be given the following explicit description:

$$\omega \to (C(x) \otimes 1 \otimes 1)\omega \qquad \omega \in \Omega^{a,b}(w) \subset K^{a,b},$$
$$\theta \to (C(x) \otimes 1 \otimes 1)\theta \qquad \theta \in \Omega^a(b).$$

Lemma 10. *For* $\omega \in K^{a,b}$, $d\omega = 0$ $(\delta\omega = 0)$ *if and only if* $d\hat{\omega} = 0$, $(\delta\hat{\omega} = 0)$.

Proof of Lemma 10. Equivalence of $d\omega = 0$ and $d\hat{\omega} = 0$ is obvious. To prove equivalence of $\delta\hat{\omega} = 0$ and $\delta\hat{\omega} = 0$, we note that $\delta\hat{\omega} = 0$ if and only if $d^*(C(x) \otimes 1 \otimes 1)\hat{\omega} = 0$. It follows from the description of $\sharp$ above, definition of $*$ and Lemma 9 that the latter is equivalent to $d(* \sharp \omega) = 0$, that is, $\delta\omega = 0$.

q.e.d.

It is clear that $H^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi) \subset K^{a,b}$ [6], and by Lemma 10 we can conclude

$$H^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi) \cong H^a(X, \Gamma, W \otimes \wedge^b V^*, \psi \otimes (\wedge^b \cdot \rho)^*) \tag{10}$$

Let $\Gamma \cdot L = G$, $L = H$ and $W = A$ in the Hochschild-Serre spectral sequence(3). Then the $E_2^{a,b}$ term of the spectral sequence is $H^a(\Gamma, H^b(L, W))$, and E_∞ term is a grading of $H^p(\Gamma \cdot L, W)$. Therefore

$$\dim H^p(\Gamma \cdot L, W) \le \dim \sum_{a+b} H^a(\Gamma, H^b(L, W)) \tag{11}$$

We know that $H^b(L, W) \simeq \wedge^b V^* \otimes W \simeq \mathrm{Hom}(\wedge^b V, W)$ and action of Γ on $H^b(L, W)$ is as follows:

To $f \in \mathrm{Hom}(\wedge^b V, W)$ corresponds the homogeneous cochain c_f given by

$$c_f(u_0, \cdots, u_b) = f(u_1 - u_0 \wedge \cdots \wedge u_b - u_0) \qquad u_i \in L.$$

Then

(3) We give a brief description of a special case of Hochschild-Serre spectral sequence which is sufficient for our purposes. Let G be an abstract group, H an abelian normal subgroup of G, A an abelian group and ρ a homomorphism of G into the group of automorphisms of A. Suppose that the restriction of ρ to H is trivial. Let $f: \underbrace{H \times \cdots \times H}_{j\text{-times}} \to A$ be a representative of an element of $H^j(H, A)$ and $x \in G$ a representative of $\bar{x} \in G/H$. Set

$$(\bar{x}f)(y_1, \cdots, y_j) = x(f(x^{-1}y_1x, \cdots, x^{-1}y_jx)).$$

Then $f \to (\bar{x}f)$ induces a homomorphism of $H^j(H, A)$ into itself, and this defines action of G/H on $H^j(H, A)$. Furthermore, there is a spectral sequence with E_2 term

$$E_2^{i,j} = H^i(G/H, H^j(H, A))$$

converging to a certain grading of $H^{i+j}(G, A)$. For more details see e.g. [7].

$$(\gamma c_f)(u_0, \cdots, u_b) = \psi(\gamma)f(\gamma^{-1}(u_1-u_0)\gamma\wedge\cdots\wedge\gamma^{-1}(u_b-u_0)\gamma)$$
$$= \psi(\gamma)f(\rho(\gamma)^{-1}(u_1-u_0)\wedge\cdots\wedge\rho(\gamma)^{-1}(u_b-u_0)),$$

that is, Γ acts on $H^b(L, W)=W\otimes\wedge^b V^*$ via $\psi\otimes(\wedge^b\cdot\rho)^*$.

Now we have

$$\begin{aligned}\dim H^p(\tilde{X}, \Gamma\cdot L, W, \psi) &\geq \dim \sum_{a+b=p} \mathcal{H}^{a,b}(\tilde{X}, \Gamma\cdot L, W, \psi)\\ &= \dim \sum_{a+b=p} \mathcal{H}^a(X, \Gamma, W\otimes\wedge^b V^*, \psi\otimes(\wedge^b\cdot\rho)^*) \quad \text{[by (10)]}\\ &= \dim \sum_{a+b=p} H^a(\Gamma, W\otimes\wedge^b V^*, \psi\otimes(\wedge^b\cdot\rho)^*)\\ &= \dim \sum_{a+b=p} H^a(\Gamma, H^b(L, W))\\ &\geq \dim H^p(\Gamma\cdot L, W) \quad \text{[by (11)]}\\ &= \dim \mathcal{H}^p(\tilde{X}, \Gamma\cdot L, W, \psi).\end{aligned}$$

Since the space of harmonic forms $\mathcal{H}^p(\tilde{X}, \Gamma\cdot L, W, \psi)$ is finite dimensional, we have shown

$$\mathcal{H}^p(\tilde{X}, \Gamma\cdot L, W, \psi) = \sum_{a+b=p} \mathcal{H}^{a,b}(\tilde{X}, \Gamma\cdot L, W, \psi)$$

completing the proof of Theorem 3.

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