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FOURIER SERIES OF SMOOTH FUNCTIONS ON COMPACT LIE GROUPS

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Introduction

The purpose of the present note is to give an elementary proof of the following theorems. Any C^{2k}-function on a compact connected Lie group G can be expanded by the absolutley and uniformly convergent Fourier series of the matricial components of irreducible representations if $2k > \frac{1}{2}\dim G$ (Theorem 1). The Fourier transform is a topological isomorphism of $C^\infty(G)$ onto the space $S(D)$ of rapidly decreasing functions on the set D of the classes of irreducible representations of G (Theorem 3 and 4).

The related results which the author found in the literature are as follows. In Séminaire Sophus Lie [1] exposé 21, it was proved that any C^∞-functions on G can be expanded by the uniformly convergent Fourier series. Zhelobenko [3] proved Theorem 3 for the group SU(2). R.A. Mayer [4] proved that the Fourier series of any C^1-function on SU(2) is uniformly convergent but there exists a C^1-function on SU(2) whose Fourier series does not converge absolutely.

1. The Fourier expansion of a smooth function

Throughout this paper we use the following notations. G: a compact connected Lie group, G_0: the commutator subgroup of G, T: a maximal toral subgroup of G, l: the rank of $G = \dim T$, p: the rank of G_0, n: the dimension of $G = l + 2m$, $\mathfrak{g}$: the Lie algebra of G, $\mathfrak{g}^c$: the complexification of $\mathfrak{g}$, $\mathfrak{t}$: the Lie algebra of T, R: the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$, dg: the Haar measure on G normalized as $\int_G dg = 1$, $L^2(G)$: the Hilbert space of the complex valued square integrable functions on G with respect to dg, $C^k(G)$: the set of all k-times continuously differentiable complex valued functions on G, $||A|| = \mathrm{Tr}(AA^*)^{1/2}$: the Hilbert-Schmidt norm of a matrix A.

In this paper, a finite dimensional continuous matricial representation of G

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is simply called a representation of G. So a representation of G is a continuous and hence analytic homomorphism of G into $GL(k, \boldsymbol{C})$ for some $k \geqq 1$. For any representation U of G, the differential dU of U is defined as

$$dU(X) = \left[\frac{d}{dt} U(\exp tX)\right]_{t=0}$$

for any X in $\mathfrak{g}$. The differential dU of U is a representation of the Lie algebra $\mathfrak{g}$. The representation $dU$of $\mathfrak{g}$ is uniquely extended to a representation of the universal enveloping algebra U($\mathfrak{g}$) of $\mathfrak{g}$. This representation of U($\mathfrak{g}$) is also denoted by dU.

For any representation U of G, all the elements in $dU(\mathfrak{t})$ can be transformed simultaneously into the diagonal matrices. That is, there exists a non singular matrix Q and pure imaginary valued linear forms $\lambda_1,\cdots,\lambda_k$ on $\mathfrak{t}$ such that

$$QdU(H)Q^{-1} = \begin{pmatrix} \lambda_1(H) & & 0 \\ & \ddots & \\ 0 & & \lambda_k(H) \end{pmatrix}$$

for all H in $\mathfrak{t}$. The linear forms $\lambda_1,\cdots,\lambda_k$ are called the weights of U.

We fix once for all a positive definite inner product (X, Y) on $\mathfrak{g}$ which is invariant under Ad G. The norm defined by the inner product is denoted by $|X| = (X, X)^{1/2}$. The inner product (X, Y) is extended to a bilinear form on the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$. A pure imaginary valued linear form(in particular a weight of a representation) λ is identified with an element h_λ in $\mathfrak{t}$ which satisfies

$$\lambda(H) = i(h_\lambda, H)$$

for all H in $\mathfrak{t}$. So we denote as $\lambda(H) = i(\lambda, H)$. Let $\Gamma = \Gamma(G) = \{H \in \mathfrak{t} \ ; \exp_G H = 1\}$ be the kernel of the homomorphisim $\exp_G : \mathfrak{t} \rightarrow T$. Then Γ is a discrete subgroup of $\mathfrak{t}$ of rank l. Let I be the set of all G–integral forms on $\mathfrak{t}$:

$$I = \{\lambda \in \mathfrak{t} : (\lambda, H) \in 2\pi \boldsymbol{Z} \text{ for all } H \in \Gamma\}.$$

Then the set I coincides with the set of all the weights of the representations of G. We choose once for all a lexicographic order $\mathcal{O}$ in $\mathfrak{t}$. Let P be the set of positive roots with respect to the order $\mathcal{O}$. Then the number m of elements in P is equal to $\frac{1}{2}(n-l)$. Let B be the set of simple roots in P, that is, B is the set of roots in P which can not be the sum of two elements in P. B consists of exactly p elements (p = rank G_0). We denote the elements of B as $\alpha_1,\cdots,\alpha_p$.

Let $\lambda_1,\cdots,\lambda_k$ be the weights of a representation U. Then the maximal element λ among λ_i's in the order $\mathcal{O}$ is called the highest weight of U. The set of all highest weights of the representations of G coincides with the set D of all dominant G–integral forms on $\mathfrak{t}$:

$$D = \{\lambda \in I\,; (\lambda, \alpha_i) \geqq 0\ (1 \leqq i \leqq p)\}.$$

Since an irreducible representation of G is uniquely determined, up to equivalence, by its highest weight (cf. Serre [2] Ch. VII Théorème 1), there exists a bijection from D onto the set $\mathfrak{D}$ of equivalence classes of irreducible representations of G. $\mathfrak{D}$ is identified with D by this bijection. We choose, once for all, an irreducible unitary representation U^λ with the highest weight λ for each λ in D. The degree $d(\lambda)$ of the representation U^λ is given by Weyl's dimension formula:

$$d(\lambda) = \prod_{\alpha \in P} \frac{(\lambda+\delta, \alpha)}{(\delta, \alpha)} \tag{1.0}$$

where $\delta = 2^{-1} \sum_{\alpha \in P} \alpha$.

If G is abelian, the right hand side of (1. 0) should be understood to express 1.

Let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$ and $g_{ij} = (X_i, X_j)$ and $(g^{ij}) = (g_{ij})^{-1}$. Then the element Δ defined by

$$-\Delta = \sum_{i,j=1}^{n} g^{ij} X_i X_j$$

in the universal enveolping algebra U($\mathfrak{g}$) of $\mathfrak{g}$ is called the Casimir operator of $\mathfrak{g}$. Δ is independent of the choice of the basis $X_1, \cdots, X_n$. As an element in U($\mathfrak{g}$), Δ is regarded as a left invariant linear differential operator on G.

Let $u^\lambda_{ij}(g)$ be the (i, j)–element of the unitary matrix $U^\lambda(g)$. Then the following Lemma is well known.

Lemma 1.1. 1) *Let dU^λ be the differential of the representation U^λ. Then we have*

$$dU^\lambda(\Delta) = (\lambda, \lambda+2\delta)\ 1.$$

2) *The matricial element u^λ_{ij} is an eigenfunction of the Casimir operator Δ regarded as a differential operator on G:*

$$\Delta u^\lambda_{ij} = (\lambda, \lambda+2\delta) u^\lambda_{ij}.$$

Proof. 1) Since the Casimir operator Δ belongs to the center of U($\mathfrak{g}$), dU^λ (Δ) is a scalar operator $c1$ by Schur's lemma. The scalar c is determined as follows. We can choose a Weyl base E_α $(\alpha \in R)$, $H_i (1 \leqq i \leqq l)$ of $\mathfrak{g}^c$ satisfying $(E_\alpha, E_{-\alpha}) = 1$, $(H_i, H_j) = \delta_{ij}$ and $E_\alpha + E_{-\alpha}$, $i(E_\alpha - E_{-\alpha})$, $H_i \in \mathfrak{g}$. Then we have

$$-\Delta = \sum_{\alpha \in R} E_{-\alpha} E_\alpha + \sum_{i=1}^{l} H_i^2 = \sum_{\alpha \in P} (2E_{-\alpha} E_\alpha + H_\alpha) + \sum_{i=1}^{l} H_i^2,$$

because $[E_\alpha, E_{-\alpha}] = H_\alpha$ where H_α is the element in the space it satisfying $(H, H_\alpha) = \alpha(H)$ for every H in $\mathfrak{t}^c$. Let $x \neq 0$ be the weight vector corresponding to the highest weight λ: $dU^\lambda(E_\alpha)\, x = 0$ $(\alpha \in P)$, $dU^\lambda(H_\alpha)\, x = \lambda(H_\alpha)\, x = -(\lambda, \alpha)x$. Then we have

$$cx = dU^\lambda(\Delta)\, x = \{\sum_{\alpha \in P}(\lambda, \alpha) - \sum_{i=1}^{l} \lambda(H_i)^2\}\, x = \{(\lambda, 2\delta) + (\lambda, \lambda)\}\, x$$
$$= (\lambda, \lambda + 2\delta)\, x.$$

2) For any element X in $\mathfrak{g}$, we have

$$(Xu_{ij}^\lambda)(g) = \left[\frac{d}{dt} u_{ij}^\lambda(g \exp tX)\right]_{t=0} = \sum_{k=1}^{d(\lambda)} u_{ik}^\lambda(g) \left[\frac{du_{kj}^\lambda}{dt}(\exp tX)\right]_{t=0}.$$

This equality can be expressed as

$$(XU^\lambda)(g) = U^\lambda(g)dU^\lambda(X).$$

So we get

$$(\Delta U^\lambda)(g) = U^\lambda(g)dU^\lambda(\Delta) = (\lambda, \lambda + 2\delta)U^\lambda(g)$$

by 1). q.e.d.

By Peter–Weyl's theorem, the set

$$\mathfrak{B} = \{d(\lambda)^{1/2} u_{ij}^\lambda\ ;\ \lambda \in D,\ 1 \leqq i, j \leqq d(\lambda)\}$$

is an orthonormal base of $L^2(G)$. Therefore any function f in $L^2(G)$ can be expanded by a mean convergent Fourier series of $\mathfrak{B}$:

$$f = \sum_{\lambda \in D} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (f, u_{ij}^\lambda) u_{ij}^\lambda. \tag{1.1}$$

The precise meaning of (1.1) is given by

$$\lim_{n \to \infty} \| f - \sum_{|\lambda| \leqq n} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (f, u_{ij}^\lambda) u_{ij}^\lambda \|_2 = 0. \tag{1.2}$$

(1.2) is equivalent to the Parseval's equality:

$$\|f\|_2^2 = \sum_{\lambda \in D} d(\lambda) \sum_{i,j=1}^{d(\lambda)} |(f, u_{ij}^\lambda)|^2.$$

For an arbitrary function f in $L^2(G)$, the right hand side of (1.1) does not, in general, converge at every point of G. We shall show that if f is sufficiently smooth, then the series(1.1) converges uniformly on G. First we give a convenient expression of the series (1.1). Let f be a function in $L^1(G)$ and $\lambda \in D$. Then the λ-th Fourier coefficient $\mathcal{F}f(\lambda)$ of f is defined by

$$\mathcal{F}f(\lambda) = \int_G f(g) U^\lambda(g^{-1}) dg.$$

$\mathcal{F}f(\lambda)$ is a matrix of degree $d(\lambda)$ and its (i,j)–element $\mathcal{F}f(\lambda)_{ij}$ is given by

$$\mathcal{F}f(\lambda)_{ij} = \int_G f(g)u^\lambda_{ij}(g^{-1})dg \tag{1.3}$$
$$= \int_G f(g)\overline{u^\lambda_{ji}(g)} = (f, u^\lambda_{ji}).$$

Therefore we have

$$\sum_{i,j=1}^{d(\lambda)}(f, u^\lambda_{ij})u^\lambda_{ij}(g) = \sum_{i,j=1}^{d(\lambda)}\mathcal{F}f(\lambda)_{ij}u^\lambda_{ij}(g) \tag{1.4}$$
$$= \mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g)).$$

By (1.3) the Parseval equality can be expressed as

$$\|f\|_2^2 = \sum_{\lambda\in D} d(\lambda)\|\mathcal{F}f(\lambda)\|^2. \tag{1.5}$$

Lemma 1.2. *If f belongs to $C^2(G)$, then we have*

$$\mathcal{F}(\Delta f)(\lambda) = \omega(\lambda)\mathcal{F}f(\lambda)$$

where

$$\omega(\lambda) = (\lambda, \lambda+2\delta). \tag{1.6}$$

Proof. Let φ and ψ be any C^1–functions on G. Then for any element X in $\mathfrak{g}$, we have

$$(X\varphi, \psi) = \int_G\left[\frac{d}{dt}\varphi(g \exp tX)\right]_{t=0}\overline{\psi(g)}\, dg = \frac{d}{dt}\left[\int_G \varphi(g \exp tX)\overline{\psi(g)}\,dg\right]_{t=0}$$
$$= \frac{d}{dt}\left[\int_G \varphi(g)\overline{\psi(g \exp(-tX))}\,dg\right]_{t=0} = -(\varphi, X\psi).$$

Let $X_1,\cdots,X_n$ be an orthonormal base of $\mathfrak{g}$: $(X_i, X_j) = \delta_{ij}$. Then we have $\Delta = -\sum_{i=1}^n X_i^2$ and by the above equality we get

$$(\Delta\varphi, \psi) = (\varphi, \Delta\psi) \tag{1.7}$$

for any C^2–functions φ and ψ. By (1.3), (1,7) and Lemma 1.1, 2), we have

$$\mathcal{F}(\Delta f)(\lambda)_{ij} = (\Delta f, u^\lambda_{ji}) = (f, \Delta u^\lambda_{ji}) = \omega(\lambda)\mathcal{F}f(\lambda)_{ij}.$$

q. e. d.

Lemma 1.3. *Let $D_0 = D - \{0\}$. Then the series*

$$\zeta(s) = \sum_{\lambda\in D_0}(\lambda, \lambda)^{-s}$$

converges if $2s > l$.

Proof. Let I be the set of all G–integral form on $\mathfrak{t}$ and $I_0 = I-\{0\}$. It is sufficient to prove the series

$$\sum_{\lambda\in I_0} (\lambda, \lambda)^{-s}$$

converges if $2s>l$. Let $\lambda_1,\cdots,\lambda_l$, be a basis of the lattice I and $\langle x, y\rangle$ be the inner product on $\mathfrak{t}$ defined by $\langle\Sigma x_i\lambda_i, \Sigma y_i\lambda_i\rangle=\Sigma x_i y_i$. Then it is well known that the series

$$\sum_{\lambda\in I_0} \langle\lambda, \lambda\rangle^{-s} = \sum_{n\in Z^l-\{0\}} (n_1^2+\cdots+n_l^2)^{-s} \tag{1.8}$$

converges if and only if $2s>l$. On the other hand, there exists a positive definite symmetric operator A such that $(x, y) = \langle Ax, y\rangle$ for all x, y in $\mathfrak{t}$. Let a and b be the maximal and minimal eigenvalues of A. Then we have

$$b\langle\lambda, \lambda\rangle \leqq (\lambda, \lambda) \leqq a\langle\lambda, \lambda\rangle$$

for all λ in $\mathfrak{t}$. Therefore the series $\sum_{\lambda\in I_0} (\lambda, \lambda)^{-s}$ converges if and only if the series $\sum_{\lambda\in I_0} \langle\lambda, \lambda\rangle^{-s}$ converges. So we have proved that $\sum_{\lambda\in D_0} (\lambda, \lambda)^{-s}$ converges if $2s>l$.

Theorem 1. *Let f be a continuous function on a compact connected Lie group G and let $l =$ rank G, $n =$ dim $G = l+2m$. If f satisfies one of the following conditions* (1) *and* (2), *then the Fourier series of f,*

$$\sum_{\lambda\in D} d(\lambda)\mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g))$$

converges to $f(g)$ absolutely and uniformly on G:

(1) *f is $2k$-times continuously differentiable and $2k>\frac{l}{2}+m = \frac{n}{2}$,*

(2) $||\mathcal{F}f(\lambda)|| = O(|\lambda|^{-h})$ $(|\lambda|\to\infty)$ *for some integer $h>l+\frac{3}{2}m$.*

Proof. (1) Suppose f belongs to $C^{2k}(G)$. Then we have, by Lemma 1.2,

$$\mathcal{F}f(\lambda) = \omega(\lambda)^{-k}\mathcal{F}(\Delta^k f)(\lambda) \qquad (\lambda\in D_0). \tag{1.9}$$

On the other hand we have an inequality

$$\omega(\lambda) = (\lambda, \lambda+2\delta) \geqq |\lambda|^2. \tag{1.10}$$

By (1.9) and (1.10), we have

$$||\mathcal{F}f(\lambda)|| \leqq ||\mathcal{F}(\Delta^k f)(\lambda)|| \, |\lambda|^{-2k} \text{ for all } \lambda\in D_0 = D-\{0\}. \tag{1.11}$$

Since $(A, B) = \mathrm{Tr}(AB^*)$ is an inner product on the space $M_n(C)$ of the matrices of order n, we have the Schwarz inequality

$$|\mathrm{Tr}(AB)| \leqq \|A\|\,\|B\|. \tag{1.12}$$

Since $U^\lambda(g)$ is a unitary matrix of order $d(\lambda)$, the Hilbert-Schmidt norm of $U^\lambda(g)$ is equal to

$$\|U^\lambda(g)\| = d(\lambda)^{1/2}. \tag{1.13}$$

By (1.12), (1.13) and (1.11), we have

$$\begin{aligned}\sum_\lambda d(\lambda)|\mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g))| &\leqq \sum_\lambda d(\lambda)^{3/2}\|\mathcal{F}f(\lambda)\| \\ &\leqq \sum_\lambda d(\lambda)^{3/2}|\lambda|^{-2k}\|\mathcal{F}(\Delta^k f)(\lambda)\|.\end{aligned} \tag{1.14}$$

By the Schwarz inequality, the right hand side of (1.14) is

$$\leqq \Big(\sum_\lambda d(\lambda)\|\mathcal{F}(\Delta^k f)(\lambda)\|^2\Big)^{1/2}\Big(\sum_\lambda d(\lambda)^2|\lambda|^{-4k}\Big)^{1/2}. \tag{1.15}$$

Since $\Delta^k f \in C^0(G) \subset L^2(G)$, we have the Parseval equality

$$\|\Delta^k f\|_2^2 = \sum_{\lambda\in D} d(\lambda)\|\mathcal{F}(\Delta^k f)(\lambda)\|^2. \tag{1.16}$$

Moreover by Weyl's dimension formula, we have for any $\lambda \in D_0$

$$d(\lambda) \leqq C(|\lambda|+|\delta|)^m \leqq N|\lambda|^m \tag{1.17}$$

where $C = \prod_{\alpha\in P}|\alpha|(\delta,\alpha)^{-1}$ and N are positive constants. By (1.16) and (1.17), the right hand side of (1.15) is

$$\leqq \|\Delta^k f\|_2\Big(N^2\sum_\lambda |\lambda|^{2m-4k}\Big)^{1/2}. \tag{1.18}$$

Since $4k-2m > l+2m-2m = l$ by condition (1), the series in (1.18) converges (Lemma 1.3). So we have proved that the Fourier series of f converges absolutely and uniformly on G, if f satisfies the condition (1). The sum $s(g)$ of the Fourier series of f is a continuous function and equal to $f(g)$ almost everywhere on G by the Parseval equality. Since f and s are continuous, the sum $s(g)$ is equal to $f(g)$ everywhere on G.

If a function f satisfies the condition (2), then there exists a positive constant M such that

$$\|\mathcal{F}f(\lambda)\| \leqq M|\lambda|^{-h} \qquad \text{for all } \lambda \in D_0. \tag{1.19}$$

So we have

$$\begin{aligned}\sum_\lambda d(\lambda)|\mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g))| &\leqq \sum_\lambda d(\lambda)^{3/2}\|\mathcal{F}f(\lambda)\| \\ &\leqq L\sum_\lambda (|\lambda|+|\delta|)^{3m/2}|\lambda|^{-h}\end{aligned} \tag{1.20}$$

where $L = M(\prod_{\alpha\in P} |\alpha|(\delta, \alpha)^{-1})^{3/2}$ is a positive constant. Therefore the series on the right hand side of (1.20) converges if $h-3m/2>l$, i.e., $h>l+3m/2$ (Lemma 1.3). q.e.d.

Corollary to Theorem 1. *If f is a C^{2k}-function on G, then we have* $\|\mathfrak{F}f(\lambda)\| = o(|\lambda|^{-2k})$ $(|\lambda|\to\infty)$, *that is,*

$$\lim_{|\lambda|\to\infty} |\lambda|^{2k}\|\mathfrak{F}f(\lambda)\| = 0.$$

Proof. By the inequality (1.11), we have

$$|\lambda|^{2k}\|\mathfrak{F}f(\lambda)\| \leqq \|\mathfrak{F}(\Delta^k f)(\lambda)\|. \tag{1.21}$$

Since $\Delta^k f$ belongs to $C^0(G)\subset L^2(G)$, we have

$$\lim_{|\lambda|\to\infty} \|\mathfrak{F}(\Delta^k f)(\lambda)\| = 0 \tag{1.22}$$

by the Parseval equality (1.16). (1.21) and (1.22) prove the Corollary.

2. Fourier coeffcients of a smooth function

Theorem 2. *Let G be a compact connected Lie group and D be the set of all dominant G-integral forms on the Lie algebra $\mathfrak{t}$ of a maximal toral subgroup T of G. Let U^λ be an irreducible unitary representation of G with the highest weight $\lambda\in D$ and $d(\lambda)$ be the degree of U^λ. Then we have the following inequality for every X in the Lie algebra $\mathfrak{g}$ of G:*

$$\begin{aligned} &\|dU^\lambda(X)\|^2 \leqq d(\lambda)|\lambda|^2|X|^2 \quad \text{for any } \lambda\in D \text{ and} \\ &\|dU^\lambda(X)\|^2 \leqq N|\lambda|^{m+2}|X|^2 \quad \text{for any } \lambda\in D_0 \end{aligned} \tag{2.1}$$

where N is a positive constant and m is the number of the positive roots.

Proof. First we show that the inequality (2.1) is valid for every X in $\mathfrak{g}$ if (2.1) is valid for every X in the Cartan subalgebra $\mathfrak{t}$. Since every element X in $\mathfrak{g}$ is conjugate to an element H in $\mathfrak{t}$, that is, there exists an element g in G such that $(\mathrm{Ad}\, g)X= H$, we have

$$\|dU^\lambda(H)\| = \|U^\lambda(g)dU^\lambda(X)U^\lambda(g^{-1})\| = \|dU^\lambda(X)\| \quad \text{and} \tag{2.2}$$

$$|H| = |X|. \tag{2.3}$$

The equalities (2.2) and (2.3) prove that if the inequality (2.1) is valid for any H in $\mathfrak{t}$, then (2.1) is valid for every X in $\mathfrak{g}$.

Now let X be any element in $\mathfrak{t}$ and $W(\lambda)$ be the set of weights in the representation U^λ. Then the linear transformation $dU^\lambda(X)$ is represented by a diagonal matrix whose diagonal elements are $\{i(\mu, X): \mu \in W(\lambda)\}$ with respect to some orthonormal base of the representation space. Therefore we have

$$(2.4) \qquad ||dU^\lambda(X)||^2 = \sum_{\mu\in W(\lambda)} |i(\mu, X)|^2 \leqq \sum_{\mu\in W(\lambda)} |\mu|^2\, |X|^2.$$

On the other hand every weight μ in $W(\lambda)$ has the form

$$(2.5) \qquad \mu = \lambda - \sum_{i=1}^{p} m_i\alpha_i,$$

where m_i's are non negative integers. (cf. Serre [2] Ch. VII Théorème 1). If $\mu \in W(\lambda)$ is dominant, that is, $(\mu, \alpha_i) \geqq 0$ $(l \leqq i \leqq p)$, then we have by (2.5)

$$(2.6) \qquad |\mu|^2 \leqq |\mu|^2 + \sum_{i=1}^{p} m_i(\mu, \alpha_i) = (\lambda, \mu) = |\lambda|^2 - \sum_{i=1}^{p} m_i(\lambda, \alpha_i) \leqq |\lambda|^2.$$

Since every weight μ in $W(\lambda)$ is conjugate to a dominant weight in $W(\lambda)$ under the Weyl group, (cf. Serre [2] Ch. VII-12 Remarque), we have the inequality

$$(2.7) \qquad |\mu| \leqq |\lambda| \qquad \text{for all } \mu \in W(\lambda)$$

by (2.6). The inequalities (2.4) and (2.7) prove the inequality

$$(2.8) \qquad ||dU^\lambda(X)||^2 \leqq d(\lambda)\,|\lambda|^2 |X|^2.$$

Since the degree $d(\lambda)$ of U is given by Weyl's dimension formula

$$d(\lambda) = \prod_{\alpha\in P} (\lambda+\delta, \alpha)(\delta, \alpha)^{-1},$$

$d(\lambda)$ is estimated by (1.17) as

$$(2.9) \qquad d(\lambda) \leqq C(|\lambda| + |\delta|)^m \leqq N|\lambda|^m \quad \text{for any } \lambda \in D_0$$

where C and N are positive constants and m is the mumber of positive roots. So we have proved Theorem 2 completely.

Lemma 2.1. *Let G be a connected Lie group and $\mathfrak{g}$ be the Lie algebra of G. Moreover let f be a complex valued function on G, and k be a positive integer. Then the function f belongs to $C^k(G)$ if and only if*

$$(Xf)(g) = \left[\frac{d}{dt} f(g \exp tX)\right]_{t=0}$$

can be defined for every X in $\mathfrak{g}$ and g in G, and it belongs to $C^{k-1}(G)$.

Proof. JI a function f belongs to $C^k(G)$, then $\varphi(g, t)=f(g\exp tX)$ belongs to $C^k(G\times \boldsymbol{R})$. So $(Xf)(g)=\frac{\partial\varphi}{\partial t}(g, 0)$ exists and belongs to $C^{k-1}(G)$.

Conversely suppose that Xf is defined and belongs to $C^{k-1}(G)$ for every $X\in \mathfrak{g}$. Then for any real number t, $(df/dt)(g\exp tX)$ exists and is equal to $(Xf)(g\exp t X)$. Moreover for any element h in G, $(df/dt)(g\exp t Xh)$ exists and is equal to

$$(2.10)\qquad \frac{d}{dt}f(g\exp t Xh)=\frac{d}{dt}f(gh\exp(t\,\mathrm{Ad}\,h^{-1}X))$$
$$=((\mathrm{Ad}\,h^{-1}X)f)(g\exp t Xh).$$

Let $X_1, X_2,\cdots,X_n$ be a base of $\mathfrak{g}$ and

$$\varphi(t)=\varphi(t_1,\cdots,t_n)=\exp tX_1\cdots\exp t_nX_n.$$

Then φ is an analytic diffeomorphism of an open neighbourhood W of 0 in $\boldsymbol{R}^n$ onto an open neighbourhood V of the identity element e in G. Let

$$(2.11)\qquad (\mathrm{Ad}(\exp t_1X_1\cdots\exp t_nX_n)^{-1})X_i=\sum_{j=1}^{n}a_{ij}(t)X_j.$$

Then $a_{ij}(t)=a_{ij}(t_1,\cdots,t_n)$ is an analytic function on $\boldsymbol{R}^n$. Let g be a fixed element in G. Then the mapping $g\varphi(t)\mapsto t=(t_1,\cdots,t_n)$ defines a local coordinates on gV, the canonical coordinates of the second kind. Let $\partial/\partial t_i$ be the partial derivalive with respect to t_i just introduced. Then by the equalities (2.10) and (2.11), $\frac{\partial f}{\partial t_i}(g\varphi(t))$ exists and is equal to

$$(2.12)\qquad \frac{\partial}{\partial t_i}f(g\varphi(t))=[\mathrm{Ad}(\exp t_{i+1}X_{i+1}\cdots\exp t_nX_n)^{-1}X_i]f(g\varphi(t))$$
$$=\sum_{j=1}^{n}a_{ij}(0,\cdots,0,t_{i+1},\cdots,t_n)(X_jf)(g\varphi(t)).$$

By the assumption, the right hand side of (2.12) regarded as a function of t is a C^{k-1}-function on W. So f is a C^k-fucntion on gV. Since g is arbitrary, this proves that f is a C^k-function on G.

Lemma 2.2. *Let G, $\mathfrak{g}$, f, k be as in Lemma* 2.1. *Then f is a C^k-function on G if and only if $X_kX_{k-1}\cdots X_1f$ can be defined and is continuous for any k elements $X_1,\cdots,X_k$ in $\mathfrak{g}$.*

Proof. This Lemma is easily proved by the induction with respect to k using Lemma 2.1.

Theorem 3. *For any continuous function f on a compact connected Lie group G, the following two conditions* (1) *and* (2) *are mutually equivalent.*

(1) *f is a C^∞–function on G.*
(2) *The Fourier coefficients $\mathcal{F}f(\lambda)$ is rapidly decreasing: $\lim_{|\lambda|\to\infty} |\lambda|^h \|\mathcal{F}f(\lambda)\| = 0$ for every non negative integer h.*

Proof. (1)⇒(2). This part of Theorem 3 is proved in Corollary to Theorem 1.
(2)⇒(1). Suppose that $\mathcal{F}f(\lambda)$ is rapidly decreasing. Then f satisfies the condition (2) in Theorem 1. So the Fourier series of f converges uniformly to f. Thus for every $g\in G$, $X\in\mathfrak{g}$ and $t\in\boldsymbol{R}$ we have

$$f(g \exp tX) = \sum_{\lambda\in D} d(\lambda)\mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g \exp tX)). \tag{2.13}$$

The series obtained from the right hand side of (2.13) by termwise differentiation with respect to the variable t is

$$\sum_{\lambda\in D} d(\lambda)\mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g \exp tX)dU^\lambda(X)). \tag{2.14}$$

By Theorem 2 and the rapidly decreasingness of $\mathcal{F}f(\lambda)$, the series (2.14) converges absolutely and uniformly with respect to t, when t runs through any bounded set in $\boldsymbol{R}$. Therefore the series (2.13) can be differentiated termwise and the function $f(g \exp tX)$ is differentiable with respect to t. So

$$(Xf)(g) = \left[\frac{d}{dt}f(g \exp tX)\right]_{t=0}$$

is defined and equal to

$$\sum_{\lambda\in D} d(\lambda)\mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g)dU^\lambda(X)). \tag{2.15}$$

Since (2.15) is uniformly convergent on G, the sum Xf is a continuous function on G. Therefore f is a C^1–function by Lemma 2.1.

By the same argument, $X_1\cdots X_k f$ is defined and continuous for any $k\in\boldsymbol{N}$ and $X_1,\cdots,X_k\in\mathfrak{g}$ and it has the following uniformly convergent expansion;

$$(X_1\cdots X_k f)(g) = \sum_{\lambda\in D} d(\lambda)\mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g)dU^\lambda(X_1)\cdots dU^\lambda(X_k)). \tag{2.15}$$

So f is a C^k-function for any $k\in\boldsymbol{N}$ by Lemma 2.2, i.e., f is a C^∞–function on G.

3. The topology of $\mathrm{C}^\infty(G)$ and $S(D)$

Let G be a compact connected Lie group as before. The space $C^\infty(G)$ of all complex valued C^∞–functions on G is topologized by the family of seminorms:

$$\{p_U(f) = \|Uf\|_\infty : U\in \mathrm{U}(\mathfrak{g})\}. \tag{3.0}$$

$C^\infty(G)$ is a complete locally convex topological vector space by this topology. It is clear that the topology of $C^\infty(G)$ is coincides with the one which is determined by the subfamily of seminorms:

$$\{p_{X_1}\cdots_{X_k}(f)=\|X_1\cdots X_k f\|_\infty: k=0,1,2;\cdots, X_1,\cdots X_k\in\mathfrak{g}\}. \tag{3.1}$$

Let $S(D)$ be the space of matrix valued functions F on the lattice D which satisfies the following two conditions:
(1) $F(\lambda)$ belongs to the space $M_{d(\lambda)}(\boldsymbol{C})$ of complex matrices of order $d(\lambda)$ for each $\lambda\in D$.

(2) $F(\lambda)$ is a rapidly decreasing function of λ: i.e., $\lim_{|\lambda|\to\infty}|\lambda|^k\|F(\lambda)\|=0$ for all $k\in\boldsymbol{N}$.

In the following, we use the inner product (X, Y) which satisfies the following condition:

$$(\lambda,\lambda)\geqq 1 \text{ for all } \lambda\in D_0=D-\{0\}. \tag{3.3}$$

The vector space $S(D)$ is topologized by the family of seminorms

$$\{q_s(F)=\operatorname*{Max}_{\lambda\in D}|\lambda|^s\|F(\lambda)\|:;\ \geqq 0\}.$$

By the condition (3.3), we get the following inequality for the seminorms on $S(D)$:

$$q_s(F)\leqq q_t(F)\qquad \text{if } 0<s\leqq t \tag{3.4}$$

for all F in $S(D)$.

Using these topologies, the result in Theorem 3 can be reformulated more precisely in the following Theorem 4.

Theorem 4. *The Fourier transform $\mathcal{F}:f\to\mathcal{F}f$ is a topological isomorphism of $C^\infty(G)$ onto $S(D)$.*

Proof. By Theorem 3, the Fourier transform $\mathcal{F}$ mapps $C^\infty(G)$ into $S(D)$. Since any continuous function f on G is uniquely determined by its Fourier coefficients $\mathcal{F}f(\lambda)$ by (1.5), the mapping $\mathcal{F}$ is injective. The mapping $\mathcal{F}$ is also surjective. Let F be a function in $S(D)$. Then the series

$$\sum_{\lambda\in D} d(\lambda)\operatorname{Tr}(F(\lambda)U^\lambda(g)) \tag{3.5}$$

converges uniformly on G, because the function F satisfies the condition (2) in Theorem 1. Let $f(g)$ be the sum of the series (3.5). Then f is a continuous function on G and the Fourier transform $\mathcal{F}f$ of f coincides with the original function

F by the orthogonality relations. Since $F(\lambda) = \mathcal{F}f(\lambda)$ is rapidly decreasing, the function f is a C^∞-function on G by Theorem 3. Thus we have proved that the Fourier transform $\mathcal{F}$ is a linear isomorphism of $C^\infty(G)$ onto $S(D)$.

Now we shall prove that the Fourier transform $\mathcal{F}$ is a homeomorphism. First we show that $\mathcal{F}$ is continuous. Since $(\Delta^k f)(\lambda) = \omega(\lambda)^k \mathcal{F}f(\lambda)$ (Lemma 1.2), we have

$$(3.6)\qquad \omega(\lambda)^k||\mathcal{F}f(\lambda)|| = ||\mathcal{F}(\Delta^k f)(\lambda)|| \leqq \int_G |\Delta^k f(g)|\ ||U^\lambda(g^{-1})||dg \leqq d(\lambda)^{1/2}||\Delta^k f||_\infty.$$

Since $|\lambda|^2 \leqq \omega(\lambda)$ and there exists a constant $M>0$ such that $d(\lambda)^{1/2} \leqq M|\lambda|^{m/2}$ for all $\lambda \in D_0$, we have

$$(3.7)\qquad |\lambda|^{2k-m/2}||\mathcal{F}f(\lambda)|| \leqq M||\Delta^k f||_\infty$$

by (3.6). Therefore we have

$$(3.8)\qquad q_{2k-m/2}(\mathcal{F}f) \leqq M||\Delta^k f||_\infty$$

for all f in $C^\infty(G)$ and all $k>\frac{1}{4}m$. Since k can be taken arbitrarily large, we have proved by (3,4) and (3,8) that for any $s>0$ there exists an integer $k>0$ such that the inequality

$$(3.9)\qquad q_s(\mathcal{F}f) \leqq M||\Delta^k f||_\infty.$$

is valid. On the other hand, since $||\mathcal{F}f(0)|| \leqq ||f||_\infty$ by the definition of $\mathcal{F}f$, we have

$$(3.10)\qquad q_0(\mathcal{F}f) \leqq ||\mathcal{F}f(0)|| + \operatorname*{Max}_{\lambda \in D_0} ||\mathcal{F}f(\lambda)|| \leqq ||f||_\infty + M||\Delta^k f||_\infty$$

for $k>\frac{1}{4}m$ by (3.3) and (3.7). The inequalities (3.8) and (3.10) prove that the Fourier transform $\mathcal{F}$ is a continuous mapping of $C^\infty(G)$ into $S(D)$.

Next we shall prove that the inverse Fourier transform $\mathcal{F}^{-1}$: $\mathcal{F}f \to f$ is continuous. Since $|\lambda|^2 \leqq \omega(\lambda)$ and there exists a constant $M>0$ such that $d(\lambda) \leqq M^2|\lambda|^m$, the series

$$(3.11)\qquad \sum_{\lambda \in D_0} d(\lambda)^{(3+k)/2}\omega(\lambda)^{-s}$$

converges to a positive real number K if $s>2^{-1}l+4^{-1}(k+3)m$ by Lemma 1.3. Let k be a positive integer and $X_1, \cdots, X_k$ be k elements in $\mathfrak{g}$. Then by (2.15) and Theorem 2, we have the inequality

$$(3.12)\qquad ||X_1 \cdots X_k f||_\infty \leqq \sum_{\lambda \in D_0} d(\lambda)^{(3+k)/2}|\lambda|^k||\mathcal{F}f(\lambda)||\ |X_1| \cdots |X_k|$$

$$= |X_1|\cdots|X_k|\sum_{\lambda\in D_0} d(\lambda)^{(3+k)/2}\omega(\lambda)^{-s}|\lambda|^k||\mathcal{F}(\Delta^s f)(\lambda)||$$
$$\leqq K|X_1|\cdots|X_k|q_k(\mathcal{F}(\Delta^s f))$$
$$= K|X_1|\cdots|X_k|\operatorname*{Max}_{\lambda\in D}\omega(\lambda)^s|\lambda|^s||\mathcal{F}f(\lambda)||.$$

if $s>\frac{1}{2}l+\frac{1}{4}(k+3)m$. Since $\omega(\lambda)=(\lambda,\lambda+2\delta)\leqq|\lambda|^2+2|\lambda||\delta|$

and $\omega(\lambda)^s\leqq\sum_{r=0}^{s}{}_sC_r|\lambda|^{s+r}(2|\delta|)^{s-r}$,

we have

$$\text{(3.13)}\qquad ||X_1\cdots X_k f||_\infty\leqq K|X_1|\cdots|X_k|\sum_{r=0}^{s}{}_sC_r(2|\delta|)^{s-r}q_{s+n+r}(\mathcal{F}f).$$

Similarly we have the inequality

$$\text{(3.14)}\qquad ||f||_\infty\leqq||\mathcal{F}f(0)||+\sum_{\lambda\in D_0}d(\lambda)|\mathrm{Tr}(\mathcal{F}f(\lambda)U^\lambda(g)|$$
$$\leqq q_0(\mathcal{F}f)+\sum_{\lambda\in D_0}d(\lambda)^{3/2}||\mathcal{F}f(\lambda)||$$
$$\leqq q_0(\mathcal{F}f)+\sum_{\lambda\in D_0}d(\lambda)^{3/2}\omega(\lambda)^{-s}||\mathcal{F}(\Delta^s f)(\lambda)||$$
$$\leqq q_0(\mathcal{F}f)+K\operatorname*{Max}_{\lambda\in D_0}\omega(\lambda)^s||\mathcal{F}f(\lambda)||$$
$$\leqq q_0(\mathcal{F}f)+K\sum_{r=0}^{s}{}_sC_r(2|\delta|)^{s-r}q_{s+r}(\mathcal{F}f).$$

for $s>\frac{1}{2}l+\frac{3m}{4}$.

The inequalities (3.13) and (3.14) prove that the inverse Fourier transform $\mathcal{F}^{-1}$: $\mathcal{F}f\to f$ is a continuous mapping from $S(D)$ into $C^\infty(G)$. q.e.d.

Corollary to Theorem 4. *The topology of $C^\infty(G)$ defined by the family of seminorms (3.0) (or (3.1)) coincides with the topology defined by the family of seminorms*

$$\{r_m(f)=||\Delta^m f||_\infty;\ m=0,1,2,\cdots\}.$$

Proof. This Corollary is clear from the inequalities (3.10) and (3.9) and Theorem 4.

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Added in proof

The Fourier series in Theorem 1 is obtained from the series (1.1) by first taking the partial sum $\sum_{i,j=1}^{d(\lambda)}$. However we can prove that the original series (1.1) converges absolutely and uniformly if f belongs to $C^{2k}(G)$ and $2k>\frac{n}{2}$.

This fact can be seen from the following inequalities:

$$\sum_{\lambda\in D_0}\sum_{i,j=1}^{d(\lambda)} d(\lambda)|(f,u_{ij}^{\lambda})|\,|u_{ij}^{\lambda}(g)| \leqq \sum_{\lambda\in D_0}\sum_{i,j\in 1}^{d(\lambda)} d(\lambda)|\lambda|^{-2k}|(\Delta^k f, u_{ij}^{\lambda})|\,|u_{ij}^{\lambda}(g)|$$

$$\leqq \Big(\sum_{\lambda\in D_0}\sum_{i,j=1}^{d(\lambda)} d(\lambda)|(\Delta^k f, u_{ij}^{\lambda})|^2\Big)^{1/2}\Big(\sum_{\lambda\in D_0}\sum_{i,j} d(\lambda)|\lambda|^{-4k}|u_{ij}^{\lambda}(g)|^2\Big)^{1/2}$$

$$\leqq ||\Delta^k f||_2\Big(\sum_{\lambda\in D_0} d(\lambda)^2|\lambda|^{-4k}\Big)^{1/2} \leqq ||\Delta^k f||_2 N\Big(\sum_{\lambda\in D_0}|\lambda|^{2m-4k}\Big)^{1/2}.$$