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ON THE BORSUK-ULAM THEOREM FOR Z_{p^a} ACTIONS ON S^{2n-1} AND MAPS $S^{2n-1}\to R^m$

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1. Introduction

In [3] we raised the following question:

Let G be a finite group acting properly (as a group of *homeomorphisms*) on the n-sphere. For any (topological) m-manifold M and any map $f: S^n\to M$ let $A(f)=\{x\in S^n \mid f(x)=f(xg), \text{ all } g\in G\}$. What can be deduced about $\dim A(f)$?

In case $M=R^m$, euclidean m-space, $A(f)$ is the set of solutions of $(|G|-1)m+1$ equations in $n+1$ unknowns so one might hope to get

$$\dim A(f)\geq n-(|G|-1)m\,. \tag{1.1}$$

If G is cyclic of prime order then (1.1) actually holds even for maps $f: S^n\to M^m$ provided M^m is compact (for $G=Z_2$ and $m=n$ assume also that $f_*=0: H_n(S^n; Z_2)\to H_n(M^n; Z_2)$), see [3]. In this note we consider $G=Z_{p^a}$, cyclic of *odd* prime power order, and we restrict attention to maps into R^m. Our results are expressed in two theorems:

Mod p^a Borsuk-Ulam theorem: *For any proper action of Z_{p^a} on S^{2n-1}, p an odd prime, and any map $f: S^{2n-1}\to R^m$ one has*

$$\dim A(f)\geq(2n-1)-(p^a-1)m-[m(a-1)p^a-(ma+2)p^{a-1}+m+3]\,.$$

Mod p^a Borsuk-Ulam anti-theorem: *Consider the standard linear action of Z_{p^a} on S^{2n-1}. Assume $a>1$ and $p^a\neq 9$. If $2n-1\leq(p^a-1)m+(2p-3)m-1$ then there exists a map $f: S^{2n-1}\to R^m$ with $A(f)=\phi$.*

Notice that the anti-theorem says that (1.1) fails whenever $a>1$ and $p^a\neq 9$; the theorem gives $m(a-1)p^a-(ma+2)p^{a-1}+m+3$ as an upper bound for this failure. For $a=1$ this upper bound is 1, so for $G=Z_p$ we are 1 off our previous results [3].

REMARKS. 1. dim means covering dimension.

2. For $p^a=9$ and $m>1$ there is a result similar to the anti-theorem. We leave that to the interested reader.

3. In private correspondence with M. Nakaoka I have recently learned that (1.1) holds for Z_p-actions on mod p homology spheres $\mathcal{S}^n$ and maps f: $\mathcal{S}^n \to M^m$ without the restriction of niceness of f imposed by me in [3].

2. Proof of theorem

Let μ: $S \times G \to S$ be a proper action of the cyclic group G of *odd* prime power order $p^a = q = 2k+1$ on the $(2n-1)$ sphere S. Denote by η the corresponding principal G-bundle $S \to S/\mu$ over the orbit space S/μ. For a complex G-module M let $\eta[M]$ be the complex vector bundle $S \times_G M \to S/\mu$. The correspondence $M \mapsto \eta[M]$ gives rise to a ring homomorphism α: $\mathcal{R}G \to K^0(S/\mu)$ where $\mathcal{R}G$ is the complex representation ring for G while K^0 denotes complex K-theory. Denote by L the standard 1–dimensional complex G-module, i.e. $L = C$, the field of complex numbers, and, fixing a generator g_0 for G, $g_0 c = \exp(2\pi q^{-1}\sqrt{-1})c$. Then $\mathcal{R}G = Z[\rho]/(\rho^q - 1)$ where ρ is the class of L. Finally, put $\lambda = \eta[L]$ and for any map f: $S \to R^m$ let λ_f be the restriction of λ to $A(f)/\mu \subseteq S/\mu$.

Now the mod p^a Borsuk-Ulam theorem is essentially contained in

Lemma 1. *If $d\lambda_f$ has a never vanishing section then $d \geq n-1+p^{a-1} - \frac{1}{2}am(p^a - p^{a-1})$.*

Proof. Assume that $d\lambda_f$ has a never vanishing section. We first show

$$(2.1)\quad P(\rho) := (\rho-1)^d[(\rho-1)(\rho^2-1)\cdots(\rho^k-1)]^m \in (\rho-1)^n \cdot Z[\rho]/(\rho^q-1)\,.$$

Recall that the i^{th} Atiyah class $a_i(\xi)$ of an n-dimensional complex vector bundle ξ is given by $a_i(\xi) = \gamma^i(\xi - n)$ with γ^i as in [1]. Then we have the usual Whitney duality, namely $a_i(\xi_1 \oplus \xi_2) = \sum_{j+k=i} a_j(\xi_1)a_k(\xi_2)$, also for any line bundle ξ, $a_1(\xi) = \xi - 1$. Therefore it is immediate that $\alpha P(\rho) = a_{d+mk}(\Lambda)$ where Λ is the vector bundle $d\lambda \oplus m[\lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^k]$, and so (2.1) follows from

$$(2.2)\qquad \mathrm{Ker}\,(\alpha\colon \mathcal{R}G \to K^0(S/\mu)) \subseteq (\rho-1)^n \cdot \mathcal{R}G\,,$$

(2.3) $\qquad \Lambda$ admits a never vanishing section.

To get (2.2) we compare μ with the standard linear action μ_0: $S \times G \to S$ obtained by viewing S as the unit sphere in $nL = L \oplus \cdots \oplus L$. S^{2n-1}/μ_0 is a $(2n-1)$-dimensional cell complex and η: $S \to S/\mu$ is $(2n-1)$-universal in the sense of [5]. Hence there is a bundle map

$$\begin{array}{ccc} S & \xrightarrow{\;b\;} & S \\ \downarrow{\scriptstyle \eta_0} & \bar{b} & \downarrow{\scriptstyle \eta} \\ S/\mu_0 & \longrightarrow & S/\mu\,. \end{array}$$

Furthermore, it is obvious that

$$\begin{tikzcd} \mathcal{R}G \arrow[r, "\alpha"] \arrow[dr, "\alpha_0"'] & K^0(S/\mu) \arrow[d, "\bar{b}^*"] \\ & K^0(S/\mu_0) \end{tikzcd}$$

commutes, so Ker $\alpha \subseteq$ Ker α_0. But α_0 fits into an exact sequence (see [1])

$$\mathcal{R}G \xrightarrow{\varphi} \mathcal{R}G \xrightarrow{\alpha_0} K^0(S/\mu_0)$$

where φ is multiplication by $\lambda_{-1}(n\rho)=(1-\rho)^n$, so Ker $\alpha_0 \subseteq (\rho-1)^n \cdot \mathcal{R}G$.

In [3] it is shown that f gives rise to a section S of $m(\lambda \oplus \lambda^2 \cdots \oplus \lambda^k)$ which vanishes precisely on $A(f)/\mu$ (see especially Digression 1, p. 171–2 and Step 3, p. 180–1 of [3]). s and the given section s_0 of $d\lambda_f$ go together to prove (2.3). This completes the proof of (2.1).

Our next step is to show that (2.1) is actually equivalent to the inequality $d \geq n-1+p^{a-1}-\frac{1}{2}am(p^a-p^{a-1})$. The equivalence is obvious if $n<d+mk$, so assume $n \geq d+mk$. Lift (2.1) to the polynomial ring $Z[x]$ to get the equivalent

(2. 1. 1) $$\exists g,\, h \in Z[x] \colon P(x) = g(x)(x-1)^n + h(x)(x^q-1)\,.$$

Now $P(x)=(x-1)^{d+mk} \cdot \prod_{j=2}^{k} f_j(x)^{m[k/j]}$; $(x^q-1)=(x-1)\cdot\prod_{i=1}^{a} f_{p^i}(x)$ where f_j is the j^{th} cyclotomic polynomial and $[k/j]$ is the integral part of k/j. Hence, if (2.1.1) holds then g is divisible by $\prod_{i=1}^{a-1} f_{p^i}(x)$ and h is divisible by $(x-1)^{d+km-1}$. So, putting $\varepsilon_j=0$ if $j \nmid p^a$, $\varepsilon_j=1$ if $j \mid p^a$, (2.1.1) implies (and is clearly implied by)

(2. 1. 2) $$\exists \bar{g},\, \bar{h} \in Z[x] \colon \prod_{j=2}^{k} f_j(x)^{m[k/j]-\varepsilon_j} = \bar{g}(x)\cdot(x-1)^{n-d-km} + \bar{h}(x)\cdot f_q(x)\,.$$

Let γ be a primitive q^{th} root of unity and consider the projection $Z[x] \to Z[\gamma] \subseteq C$. Its kernel is the ideal generated by $f_q(x)$ so (2.1.2) is equivalent to

(2. 1. 3) $$(\gamma-1)^{n-d-km} \mid \prod_{j=2}^{k} f(\gamma)^{m[k/j]-\varepsilon_j} \qquad \text{in } Z[\gamma]\,.$$

Now $Z[\gamma]$ is precisely the algebraic integers of the field $Q(\gamma)$ and $(\gamma-1)Z[\gamma]$ is the *unique* prime ideal in $Z[\gamma]$ lying above pZ, see e.q. [6]. Let $\mathcal{N} \colon Q(\gamma) \to Q$ be the norm map for the extension $Q(\gamma)/Q$. It is then an immediate consequence of classical ideal theory for Dedekind extensions that (2.1.3) is equivalent to

(2. 1. 4) $$\mathcal{N}(\gamma-1)^{n-d-km} \mid \prod_{j=1}^{k} \mathcal{N}(f_j(\gamma))^{m[k/j]-\varepsilon_j} \qquad \text{in } Z.$$

The norms involved here are not hard to compute, so rearranging (2.1.4) slightly it takes the desired form $d \geq n-1+p^{a-1}-\frac{1}{2}am(p^a-p^{a-1})$.

If $A(f)/\mu$ happens to be a CW complex then of course we have $(\dim A(f)/\mu<2d)\Rightarrow(d\lambda_f$ has a never vanishing section), and the above lemma can then be translated into a condition on $\dim A(f)/\mu$. Since also $\dim A(f)=\dim A(f)/\mu$ (because $A(f)\to A(f)/\mu$ is a finite covering and dim has the monotonicity and sum-properties, see [4]) this completes the proof of the mod p^a Borsuk-Ulam theorem. $A(f)/\mu$, however, need not be a CW complex so we need to know the following

Lemma 2. *If λ is a complex line bundle over a compact metric space X of covering dimension $<2d$ then $d\lambda$ admits a never vanishing section.*

I certainly do not believe that this lemma is unknown. However, nor do I know of any reference for it, so a proof of it is given as an appendix.

3. Proof of the anti-theorem

Consider the standard linear action μ_0 of $G=Z_{p^a}$ on S^{2N-1}, N big, i.e. view S^{2N-1} as the unit sphere in $NL=L\oplus\cdots\oplus L$. S^{2N-1}/μ_0 is a CW-complex with S^{2N-1}/μ_0 as $(2n-1)$-skeleton. Let ξ be the vector bundle $S^{2N-1}\times_G IG\to S^{2N-1}/\mu_0$ where IG is the augmentation ideal of the real group algebra RG. We notice that the anti-theorem is a consequence of

(3. 1) $m\xi$ admits a never vanishing section over the $[(p^a-1)m+(2p-3)m-1]$-skeleton.

Indeed, it is well known how a section s of $m\xi$ over the $(2n-1)$-skeleton corresponds to an equivariant map F: $S^{2n-1}\to m(IG)=IG\oplus IG\oplus\cdots\oplus IG$. If i: $IG\to RG$ is the inclusion then equivariance of F means that $(i\oplus\cdots\oplus i)F$ has the form $(i\oplus\cdots\oplus i)F(x)=(\Sigma_g f_1(xg^{-1})g,\cdots,\Sigma_g f_m(xg^{-1})g)$ for well defined continuous maps f_i: $S^{2n-1}\to R$. Put $f=(f_1,\cdots,f_m)$ and notice that $A(f)=\phi$ is equivalent to s having no zeros.

If we have shown (3.1) for $m=1$ then it follows for general m by noticing that $m\xi\cong\Delta^*(\xi\times\cdots\times\xi)$ for any skeletal approximation Δ: $S^{2N-1}/\mu_0\to S^{2N-1}/\mu_0\times\cdots\times S^{2N-1}/\mu_0$ to the m-fold diagonal. Hence, assume $m=1$. In [3] we showed that the mod p Euler class of ξ vanishes whenever $a>1$. If we further exclude the case $p^a=9$ then the same proof shows that the integral Euler class vanishes. Hence ξ does have a never vanishing section over the $2k$ skeleton. The obstructions to extending this section over the succesive skeleta lie in $H^{2k+i}(S^{2N-1}/G;\pi_{2k-1+i}(S^{2k-1}))\cong H^{2k+i}(G;\pi_{2k-1+i}(S^{2k-1}))$. For $0<i<2p-3$ the homotopy group in question has vanishing p-primary component so the obstructions vanish and we do have our desired section over the $(2k+2p-4)$-skeleton.

REMARK. In the above we have made strong use of the fact that ξ admits a complex structure so that ξ is orientable and hence no twisting of coefficients occur.

4. Remarks on the case $G=Z_{25}$, $m=1$, linear action

For $G=Z_{25}$ and $m=1$ our results show that there exists a map $f\colon S^{29}\to R$ with $A(f)=\phi$ whereas every map $f\colon S^{33}\to R$ has $A(f)\neq\phi$. In fact every map $f\colon S^{31}\to R$ has $A(f)\neq\phi$ as we now show. Suppose that $A(f_0)=\phi$ for some $f_0\colon S^{31}\to R$. Then

$$s_0(xG)=(x,\ \pi(\Sigma_g f_0(xg^{-1})g))G$$

defines a cross-section s_0 of ξ over the 31–skeleton. ($\pi\colon RG\to IG$ is given by $\pi(\Sigma r_g/g)=\Sigma r_g(g-1)$). The obstruction to extending s_0 further lie in $H^{32+i}(S^{2N-1}/Z_{25};\ \pi_{31+i}(S^{23}))$. Since the 5–primary component of $\pi_{31+i}(S^{23})$ is zero for $0\leq i<6$ we get a never vanishing section over the 37–skeleton. As in §3 this gives an $f\colon S^{37}\to R$ with $A(f)=\phi$. But that contradicts the above result for maps $S^{33}\to R$.

Unfortunately for $p^a>25$ our positive and negative results are too far apart to close the gap between them by means as trivial as the above.

Appendix. Proof of lemma 2

Let Δ be the abstract $4d$–1 simplex and $|\Delta|$ its standard realization in R^{4d}. By the general embedding theorem for compacta (see e.g. p. 139 of [2]) X can be taken as a closed subspace of $|\Delta|$. Let K_n be the subcomplex of $\Delta^{(n)}$ ($=n^{th}$ barycentric subdivision of Δ) spanned by all $4d-1$ simplices τ for which $|\tau|\cap X\neq\phi$. Then K_n is a subcomplex of the barycentric subdivision of K_{n-1} so the inclusion $i_n\colon |K_n|\to|K_{n-1}|$ admits a simplicial approximation $\varphi_n\colon K_n\to K_{n-1}$. Also $\{|K_n|\}$ is cofinal in the (downward) directed set of all neighborhoods of X in $|\Delta|$, so for any abelian group A we have $\check{H}^*(X;A)\cong\lim_{\to n} H^*(|K_n|;A)$, where as usual $\check{H}^*$ is Cech cohomology, while H^* can be taken as any ordinary cohomology theory. Since line bundles are characterized by the first Chern class $c_1\in\check{H}^2(-;Z)$ it follows that λ admits an extension λ_N over $|K_N|$ for N sufficiently large. Fix such an N and define (inductively, for $n>N$) $\lambda_n=|\varphi_n|^*\lambda_{n-1}$. Let σ_n be the sphere bundle associated with $d\lambda_n$. Since $\lambda\cong\lambda_n|X$, $n\geq N$, it is clearly sufficient to show that σ_n admits a cross-section when n is sufficiently large, in other words, if we let k be the maximal number such that for some $n\geq N$ σ_n admits a cross-section over the k-skeleton $|K_n^k|$ of K_n, then we must show $k\geq 4d-1$. Suppose $k<4d-1$. Choose $n\geq N$ such that $\sigma_n\big||K_n^k|$ has a cross-section, s, say. Consider the restriction s' of s to the $(k-1)$-skeleton and the obstruction c to extending s' over the $(k+1)$-skeleton (obstruction in the sense of [5]). $c\in H^{k+1}(|K_n|;\pi)$ where $\pi=\pi_k(S^{2d-1})$, and —

by maximality of $k-$ $c \neq 0$. Since k is clearly $\geq 2d-1$ our assumption on $\dim X$ assures that $H^{k+1}(X; \pi) = \lim_{\to j} H^{k+1}(|K_j|; \pi)$ vanishes so there is an $m > n$ such that $c \big| |K_m| = |\varphi|^* c = 0$; here φ is an abbreviation for $\varphi_{n+1}\varphi_{n+2}\cdots \varphi_m \colon K_m \to K_n$. Now $\sigma_m = |\varphi|^* \sigma_n$ and $|\varphi|$ is skeleton preserving so s gives rise to a cross-section s_1 of $\sigma_m \big| |K_m^k|$. Moreover, if s_1' is the restriction of s_1 to $|K_m^{k-1}|$ then the obstruction to extending s_1' over $|K_m^{k+1}|$ is precisely $|\varphi|^* c$. But $|\varphi|^* c = 0$ so s_1' *does* extend over $|K_m^{k+1}|$, thus contradicting the maximality of k.

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