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Osaka, J. Math.
8 (1971), 21-28

ON THE SPACES OF GENERALIZED CURVATURE TENSOR FIELDS AND SECOND FUNDAMENTAL FORMS

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(Received October 12, 1970)

For a Riemannian manifold M let $\mathfrak{A}(M)$ be the vector space of all tensor fields A of type (1,1) that satisfy the following three conditions: (1) A is symmetric as an endomorphism of each tangent space $T_x(M)$, $x \in M$; (2) Codazzi's equation holds, that is, $(\nabla_X A)Y = (\nabla_Y A)(X)$ for all vector fields X and Y; (3) trace A is constant on M. It is hardly necessary to note that an isometric immersion of M into a space of constant sectional curvature as a hypersurface with constant mean curvature gives rise to such a tensor field A (namely, the second fundamental form), which furthermore satisfies the equation of Gauss. Now Y. Matsushima has shown (unpublished) that if M is a compact Riemannian manifold, then $\mathfrak{A}(M)$ is finite-dimensional. This is obtained as an application of the theory of vector bundle-valued harmonic forms (see [2] for other applications to the study of isometric immersions).

The purpose of the present paper is to prove two results (Theorems 1 and 2) of a similar nature. Theorem 1 generalizes the above result of Matsushima to the space of generalized second fundamental forms, which, geometrically, arise from isometric immersions of higher codimension. Theorem 2 shows finite-dimensionality of the space of generalized curvature tensor fields, which, as a matter of fact, implies the above result of Matsushima as we show in [3].

1. Forms with values in a Riemannian vector bundle

By a Riemannian vector bundle we shall mean a (real) vector bundle E over a Riemannian manifold M which has a fiber metric and a mtric connection ([1], Vol. I. pp. 116–7). The Riemannian metric on M and the fiber metric in E are denoted by $\langle\ ,\ \rangle$, whereas the Riemannian connection on M is denoted by ∇ and the metric connection in E by ∇'. If φ and ψ are sections of E and X is a vector field on M, then

$$X\langle\varphi, \psi\rangle = \langle\nabla'_X\varphi, \psi\rangle + \langle\varphi, \nabla'_X\psi\rangle.$$

Work supported by a grant of the National Science Foundation.

We denote by $C^p(E)$ the real vector space of all E-valued p-forms on M. If $\omega\in C^p(M)$, then, for each $x\in M$, ω_x is a skew-symmetric p-linear mapping of $T_x(M)\times\cdots\times T_x(M)$ (p times) into the fiber F_x over x. For $X\in T_x(M)$, the covariant derivative $\tilde{\nabla}_X\omega$ is defined as follows: if $X_1,\cdots,X_p\in T_x(M)$, then extending them to vector fields $\tilde{X}_1\cdots\tilde{X}_p$ on M we set

$$(\tilde{\nabla}_X\omega)(X_1,\cdots,X_p)=\nabla'_X\cdot\omega(\tilde{X}_1,\cdots,\tilde{X}_p)\\ -\sum_{i=1}^{n}\omega(X_1,\cdots,\nabla_X\tilde{X}_i,\cdots,X_p),$$

the right hand side being independent of the extensions of $X_1,\cdots,X_p$. We define the covariant differential $\tilde{\nabla}\omega$ of ω at x as a $(p+1)$-linear mapping

$$(X_1,\cdots,X_{p+1})\in T_x(M)\times\cdots\times T_x(M)\rightarrow(\tilde{\nabla}_{X_1}\omega)(X_2,\cdots,X_{p+1})\in F_x.$$

A differential operator ∂: $C^p(E)\rightarrow C^{p+1}(E)$ is defined essentially as an alternation of $\tilde{\nabla}\omega$. More precisely, we define

$$\partial\omega=(p+1)\,A\,(\tilde{\nabla}\omega),\quad \omega\in C^p\,(E),$$

where A is the alternation operator (see [1], Vol. I, p. 28; the present definition of $\tilde{\nabla}\omega$ differs from that in [1], Vol. I, p. 124, only in the order of $X_1,\cdots,X_{p+1}$). For our applications we note the special cases:

If $\omega\in C^1(E)$, then

$$(\partial\omega)(X,\,Y)=(\tilde{\nabla}_X\omega)\,Y-(\tilde{\nabla}_Y\omega)X.$$

If $\omega\in C^2(E)$, then

$$(\partial\omega)(X,\,Y,\,Z)=(\tilde{\nabla}_X\omega)(Y,Z)+(\tilde{\nabla}_Y\omega)(Z,\,X)+(\tilde{\nabla}_Z\omega)(X,\,Y).$$

On the other hand, we define a differential operator ∂^*: $C^p(E)\rightarrow C^{p-1}(E)$ as follows. If $\omega\in C^0(E)$, then $\partial^*\omega=0$. If $\omega\in C^p(E)$, $p\geqq 1$, and $x\in M$, let $\{e_1,\cdots,e_n\}$ be an orthonormal basis in $T_x(M)$ and set

$$(\partial^*\omega)_x(X_1,\cdots,X_{p-1})=-\sum_{i=1}^{n}(\tilde{\nabla}_{e_i}\omega)(e_i,\,X_1,\cdots,X_{p-1}),$$

the right hand side being independent of the choice of $\{e_i\}$.

The Laplacian $\Box$ of E-valued forms is defined by

$$\Box=\partial\partial^*+\partial^*\partial.$$

The following two basic facts are classical in the case where E is a trivial line bundle.

Proposition 1. *If M is compact, then $\Box\,\omega=0$ if and only if $\partial\omega=0$ and $\partial^*\omega=0$.*

Proposition 2. *If M is compact, then $\square$ is elliptic so that the vector space $\{\omega\in C^p(E);\ \square\omega=0\}$ is finite-dimensional.*

For the proof of Proposition 1, we introduce (assuming that M is orientable) an inner product in $C^p(E)$ by

$$(\theta,\omega)=\int_M\langle\theta,\omega\rangle dv,$$

where dv is the volume element of M and $\langle\theta,\omega\rangle_x$ is the natural inner product in the space of p-forms at x, that is,

$$\langle\theta,\omega\rangle_x=\sum_{i_1<\cdots<i_p}\langle\theta(e_{i_1},\cdots,e_{i_p}),\ \omega(e_{i_1},\cdots,e_{i_p})\rangle,$$

where $\{e_1,\cdots,e_n\}$ is an orthonormal basis in $T_x(M)$. Using this inner product we can show that ∂ and ∂^* are adjoint to each other:

$$(\partial\theta,\omega)=(\theta,\partial^*\omega)\quad\text{for}\quad\theta\in C^{p-1}(E),\ \omega\in C^p(E).$$

This fact readily implies Proposition 1.

In order to prove Proposition 2 it is sufficient to check the principal part of $\square\omega$. We shall here give the detail in the case of $\omega\in C^1(E)$; the general case is essential by similar.

At $x\in M$, let $\{e_1,\cdots,e_n\}$ be an orthonormal basis in $T_x(M)$ and extend to them to vector fields $E_1,\cdots,E_n$ such that $\nabla_{e_i}E_j=0$ for all $i.j$. Also let $X\in T_x(M)$ be extended to a vector field $\tilde{X}$ such that $\nabla_{e_i}\tilde{X}=0$ for all i. At x we find

$$(\partial^*\partial\omega)(X)=-\sum_{i=1}^n(\tilde{\nabla}_{e_i}\tilde{\nabla}_{E_i}\omega)X+\sum_{i=1}^n(\tilde{\nabla}_{e_i}\tilde{\nabla}_{\tilde{X}}\omega)\,e_i$$

and

$$(\partial\partial^*\omega)(X)=-\sum_{i=1}^n(\tilde{\nabla}_X\tilde{\nabla}_{E_i}\omega)\,e_i$$

so that

$$(\square\omega)(X)=-\sum_{i=1}^n(\tilde{\nabla}_{e_i}\tilde{\nabla}_{E_i}\omega)(X)+\sum_{i=1}^n(\tilde{\nabla}_{e_i}\tilde{\nabla}_{\tilde{X}}\omega)\,e_i$$
$$-\sum_{i=1}^n(\tilde{\nabla}_X\tilde{\nabla}_{E_i}\omega)\,e_i.$$

We have

Lemma. *For $\omega\in C^1(E)$ and for any vector fields X, Y and Z on M, we have*

$$(([\tilde{\nabla}_X,\tilde{\nabla}_Y]-\tilde{\nabla}_{[X,Y]})\omega)\,Z$$
$$=R'(X,Y)\cdot\omega(Z)-\omega(R(X,Y)Z),$$

where $R'(X,Y)$ is the curvature transformation for the connection ∇' in E defined by

$$R'(X, Y)\varphi = [\nabla'_X, \nabla'_Y]\varphi - \nabla'_{[X,Y]}\varphi, \qquad \varphi \in C^0(E).$$

Using this lemma, whose proof is straightforward, and noting that $[E_i, \tilde{X}] = 0$ at x, we obtain at x

$$(\Box\omega)X = -\sum_{i=1}^{n}(\tilde{\nabla}_{e_i}\tilde{\nabla}_{E_i}\omega)X + \sum_{i=1}^{n} R'(e_i, X)\cdot\omega(e_i)$$
$$- \omega(\sum_{i=1}^{n} R(e_i, X)e_i).$$

(We note that $\sum_{i=1}^{n} R(e_i, X)e_i$ is equal to $-S(X)$, where S is the Ricci tensor of type (1,1) of M).

Now if $\varphi_1, \cdots, \varphi_p$ are linearly independent sections of E such that $\nabla'_X \varphi_m = 0$ $X \in T_x(M)$, for all then writing $\omega = \sum_{m=1}^{p} \omega^m \varphi_m$ we get

$$\tilde{\nabla}_{E_i}\omega = \sum_{m=1}^{p}(\nabla_{E_i}\omega^m)\varphi_m + \sum_{m=1}^{p}\omega^m(\nabla'_E \varphi_m)$$

and

$$\sum_{i=1}^{n}\tilde{\nabla}_{e_i}\tilde{\nabla}_{E_i}\omega = \sum_{m=1}^{p}\sum_{i=1}^{n}\{(\nabla_{e_i}\nabla_{E_i}\omega^m)\varphi_m + \omega^m(\nabla'_{e_i}\nabla'_{E_i}\varphi_m)\}.$$

Thus the φ_m-component of the principal part of $\Box\omega$ is given by $\sum_{i=1}^{n}\nabla_{e_i}\nabla_{E_i}\omega^m$. This proves that $\Box$ is an elliptic operator.

2. Generalized second fundamental forms

Let N be a Riemannian vector bundle (whose connection is denoted by ∇') over a Riemannian manifold M. For the tangent bundle $T(M)$ and its dual bundle T^*, the vector bundle Hom $(N, T^*\otimes T)$ is a Riemannian vector bundle over M in the natural way. For a section A of Hom $(N, T^*\otimes T)$, which is expressed by $\xi \in N_x \to A_\xi \in T^*_x \otimes T_x$ at each $x \in M$, and for any vector field X on M, the covariant derivative $\tilde{\nabla}_X A$ is a section such that

$$(\tilde{\nabla}_X A)_\xi = \nabla_X(A_\xi) - A_{\nabla'_X\xi},$$

where ξ is any section of N. We shall call A a *generalized second fundamental form* if A_ξ is a symmetric endomorphism of $T_x(M)$ for every $\xi \in N_x$, $x \in M$, and if A satisfies Codazzi's equation, that is,

$$(\tilde{\nabla}_X A)_\xi Y = (\tilde{\nabla}_Y A)_\xi X$$

for every section ξ of N and for all vector fields X and Y on M. Actually, each side of the equation makes sense for $\xi \in N_x$ and $X, Y \in T_x(M)$. Geometrically, an isometric immersion of M into a space of constant sectional curvature gives rise to the normal bundle N and the seond fundamental form A which satisfies the equations of Gauss and Codazzi (for example, see [1], Vol. II, p. 14, p. 23–25).

For a section A of Hom $(N, T^* \otimes T)$ we define the *mean curvature section* η of A as follows. If $\{\xi_1 \cdots, \xi_p\}$ is an orthonormal basis of N_x, $x \in M$, then

$$\eta_x = \frac{1}{n} \sum_{i=1}^{p} (\text{trace } A_{\xi_i}) \xi_i, \quad n = \dim M,$$

We say that A has *constant mean curvature* if the mean curvature section η of A is parallel with respect to the connection ∇' in N.

For a Riemannian vector bundle N over M, let $\mathfrak{A}(M, N)$ be the set of generalized second fundamental forms A with constant mean curvature. It is a real vector space in the natural fashion. We have

Theorem 1. *If M is compact, then $\mathfrak{A}(M, N)$ is finite-dimensional.*

Proof. We consider one more vector bundle $E = \text{Hom}(N, T)$, which is a Riemannian vector bundle over M in the natural way, For a section φ of E and a vector field X on M, the covariant derivative $\nabla^*_X \varphi$ is defined by

$$(\nabla^*_X \varphi)\xi = \nabla_X(\varphi(\xi)) - \varphi(\nabla'_X \xi),$$

where ξ is any section of N.

We consider $A \in \mathfrak{A}(M, N)$, which is a section of Hom $(N, T^* \otimes T)$, as an E-valued 1-form ω as follows: for any $X \in T_x(M)$, $x \in M$, $\omega(X)$ is the element of Hom (N_x, T_x) such that

$$\omega(X) \cdot \xi = A_\xi(X).$$

The covariant derivative $\tilde{\nabla}_X \omega$ of ω is the E-valued 1-form such that

$$\begin{aligned}(\tilde{\nabla}_X \omega)(Y) \cdot \xi &= (\nabla^*_X(\omega(Y)))\xi - \omega(\nabla_X Y)\xi \\ &= \nabla_X(\omega(Y) \cdot \xi) - \omega(Y)(\nabla'_X \xi) - \omega(\nabla_X Y)\xi \\ &= \nabla_X(A_\xi Y) - A_{\nabla'_X \xi} Y - A_\xi(\nabla_X Y) \\ &= (\nabla_X A_\xi) Y - A_{\nabla'_X \xi} Y \\ &= (\tilde{\nabla}_X A)_\xi Y,\end{aligned}$$

where Y is a vector field on M and ξ is a section of N. Thus Codazzi's equation for A is equivalent to

$$(\tilde{\nabla}_X \omega) Y = (\tilde{\nabla}_Y \omega) X,$$

that is,

$$\partial \omega = 0.$$

On the other hand, $\partial^* \omega$ is an E-valued 0-form (i.e. a section of E) defined by

$$(\partial^*\omega)_x = -\sum_{i=1}^{n} (\tilde{\nabla}_{e_i}\omega)(e)_i\,, \qquad x \in M\,,$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis in $T_x(M)$. For any $\xi \in N_x$, we extend it to a section of N such that $\nabla'_X\xi = 0$ for every $X \in T_x(M)$. Then $(\tilde{\nabla}_{e_i}A)_\xi = \nabla_{e_i}(A_\xi)$ at x. Also, Codazzi's equation in this case gives $(\nabla_{e_i}A_\xi)Y = (\nabla_Y A_\xi)e_i$ for any $Y \in T_x(M)$. Also noting that $\nabla_{e_i}A_\xi$ is symmetric together with A_ξ, we have

$$\begin{aligned}
-\langle(\partial^*\omega)\cdot\xi,\ Y\rangle &= \sum_{i=1}^{n} \langle(\tilde{\nabla}_{e_i}\omega)(e_i)\cdot\xi,\ Y\rangle \\
&= \sum_{i=1}^{n} \langle(\tilde{\nabla}_{e_i}A)_\xi e_i,\ Y\rangle = \sum_{i=1}^{n} \langle(\nabla_{e_i}A_\xi)e_i,\ Y\rangle \\
&= \sum_{i=1}^{n} \langle e_i,\ (\nabla_{e_i}A_\xi)Y\rangle = \sum_{i=1}^{n} \langle e_i,\ (\nabla_Y A_\xi)e_i\rangle \\
&= \text{trace}\,(\nabla_Y A_\xi) = Y\cdot\text{trace}\, A_\xi\,.
\end{aligned}$$

As in the following lemma, this is 0 if and only if A has constant mean curvature.

Lemma. *If A has constant mean curvature, then, for a section ξ of N of unit length such that $\nabla'_X\xi = 0$ for every $X \in T_x(M)$, we have $X\cdot\text{trace}\ A_\xi = 0$ for every $X \in T_x(M)$. The converse also holds.*

To prove the lemma, let $\xi_1 = \xi$ and choose sections $\xi_2, \cdots, \xi_p$ such that they are orthonormal at each point and $\nabla'_X\xi_i = 0$ for every $X \in T_x(M)$. Then

$$\nabla'_X\eta = \frac{1}{n}\sum_{i=1}^{n} X\cdot(\text{trace}\, A_{\xi_i})\xi_i \qquad \text{at } x.$$

Thus $\nabla'_X\eta = 0$ at x if and only if $X\cdot$ trace $A_{\xi_i} = 0$, $1 \le i \le p$.

We have thus shown that, for the E-valued 1-form ω corresponding to a generalized second fundamental form A, $\partial^*\omega = 0$ if and only if A has constant mean curvature.

The mapping $\omega \to A$ is clearly a linear isomorphism of $\mathfrak{A}(M, N)$ into the vector space $\{\omega \in C^1(E);\ \Box\omega = 0\}$. By Proposition 2 we see that $\mathfrak{A}(M, N)$ is finite-dimensional. This completes the proof of Theorem 1.

3. Generalized curvature tensor fields

Let M be a Riemannian manifold. A tensor field L of type $(1,3)$ defines at each $x \in M$ a bilinear mapping

$$(X,\ Y) \in T_x(M) \times T_x(M) \to L(X,\ Y) \in \text{Hom}\,(T_x(M),\ T_x(M))\,.$$

We say that L is a *generalized curvature tensor field* if it has the following properties for all vector fields X, Y and Z:

(1) $L(Y, X)=-L(X, Y)$;

(2) $L(X, Y)$ is a skew-symmetric endomorphism of $T_x(M)$;

(3) $\mathfrak{S}L(X, Y)Z=0$, where $\mathfrak{S}$ denotes the cyclic sum over X, Y, and Z (first Bianchi identity).

We shall say that L is proper if it satisfies the second Bianchi identity: $\mathfrak{S}(\nabla_X L)(Y, Z)=0$.

For a generalized curvature tensor field L, we define its Ricci tensor field $K=K_L$ by

$$K(X)=\sum_{i=1}^{n} L(X, e_i)e_i \qquad \text{for } X\in T_x(M),$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis in $T_x(M)$. It follows from the first Bianchi identity that K is then a symmetric endomorphism of $T_x(M)$.

We shall denote by $\mathfrak{L}(M)$ the vector space of all proper generalized curvature tensor fields L whose Ricci tensor fields K satisfy Codazzi's equation: $(\nabla_X K)Y=(\nabla_Y K)X$ for all vector fields X and Y.

We shall prove

Theorem 2. *If M is a compact Riemannian manifold, then $\mathfrak{L}(M)$ is finite-dimensional.*

Proof. Let $O(M)$ be the bundle of orthonormal frames of M. The structure group $O(n)$ acts on its Lie algebra $\mathfrak{o}(n)$ of all skew-symmetric matrices of degree n through its adjoint representation. Let E be the vector bundle associated to $O(M)$ with the standard fiber $\mathfrak{o}(n)$. The Riemannian connection in $O(M)$ and the $ad(O(n))$–invariant inner product in $\mathfrak{o}(n)$ make E a Riemannian vector bundle over M. For each $x\in M$, the fiber over x can be considered as the vector space of all skew-symmetric endomorphisms of $T_x(M)$.

This being said, we consider a generalized curvature tensor field L as an E-valued 2-form: for $X, Y\in T_x(M)$, $L(X, Y)$ is an element of the fiber of E over x. In order to prove Theorem 2 we shall show that $\partial L=0$ and $\partial^* L=0$ for $L\in\mathfrak{L}(M)$. We note that for the natural connection in E the covariant derivative of a section φ of E is nothing but the covariant derivative with respect to the Riemannian connection ∇ on M of the corresponding tensor field of type $(1,1)$. With this remark, we have

$$(\partial L)(X, Y, Z)=\mathfrak{S}(\nabla_X L)(Y, Z).$$

Hence $\partial L=0$ if an only if L is proper.

As for $\partial^* L$, we have at $x\in M$

$$(\partial^* L)(X)=-\sum_{i=1}^{n}(\nabla_{e_i}L)(e_i, X), \qquad X\in T_x(M),$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis in $T_x(M)$. For the Ricci tensor field K of L we have

$$(\nabla_X K)Y = \sum_{i=1}^{n} (\nabla_X L)(Y, e_i)e_i \,.$$

If L is proper, this is equal to

$$-\sum_{i=1}^{n} (\nabla_Y L)(e_i, X)e_i - \sum_{i=1}^{n} (\nabla_{e_i} L)(X, Y)e_i \,.$$

The first term is equal to $(\nabla_Y K)X$. Since $\nabla_{e_i} L$ satisfies the first Bianchi identity, the second term is equal to

$$\begin{aligned} &\sum_{i=1}^{n} (\nabla_{e_i} L)(Y, e_i)X + \sum_{i=1}^{n} (\nabla_{e_i} L)(e_i, X)Y \\ &= (\partial^* L)(Y)X - (\partial^* L)(X)Y. \end{aligned}$$

Thus we obtain

$$(\partial^* L)(X)Y - (\partial^* L)(Y)X = (\nabla_X K)Y - (\nabla_Y K)X \,.$$

If K satisfies Codazzi's equation, the E-valued 1-form $l = \partial^* L$ satisfies

$$(*) \qquad l(X)Y = l(Y)X \,.$$

We shall show that $l=0$. (Conversely, if $l=0$, then K obviously satisfies Codazzi's equation.) Using skew-symmetry of $l(X)$ and $l(Y)$ and the property $(*)$, we get

$$\langle l(Y)Y, X \rangle = -\langle Y, l(Y)X \rangle = -\langle Y, l(X)Y \rangle = 0 \,.$$

Thus $l(Y)Y=0$ for all $Y \in T_x(M)$. By polarization we get $l(X)Y + l(Y)X = 0$. This together with $(*)$ implies $l(X)\,Y=0$ for all X and Y, that is, $l=0$. Hence $\partial^* L = 0$ for $L \in \mathfrak{L}(M)$. We have thus proved Theorem 2.

The significance of Codazzi's equation for the Ricci tensor field K as well as the relationship of Theorem 2 to the result of Matsushima (Theorem 1 for the case where N is a trivial line bundle) are discussed in [3].

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