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ON MARKOV CHAINS SIMILAR TO THE REFLECTING BARRIER BROWNIAN MOTION

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Contents

1. Introduction

The reflecting barrier (r. b.) Brownian motion in *one dimension* has been well known [12], [16]. In *higher dimensions*, the r. b. Brownian motion on a bounded domain D with *smooth boundary* ∂D is more or less classical (at least analytically) [13], [17], [18]; its transition density is given by the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u \quad \text{in D} \tag{1.1}$$

with the boundary condition

$$\frac{\partial u}{\partial n}=0 \qquad \text{on } \partial D\,, \tag{1.2}$$

where Δ is the Laplacian and $\partial/\partial n$, the normal derivative at the boundary. Recently, Fukushima [9] has constructed the r.b. Brownian motion on an *arbitrary bounded domain* in higher dimensions by solving a functional equation in a Hilbert space which is equivalent to (1.2) if ∂D is smooth but which involves neither ∂D nor $\partial/\partial n$.

In Markov chain theory, the counterpart of the heat equation in Brownian motion theory is the (backward) Kolmogorov equation:

$$\frac{\partial u(t, i)}{\partial t} = \sum_{j=1}^{\infty} A(i, j)u(t, j), \qquad i = 1, 2, \cdots, \tag{1.3}$$

where $A(i, j)$ is a kernel satisfying the conditions that $|A(i, j)| < \infty$, $A(i, j) \geqq 0$ for $i \neq j$ and $\sum_j A(i, j) = 0$. Feller [6] proposed the problem of determining all the Markov chains with the given Kolmogorov equation in terms of the lateral (or boundary) conditions at a certain ideal boundary. Feller also was particularly interested in the reflecting barrier condition at the ideal boundary. Assuming that the ideal boundary like Feller's or Martin's contains only a finite number of exit boundary points, Feller and others[1] have introduced the notion of the normal derivative analogue or the generalized normal derivative and studied something like the r.b. boundary condition. It seems to us, however, that no general solution on r.b. Markov chains has been obtained.

In the present paper we will not be involved in any boundary or boundary conditions but will construct, under a certain symmetric condition on the operator A, a Markov chain similar to the r.b. Brownian motion by rephrasing in terms of Markov chains what Fukushima did on the r.b. Brownian motion[2]. But since Fukushima's original proof for the existence of the r.b. Brownian motion does not apply to Markov chains, we will present a method which applies to Markov chains as well as to Brownian motions to obtain what may be called *reflecting barrier processes*. Although we are mainly concerned with Markov chains, it will be useful to show how our method applies to Brownian motions (see Appendix). Our method depends on making use of Dirichlet spaces in potential theory (or the functional analytic method in the theory of partial differential equations). In this connection we will mention Elliott [5] who has discussed the Dirichlet spaces associated with stable processes. We also will mention a forthcoming paper by Fukushima [10] in which he determines completely a class of Brownian motions including the r.b. Brownian motion in terms of certain Dirichlet spaces on the Martin boundary, developing a method similar to ours.

In the rest of this section, we will sketch some elements of the integration problem of the Kolmogorov equation, following Feller [6]. For the moment we will not be involved in the symmetric condition on the operator A, which is the basic assumption in our paper (see Section 3).

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1) See Dynkin [4]. See also Doob [2] for the generalized normal derivative at the Martin boundary associated with Brownian motion.

2) It should be noted that we are only interested in the first half of [9]; the latter half of [9] is devoted to investigating the behavior of the r.b. Brownian motion sample paths near an ideal boundary similar to the Kuramochi boundary in potential theory.

Let E be a denumerable space and $A(x, y)$, $x\in E$, $y\in E$, a kernel satisfying

$$(1.4)\qquad \begin{cases} -\infty< A(x, x)\leqq 0, \quad 0\leqq A(x, y)<\infty \qquad \text{for } x\neq y\,, \\ \sum_y A(x, y)\leqq 0\,. \end{cases}$$

We will call such A a *Kolmogorov-Feller kernel* (or *operator*). To simplify the argument we will confine ourselves to the case in which the following additional conditions are satisfied:[3)]

$$(1.5)\qquad A(x, x)<0, \quad \sum_y A(x, y) = 0\,.$$

For a function f over E, we will write Af for $\sum_y A(x, y) f(y)$ if the latter is well defined in the sense of absolute convergence. Note that $Af(x)$ is finite iff $\sum_y |A(x, y) f(y)|<\infty$. Define

$$(1.6)\qquad \begin{cases} q(x) = -A(x, x)\,, \qquad \Pi(x, y) = \dfrac{A(x, y)}{q(x)} \qquad \text{for } x\neq y\,, \\ \Pi(x, x) = 0\,. \end{cases}$$

Then one has

$$(1.7)\qquad 0<q(x)<\infty\,, \quad \Pi(x, y)\geqq 0\,, \quad \sum_y \Pi(x, y) = 1$$

and

$$(1.8)\qquad A(x, y) = q(x)\,[\Pi(x, y)-\delta(x, y)]$$

with $\delta(x, y)$ the Kronecker symbol.

The (backward) Kolmogorov equation is

$$(1.9)\qquad \frac{\partial u}{\partial t} = Au \quad \text{with} \quad u(0+, \cdot) = f$$

in the original form or

$$(1.10)\qquad (\alpha-A)v = f, \qquad \alpha>0$$

in the form rephrased in terms of the Laplace transform.

Let $G_\alpha(x, y)$, $\alpha>0$, $x\in E$, $y\in E$, be a *resolvent* over E, that is, a function of three variables (α, x, y) satisfying

$$(1.11)\qquad G_\alpha(x, y)\geqq 0\,, \quad \alpha\sum_y G_\alpha(x, y)\leqq 1\,,$$

$$(1.12)\qquad G_\alpha(x, y)-G_\beta(x, y)+(\alpha-\beta)\sum_z G_\alpha(x, z)G_\beta(z, y) = 0\,.$$

The resolvent $G_\alpha(x, y)$ is said to be an *A-resolvent* if, for each bounded function f, $v=G_\alpha f$ is a solution of (1.10). Such a resolvent satisfies

$$(1.13)\qquad \alpha G_\alpha(x, y) \to \delta(x, y) \quad \text{as} \quad \alpha\to\infty\,.$$

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3) All the results of this paper are valid with minor change in the general case (1.4). See footnote 7.

There are many A-resolvents in general. Due to [6; Theorem 4.1], the *minimal A-resolvent* $G_\alpha^0(x, y)$ allways exists and it is given by

$$G_\alpha^0(x, y) = [\alpha+q(y)]^{-1}\sum_{n\geqq 0}\Pi_\alpha^n(x, y)\,, \tag{1.14}$$

where $\Pi_\alpha^0(x, y)=\delta(x, y)$, $\Pi_\alpha(x, y)=q(x)[\alpha+q(x)]^{-1}\Pi(x, y)$ and $\Pi_\alpha^n(x, y)=\sum_z \Pi_\alpha^{n-1}(x, z)\Pi_\alpha(z, y)$. Any condition (usually, not containing the parameter α explicitly) which is satisfied by one and only one A-resolvent may be called a *lateral condition*. In most cases, the lateral condition is described as a boundary condition, but it sometimes can be expressed in another way. For example, the condition "*minimal*" is a lateral condition because of the existence and uniqueness of the required solution (1.14). Later (see Section 4) we will give a lateral condition, involving the Dirichlet norm associated with A (symmetric in the sense of Section 3), which determines the A-resolvent similar to the resolvent of the r.b. Brownian motion by Fukushima.

Feller [6; Theorem 3.1] also proved that an A-resolvent is the Laplace transform of one and only one transition function (continuous at the origin) $P(t, x, y)$ satisfying the Kolmogorov equation (1.9). More precisely, $P(t, x, y)$ satisfies the following conditions:

$$P(t, x, y)\geqq 0\,, \quad \sum_y P(t, x, y)\leqq 1 \qquad \text{for } t\geqq 0\,, \tag{1.15}$$

$$P(t+s, x, y) = \sum_z P(t, x, z)P(s, z, y)\,, \tag{1.16}$$

$$P(t, x, y) \to \delta(x, y) \quad \text{as} \quad t\to 0\,, \tag{1.17}$$

$$G_\alpha(x, y) = \int_0^\infty e^{-\alpha t}P(t, x, y)dt\,, \tag{1.18}$$

and, for each bounded function f,

$$\frac{\partial P_t f}{\partial t} = AP_t f\,. \tag{1.19}$$

In particular one has

$$\lim_{t\to\infty}\frac{P(t, x, y)-\delta(x, y)}{t} = A(x, y)\,. \tag{1.20}$$

In conclusion, the integration problem of the *proper* Kolmogorov equation (1.9) can be reduced to that of the *reformed* one (1.10). For this reason, in the following sections, we will be concerned only with A-resolvents with no further reference to transition functions or their associated Markov chains. (In paragraph (*e*) of Section 6 we will give a brief comment on the sample functions property of the r.b. Markov chain the resolvnt of which will be constructed in Section 4.)

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2. L^2-resolvents and their associated Dirichlet spaces

The notion of the Dirichlet space was introduced by Beurling and Deny [1]. We need several results on Dirichlet spaces in a special context. The following presentation is, mainly, due to Fukushima [10].

Let (E, μ) be a σ-finite measure space[4)] and $\mathcal{L}^2 = \mathcal{L}^2(E, \mu)$, the Hilbert space formed by all real-valued square-integrable functions over E with the norm

$$\|f\|_{L^2} = \left[\int_E |f(x)|^2 \mu(dx)\right]^{1/2} \tag{2.1}$$

and with the inner product

$$(f, g) = \int_E f(x)g(x)\mu(dx)\,. \tag{2.2}$$

Two functions f and g are assumed to be indentical if $f=g$ almost everywhere (μ). An L^2-*resolvent* is a family of symmetric bounded operators $\{G_\alpha, \alpha>0\}$ on $\mathcal{L}^2$ such that $0 \leqq \alpha G_\alpha f \leqq 1$ for every $0 \leqq f \leqq 1$ in $\mathcal{L}^2$ and such that

$$G_\alpha - G_\beta + (\alpha-\beta) G_\alpha G_\beta = 0\,. \tag{2.3}$$

Such a family $\{G_\alpha, \alpha>0\}$ enjoys the following properties:

$$\|\alpha G_\alpha\|_{L^2} \leqq 1\,, \quad \text{i.e.,} \quad (\alpha G_\alpha f, \alpha G_\alpha f) \leqq (f, f), \tag{2.4}$$

$$(f - \beta G_{\beta+\alpha} f, f) \geqq 0 \qquad \text{for } \alpha \geqq 0\,, \tag{2.5}$$

$$(G_\alpha f, f) \geqq 0\,. \tag{2.6}$$

Let f be in $\mathcal{L}^2$ and bounded and g, in $\mathcal{L}^2$ and $0 \leqq g \leqq 1$. Then the Schwarz inequality

$$[G_\alpha(fg)]^2 \leqq G_\alpha f^2 \cdot G_\alpha g^2$$

follows from $G_\alpha(af+bg)^2 \geqq 0$. Therefore

$$(\alpha G_\alpha(fg), \alpha G_\alpha(fg)) \leqq (\alpha G_\alpha f^2, \alpha G_\alpha g^2) = (f^2, \alpha^2 G^2 g^2) \leqq (f^2, 1) = (f, f)\,,$$

which proves (2.4) since G_α is a bounded operator. The inequality (2.5) is obvious from (2.4). Introduce a collection of symmetric and nonnegative bilinear forms on $\mathcal{L}^2$ by

4) We will omit the σ-algebra from the notation.

$$\mathrm{H}_\alpha^\beta(u, v) = \beta(u-\beta G_{\beta+\alpha}u, v)\,, \quad \alpha\geqq 0\,, \quad \beta>0\,. \tag{2.7}$$

One has

$$\begin{aligned} 0\leqq \mathrm{H}_\alpha^\beta(G_\alpha f, G_\alpha f) &= \beta(G_\alpha f-\beta G_{\beta+\alpha}G_\alpha f, G_\alpha f) \\ &= \beta(G_{\beta+\alpha}f, G_\alpha f) = (\beta G_{\beta+\alpha}G_\alpha f, f) \\ &= (G_\alpha f, f)-(G_{\beta+\alpha}f, f) \\ &\to (G_\alpha f, f) \quad \text{as} \quad \beta\to\infty\,, \end{aligned} \tag{2.8}$$

proving (2.6).

By the resolvent equation (2.3) one has

$$\frac{d}{d\beta}\mathrm{H}_\alpha^\beta(u, u) = (u-\beta G_{\beta+\alpha}u, u-\beta G_{\beta+\alpha}u)\geqq 0\,, \tag{2.9}$$

$$\frac{d^2}{d\beta^2}\mathrm{H}_\alpha^\beta(u, u) = -2(v, G_{\beta+\alpha}v)\leqq 0 \tag{2.10}$$

with $v=u-\beta G_{\beta+\alpha}u$. $\mathrm{H}_\alpha^\beta(u, u)$ increases with respect to β. Set

$$\mathrm{H}(u, u) = \lim_{\beta\to\infty}\mathrm{H}_0^\beta(u, u)\,, \quad \text{and} \quad \mathrm{H}_\alpha(u, u) = \lim_{\beta\to\infty}\mathrm{H}_\alpha^\beta(u, u)\,, \qquad \alpha>0\,.$$

Since

$$\begin{aligned} \mathrm{H}_\alpha^\beta(u, u) &= \frac{\beta^2}{\beta+\alpha}(u-(\beta+\alpha)G_{\beta+\alpha}u, u)+\frac{\alpha\beta}{\beta+\alpha}(u, u) \\ &= \frac{\beta^2}{(\beta+\alpha)^2}\mathrm{H}_0^{\beta+\alpha}(u, u)+\frac{\alpha\beta}{\beta+\alpha}(u, u)\,, \end{aligned} \tag{2.11}$$

one has $\mathrm{H}_\alpha(u, u)=\mathrm{H}(u, u)+\alpha(u, u)$.

Let $\mathcal{X}$ denote the space of all functions in $\mathcal{L}^2$ satisfying $\mathrm{H}(u, u)<\infty$. For u and v in $\mathcal{X}$, the finite limits

$$\begin{cases} \mathrm{H}_\alpha(u, v) = \lim_{\beta\to\infty}\mathrm{H}_\alpha^\beta(u, v),\ \alpha>0\,, \\ \mathrm{H}(u, v) = \lim_{\beta\to\infty}\mathrm{H}_0^\beta(u, v) \end{cases} \tag{2.12}$$

exist and

$$\mathrm{H}_\alpha(u, v) = \mathrm{H}(u, v)+\alpha(u, v)\,. \tag{2.13}$$

Proposition 2.1. *The above constructed pair* ($\mathcal{X}$, H) *enjoys the following properties*:

(a) *$\mathcal{X}$ is a linear subset of $\mathcal{L}^2$ and* H *is a symmetric, nonnegative bilinear form on $\mathcal{X}$.*

(b) *For each $\alpha>0$, $\mathcal{X}$ forms a real Hilbert space with respect to the inner product* $\mathrm{H}_\alpha=\mathrm{H}+\alpha(\ ,\ )$.

(c) *If $u\in\mathcal{X}$ and if v is a normal contraction of u,* i.e., (2.14) $$|v(x)| \leqq |u(x)| \quad \textit{and} \quad |v(x)-v(y)| \leqq |u(x)-u(y)|,$$

then $v \in \mathcal{X}$ *and* $\mathrm{H}(v, v) \leqq \mathrm{H}(u, u)$.

(d) *If* $u \in \mathcal{X}$, $\beta G_\beta u$ *converges to* u *in the* L^2*-norm.*

(e) *For every* $\alpha > 0$, G_α *maps* $\mathcal{L}^2$ *into* $\mathcal{X}$ *and for each* f *of* $\mathcal{L}^2$, *one has*

(2.15) $$\mathrm{H}_\alpha(G_\alpha f, v) = (f, v) \quad \textit{for every} \quad v \in \mathcal{X}.$$

(f) *If* $\mathcal{L}_0$ *is a dense linear subset of* $\mathcal{L}^2$, *then its range by* G_α, $\{G_\alpha f; f \in \mathcal{L}_0\}$, *is dense in* $\mathcal{X}$ *with respect to* H_α.

If $\{u_n\}$ is a Cauchy sequence relative to H_α, it is so relative to H_α^β and therefore, relative to the L^2-norm by (2.11). If u is the L^2-limit of $\{u_n\}$, it is the H_α^β-limit by (2.7) and one has $\mathrm{H}_\alpha^\beta(u-u_n, u-u_n) = \lim_{m\to\infty} \mathrm{H}_\alpha^\beta(u_m-u_n, u_m-u_n) \leqq \liminf_{m\to\infty} \mathrm{H}_\alpha(u_m-u_n, u_m-u_n)$. Hence, $\mathrm{H}_\alpha^\beta(u-u_n, u-u_n)$ is uniformly small in β for n large enough. This proves that $u \in \mathcal{X}$ and u is the H_α-limit of $\{u_n\}$.

For (c) it is enough to show that

(2.16) $$\mathrm{H}_0^\beta(u, u) \geqq \mathrm{H}_0^\beta(v, v) \qquad \text{for every} \quad \beta > 0.$$

An elementary computation shows that, for any bounded u in $\mathcal{L}^2$ and $0 \leqq w \leqq 1$ in $\mathcal{L}^2$,

$$\begin{aligned}\mathrm{H}_0^\beta(uw, uw) &= \frac{1}{2}\beta^2 \int_E w(x) G_\beta[(u(x)-u)^2 w](x) \mu(dx) \\ &\quad + \beta(1-\beta G_\beta w, u^2 w) - \beta(1-w, u^2 w).\end{aligned}$$

Let u be any function in $\mathcal{L}^2$ and v, a normal contraction of u. Let u_n, v_n be the restriction of u, v to the set $\{x; |u(x)| \leqq n\}$ with $u(x) = v(x) = 0$ for $|u(x)| > n$, so that v_n is a normal contraction of u_n. By the positivity and sub-Markov property of G_α, one has

$$\mathrm{H}_0^\beta(u_n w, u_n w) - \mathrm{H}_0^\beta(v_n w, v_n w) \geqq \beta(1-w, v_n^2 w - u_n^2 w).$$

Letting $w \uparrow 1$ and $n \to \infty$, one proves (2.16).

By (2.9),

$$\mathrm{H}(u, u) - \mathrm{H}_0^1(u, u) = \int_1^\infty (u - \beta G_\beta u, u - \beta G_\beta u) d\beta.$$

But since the integrand is decreasing by (2.10), it must converge to zero as $\beta \to \infty$, if $\mathrm{H}(u, u) < \infty$. This proves (*d*).

By (2.8), $\mathrm{H}_\alpha(G_\alpha f, G_\alpha f) = (G_\alpha f, f) < \infty$ for any f of $\mathcal{L}^2$. If $v \in \mathcal{X}$, by (*d*),

$$\begin{aligned}\mathrm{H}_\alpha(G_\alpha f, v) &= \lim_{\beta\to\infty} \mathrm{H}_\alpha^\beta(G_\alpha f, v) = \lim_{\beta\to\infty} (\beta G_{\beta+\alpha} f, v) \\ &= \lim_{\beta\to\infty} (f, \beta G_{\beta+\alpha} v) = (f, v),\end{aligned}$$

proving (2.15). Finally, (f) is direct from (2.15).

In general, a pair $(\mathcal{X}, \mathrm{H})$ is said to be an L^2-*Dirichlet space* if it satisfies conditions (a), (b) in Proposition 2.1 and the following condition:

(c′) If $u\in\mathcal{X}$ and if v is a particular kind of normal contraction of u such that

$$v=(u\vee 0)\wedge 1\,, \tag{2.17}$$

then $v\in\mathcal{X}$ and $\mathrm{H}(v, v)\leqq\mathrm{H}(u, u)$, where $(u\vee 0)\wedge 1(x)=\min\{\max(u(x), 0), 1\}$. One has seen that an L^2-resolvent generates an L^2-Dirichlet space satisfying condition (c), appearing much stronger than (c′). By the fundamental theorem of Beurling and Deny [1] it follows that, *for any* L^2-*Dirichlet space, there is an (obviously unique)* L^2-*resolvent* G_α *satisfying* (2.15). Such an L^2-resolvent satisfies conditions (d) and (f). One proves

Proposition 2.2. *Let* $(\mathcal{X}, \mathrm{H})$ *be an* L^2-*Dirichlet space and* G_α, *its associated* L^2-*resolvent through* (2.15). *Then,* $(\mathcal{X}, \mathrm{H})$ *coincides with the* L^2-*Dirichlet space* (say, $(\tilde{\mathcal{X}}, \tilde{\mathrm{H}})$) *generated by* G_α.

Corollary 1. *Any* L^2-*Dirichlet space satisfies condition* (c) *in Proposition* 2.1.

Corollary 2. *The class of all* L^2-*resolvents is in one-one correspondence with the class of all* L^2-*Dirichlet spaces through the functional equation* (2.15).

The family $\{G_\alpha f; f\in\mathcal{L}^2\}$ is dense in both $\mathcal{X}$ (relative to H_α) and $\tilde{\mathcal{X}}$ (relative to $\tilde{\mathrm{H}}_\alpha$). Moreover, $\mathrm{H}_\alpha(G_\alpha f, G_\alpha f)=(f, G_\alpha f)=\tilde{\mathrm{H}}_\alpha(G_\alpha f, G_\alpha f)$, so that $\mathcal{X}=\tilde{\mathcal{X}}$ and $\mathrm{H}_\alpha=\tilde{\mathrm{H}}_\alpha$.

Finally we will give some general remarks on resolvents. A *resolvent* on E is a family of kernels[5] $\{G_\alpha(x, B), \alpha>0\}$ such that $\alpha G_\alpha(x, E)\leqq 1$ and $G_\alpha(x, B)-G_\beta(x, B)+(\alpha-\beta)\int_E G_\alpha(x, dy)G_\beta(y, B)=0$. A resolvent is said to be μ-*symmetric* if, for any positive (measurable) functions f, g,

$$(G_\alpha f, g)=(f, G_\alpha g)\leqq+\infty, \tag{2.18}$$

where $G_\alpha f(x)=\int_E G_\alpha(x, dy)f(y)$. Then the μ-symmetric resolvent $\{G_\alpha(x, B), \alpha>0\}$ generates the L^2-resolvent $\{G_\alpha, \alpha>0\}$, since one has for every positive function f

$$(G_\alpha f)^2\leqq G_\alpha 1\cdot G_\alpha f^2$$

and therefore

5) A *kernel* $K(x, B)$ over E stands for a function of $x\in E$, $B\in$(the σ-algebra of subsets of E) such that $K(x, \cdot)$ is a measure for each x and $K(\cdot, B)$ is a measurable function for each B.

$$(\alpha G_\alpha f,\ \alpha G_\alpha f) \leqq (\alpha G_\alpha 1,\ \alpha G_\alpha f^2) \leqq (1,\ \alpha G_\alpha f^2) \leqq (\alpha G_\alpha 1, f^2) \leqq (1, f^2) = (f, f)\,.$$

In particular, if $G_{\alpha_0}(x, \cdot)$ is absolutely continuous with respect to μ for each x, then every $G_\alpha(x, \cdot)$ is so and it has the *symmetric density* $g_\alpha(x, y)$[6] with respect to μ (see [14; p. 497]). In general, however, an L^2-resolvent may not be generated by a μ-symmetric resolvent and the latter, by a symmetric resolvent density.

NOTE. The above remarks on resolvents are concerned only with Appendix. In case of Markov chains, E is a denumerable space and $0<\mu(x)<\infty$ for every $x \in E$. The trouble which is caused by sets of (μ) measure zero disappears. An L^2-resolvent determines uniquely the μ-symmetric resolvent and the symmetric resolvent density by

$$G_\alpha(x, y) = G_\alpha I_{\{y\}}(x) \quad \text{and} \quad g_\alpha(x, y) = \frac{G_\alpha(x, y)}{\mu(y)}\,,$$

where $I_{\{y\}}$ is the indicator of the singleton $\{y\}$.

3. Decomposition of L^2-Dirichlet spaces generated by μ-symmetric A-resolvents

Throughout the rest of the paper (except Appendix) we will make the following hypotheses. *E is a denumerable space. The measure μ over E is strictly positive and finite at each state of E: $0<\mu(x)<\infty$ for every $x \in E$.* (We do not assume, however, that μ is a finite measure.) *$A(x, y)$ is a Kolmogorov-Feller kernel satisfying condition* (1.5) *and the condition that*

(S) *the density* $a(x, y) = A(x, y)/\mu(y)$ *is symmetric.*

The bilinear form defined by

$$(3.1) \qquad \langle u, v\rangle = \frac{1}{2} \sum\nolimits_{x,y} (u(x)-u(y))(v(x)-v(y))a(x, y)\mu(x)\mu(y)$$ [7]

is an analogue of the classical Dirichlet form (up to constant $\frac{1}{2}$) in a locally Euclidean space D:

$$\frac{1}{2}\int_D \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx\,,$$

so that the form (3.1) may be called the *Dirichlet form with respect to* (μ, A)

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6) The collection of such symmetric densities, $\{g_\alpha(x, y), \alpha<0\}$, will be called a *symmetric resolvent density.*

7) Unless condition (1.5) is satisfied, the right side of (3.1) should be added by the term $\sum_x u(x)v(x)(-A1)(x)\mu(x)$. Then all the results are valid without any change except Proposition 4.2.

and $\langle u, u\rangle^{1/2}$, the *Dirichlet norm with respect to* (μ, A). One also uses the following notation:

$$\langle u, v\rangle_\alpha = \langle u, v\rangle + \alpha(u, v) . \tag{3.2}$$

One starts with some preliminary remarks. Let $\mathcal{B}(\mathcal{B}^+)$ be the collection of all real-valued (nonnegative) bounded functions and $C_0(C_0^+)$, the subcollection of $\mathcal{B}(\mathcal{B}^+)$ formed by functions f such that the support of f, $\{x; f(x)\neq 0\}$, is a finite set. Note that $\langle u, u\rangle$ is well defined (allowing the value infinity) only if u is finite everywhere. One also notes that, if $\langle u, u\rangle <+\infty$, then Au^2 and Au are finite everywhere. Indeed, $\langle u, u\rangle <+\infty$ implies that $\sum_{y\neq x}(u(x)-u(y))^2 a(x, y)\mu(y) = \sum_{y\neq x}(u(x)-u(y))^2 A(x, y) <+\infty$ for each x. By the inequality $u^2(y) \leqq 2[u^2(x)+(u(x)-u(y))^2]$ it follows that $\sum_{y\neq x} u^2(y)A(x, y) <+\infty$. By the Schwarz inequality, $[\sum_{y\neq x}|u(y)|A(x, y)]^2 \leqq \sum_{y\neq x}A(x, y)\cdot\sum_{y\neq x}u^2(y)\cdot A(x, y) <+\infty$. In the same way one can easily prove that, if $\langle u_n, u_n\rangle \to 0$, then $u_n(x)\to 0$ and $Au_n(x)\to 0$ for every x.

Proposition 3.1. *Suppose that Au is finite*[8] *and v is in C_0. Then, $\langle u, v\rangle$, and (u, Av) are finite and*

$$\langle u, v\rangle = -(Au, v) = -(u, Av) . \tag{3.3}$$

Use the Fubini theorem and the symmetry of $a(x, y)$ to verify the finiteness of $\sum_{x,y}|u(x)v(x)a(x, y)|\mu(x)\mu(y)$, $\sum_{x,y}|u(x)v(y)a(x, y)|\mu(x)\mu(y)$ and so on. Use again the Fubini theorem and the hypotheses on A to show that $\sum_{y,y}u(x)v(x)a(x, y)\mu(x)\mu(y) = \sum_{x,y}u(y)v(y)a(x, y)\mu(x)\mu(y) = 0$ and that $(u, Av) = \sum_{x,y}u(x)v(y)a(x, y)\mu(x)\mu(y) = \sum_{x,y}u(y)v(x)a(x, y)\mu(x)\mu(y) = (Au, v)$.

Let $\{G_\alpha(x, B), \alpha>0\}$ be a μ-symmetric A-resolvent, i.e., a resolvent such that, for every $f\in C_0$ (and therefore, every $f\in\mathcal{B}$), $u=G_\alpha f$ is a solution of the Kolmogorov equation

$$(\alpha - A)u = f \tag{3.4}$$

and such that

$$(G_\alpha f, g) = (f, G_\alpha g) \qquad \text{for any} \quad f, g\in C_0 . \tag{3.5}$$

The corresponding L^2-resolvent and the symmetric resolvent density are denoted by $\{G_\alpha, \alpha>0\}$ and $\{g_\alpha(x, y), \alpha>0\}$ (see Note in the tail of Section 2).

Proposition 3.2. *Let $(\mathcal{X}, \mathrm{H})$ be the L^2-Dirichlet space generated by a μ-symmetric A-resolvent.* (i) *If u and v are in $\mathcal{B}$ and if either u or v is in C_0, then*

8) Note that this condition is valid if either $\langle u, u\rangle <+\infty$ or $u\in\mathcal{B}$.

$$\lim_{\beta\to\infty} H_0^\beta(u, v) = \langle u, v\rangle . \tag{3.6}$$

(ii) *Space* $\mathcal{X}$ *contains* C_0 *and therefore,* $\{G_\alpha, \alpha>0\}$ *is a strongly continuous* L^2-*resolvent on* $\mathcal{L}^2$. (iii) *For every* $u\in\mathcal{X}$,

$$H(u, u)\geqq\langle u, u\rangle . \tag{3.7}$$

(iv) *If* $u\in\mathcal{X}$ *and if* $v\in C_0$,

$$H(u, v) = \langle u, v\rangle = (-Au, v) . \tag{3.8}$$

If $|f|\leqq 1$, then $|G_\beta f|\leqq 1/\beta\to 0$ as $\beta\to\infty$. Hence the Kolmogorov equation (3.4) implies that $\beta G_\beta f\to f$ boundedly and therefore, that

$$\beta(f-\beta G_\beta f) = -A(\beta G_\beta f)\to -Af \tag{3.9}$$

in the pointwise sense. In particular one has

$$\beta^2 G_\beta(x, y)\to A(x, y) \quad \text{or} \quad \beta^2 g_\beta(x, y)\to a(x, y) \qquad \text{for} \quad x\neq y \tag{3.10}$$

(with $f=I_{\{y\}}$) and

$$0\leqq\beta(1-\beta G_\beta 1(x))\to -A1(x) = 0 . \tag{3.11}$$

To prove (3.6) one may assume that $v\in C_0$. Then

$$H_0^\beta(u, v) = \beta(u-\beta G_\beta u, v)\to(-Au, v) = \langle u, v\rangle$$

by (3.9) and the preceding proposition. If $u\in C_0$, therefore, one has

$$H(u, u) = \lim_{\beta\to\infty} H_0^\beta(u, u) = \langle u, u\rangle <+\infty,$$

proving that $u\in\mathcal{X}$.

To prove (3.7) it is enough to rewrite H_0^β in the form

$$H_0^\beta(u, u) = \frac{1}{2}\beta^2 \sum_{x,y} g_\beta(x, y)(u(x)-u(y))^2\mu(x)\mu(y) + \beta\sum_x (1-\beta G_\beta 1(x))u^2(x)\mu(x) , \tag{3.12}$$

to use (3.10), (3.11) and the Fatou lemma.

By (3.6) the equality (3.8) is valid for $u\in\mathcal{X}\cap\mathcal{B}$. Since the collection $\{G_\alpha f; f\in C_0\}\subset\mathcal{X}\cap\mathcal{B}$ is dense in $\mathcal{X}$ both in the H-norm and the Dirichlet norm (by (3.7)), the first equality of (3.8) extends to every $u\in\mathcal{X}$; the second equality is due to (3.7) and Proposition 3.1.

By the formula (1.14) it follows that the *minimal A-resolvent* $\{G_\alpha^0(x, B), \alpha>0\}$ *is* μ-*symmetric.* Let $(\mathcal{X}^{(0)}, H^{(0)})$ be the L^2-Dirichlet space generated by $\{G_\alpha^0, \alpha>0\}$.

Proposition 3.3. *Space C_0 is dense in $\mathcal{X}^{(0)}$ with respect to the $H_\alpha^{(0)}$-norm (and hence the Dirichlet norm $\langle\ ,\ \rangle^{1/2}$.)*

It is enough to show that any function u of the form $G_\alpha^0 f$, $f\in C_0^+$, can be approximated in the $H_\alpha^{(0)}$-norm by a spequence $\{u_k\}$ in C_0. First one constructs $\{u_k\}$. Let ${}_kE$ be an increasing sequence of finite subsets exhausting E and ${}_kA(x,y)$, the restriction of $A(x,y)$ to ${}_kE$. The kernel ${}_kA(x,y)$ is a Kolmogorov-Feller kernel on ${}_kE$. Let ${}_kG_\alpha(x,y)$, $\alpha>0$, be the minimal ${}_kA$-resolvent. (The kernel ${}_kA$ does not satisfy condition (1.5). In construction of the minimal resolvent, however, condition (1.5) is irrelevant. Moreover, ${}_kG_\alpha(x,y)$ is the unique ${}_kA$-resolvent.) By the formula (1.14) it follows that ${}_kG_\alpha(x,y)$ increases to $G_\alpha^0(x,y)$ as $k\to\infty$. With no danger of confusion, use the same notation ${}_kG_\alpha(x,y)$ to denote the kernel on E defined by zero outside ${}_kE\times{}_kE$. Then, writing u_k for ${}_kG_\alpha f$ one has for all x in ${}_kE$

$$Au_k(x) = {}_kAu_k(x) = \alpha u_k(x)-f(x)\,,$$

so that, by the preceding propositions,

$$\begin{aligned}\text{(3.13)}\qquad H_\alpha^{(0)}(u_k, u_k) &= \langle u_k, u_k\rangle+\alpha(u_k, u_k)\\ &= (-Au_k, u_k)+\alpha(u_k, u_k)\\ &= (f, u_k)\,.\end{aligned}$$

Using (2.15) and (3.13) one can evaluate $H_\alpha^{(0)}(G_\alpha^0 f-u_k, G_\alpha^0 f-u_k)$ as follows:

$$\begin{aligned}\text{(3.14)}\qquad &H_\alpha^{(0)}(G_\alpha^0 f-u_k, G_\alpha^0 f-u_k)\\ &= H_\alpha^{(0)}(G_\alpha^0 f, G_\alpha^0 f)-2H_\alpha^{(0)}(G_\alpha^0 f, u_k)-H_\alpha^{(0)}(u_k, u_k)\\ &= (f, G_\alpha^0 f)-2(f, u_k)+(f, u_k)\\ &= (f, G_\alpha^0 f-u_k)\to 0 \qquad \text{as} \quad k\to\infty,\end{aligned}$$

since $f\geqq 0$ and $u_k={}_kG_\alpha f\uparrow G_\alpha^0 f$.

Proposition 3.4. *Let $(\mathcal{X}, H)$ be the L^2-Dirichlet space generated by an arbitrary μ-symmetric A-resolvent. If $u\in\mathcal{X}^{(0)}$, then $u\in\mathcal{X}$ and*

$$\text{(3.15)}\qquad H(u, u) = H^{(0)}(u, u) = \langle u, u\rangle\,.$$

One proves, instead of (3.15),

$$\text{(3.16)}\qquad H_\alpha(u, u) = H_\alpha^{(0)}(u, u) = \langle u, u\rangle_\alpha\,.$$

Let $\{u_n\}$ be a sequence in C_0 such that $H_\alpha^{(0)}$-$\lim u_n=u$. Since (3.16) is true for every C_0-function, $\{u_n\}$ is a Cauchy sequence in the H_α-norm. Write $v=H_\alpha$-$\lim u_n$. Since v is also the L^2-limit of u_n, one has $v=u$. Therefore, $H_\alpha(u,u)=\lim H_\alpha(u_n, u_n)=\lim H_\alpha^{(0)}(u_n, u_n)=H_\alpha^{(0)}(u, u)$. The second equality follows from $H_\alpha^{(0)}(u-u_n, u-u_n)\geqq\langle u-u_n, u-u_n\rangle_\alpha$.

NOTE. Proposition 3.4 implies that $\mathcal{X}^{(0)}$ is closed both in $\mathcal{X}$ with respect to H_α and in the space of all L^2-functions with finite Dirichlet norm with respect to the norm $\langle\, ,\, \rangle_\alpha^{1/2}$.

A function u, finite everywhere, is said to be α-*harmonic* (*with respect to* A) if $(\alpha - A)u = 0$, or equivalently, if $\Pi_\alpha u = u$ (for the definition of Π_α, see (1.14)).

Proposition 3.5. *If u is α-harmonic, in $\mathcal{L}^2$ and of finite Dirichlet norm, then $Au^2 \geqq 0$ and*

$$\langle u, u\rangle_\alpha = \frac{1}{2}\sum_x (Au^2)(x)\mu(x)\,. \tag{3.17}$$

As was noted already, $\langle u, u\rangle < +\infty$ implies that Au^2 and Au are finite. Hence one has for each x

$$\begin{aligned}0 \leqq \sum_{y\neq x}(u(x)-u(y))^2 a(x, y)\mu(y) &= \sum_y (u(x)-u(y))^2 a(x, y)\mu(y)\\ &= A(u(x)-u)^2(x)\\ &= u^2(x)A1(x) - 2u(x)Au(x) + Au^2(x)\\ &= -2\alpha u^2(x) + Au^2(x)\,.\end{aligned}$$

Integrate by the measure $\frac{1}{2}\mu(x)$ to obtain

$$\langle u, u\rangle = -\alpha(u, u) + \frac{1}{2}(Au^2, 1)\,.$$

In the same way one has a little more general result.

Proposition 3.6. *If $u \in \mathcal{L}^2$, $\langle u, u\rangle < \infty$ and if $Au \in \mathcal{L}^2$, then Au^2 is integrable (μ) and*

$$\langle u, u\rangle + (Au, u) = \frac{1}{2}(Au^2, 1)\,. \tag{3.18}$$

Theorem 1. *Let $(\mathcal{X}, \mathrm{H})$ be the L^2-Dirichlet space generated by an arbitrary μ-symmetric A-resolvent $\{G_\alpha(x, B),\ \alpha > 0\}$. Let $\mathcal{H}_\alpha$ be the space of all α-harmonic functions in $\mathcal{X}$. Then one has the orthogonal decompostion of $\mathcal{X}$ into $\mathcal{X}^{(0)}$ and $\mathcal{H}_\alpha$:*

$$\mathcal{X} = \mathcal{X}^{(0)} \oplus \mathcal{H}_\alpha \quad \textit{with respect to the } \mathrm{H}_\alpha\textit{-norm.} \tag{3.19}$$

Moreover, the operator

$$R_\alpha = G_\alpha - G_\alpha^0$$

maps $\mathcal{L}^2$ into $\mathcal{H}_\alpha$ and for each f in $\mathcal{L}^2$, one has

$$\mathrm{H}_\alpha(R_\alpha f, v) = (f, v) \qquad \textit{for every} \quad v \in \mathcal{H}_\alpha\,. \tag{3.20}$$

In particular, $\{R_\alpha f;\ f \in C_0\}$ is dense in $\mathcal{H}_\alpha$ with respect to H_α.

By (3.8), if $v\in\mathcal{C}_0$,

$$\mathrm{H}_\alpha(u, v) = \langle u, v\rangle_\alpha = (-Au, v)+\alpha(u, v) \qquad \text{for any} \quad u\in\mathcal{X}. \tag{3.21}$$

One has already seen (the note for Proposition 3.4) that $\mathcal{X}^{(0)}$ is a closed subspace of $\mathcal{X}$ in the H_α-norm. By (3.21), $u\in\mathcal{H}_\alpha$ iff u is orthogonal to $\mathcal{C}_0$. Since $\mathcal{C}_0$ is dense in $\mathcal{X}^{(0)}$ in the $\mathrm{H}_\alpha^{(0)}$-norm and since $\mathrm{H}_\alpha=\mathrm{H}_\alpha^{(0)}$ in $\mathcal{X}^{(0)}$, $u\in\mathcal{H}_\alpha$ iff u is orthogonal to $\mathcal{X}^{(0)}$, which shows that $\mathcal{H}_\alpha$ is the orthogonal complement of $\mathcal{X}^{(0)}$.

If $f\in\mathcal{L}^2$, then for every $v\in\mathcal{C}_0$

$$\begin{aligned}\mathrm{H}_\alpha(R_\alpha f, v) &= \mathrm{H}_\alpha(G_\alpha f, v)-\mathrm{H}_\alpha^{(0)}(G_\alpha^0 f, v)\\ &= (f, v)-(f, v) = 0,\end{aligned}$$

so that $R_\alpha f\in\mathcal{H}_\alpha$.

NOTE 1. Theorem 1 implies that $\mathcal{H}_\alpha=\{0\}$ *or not simultaneously for every* $\alpha>0$.

NOTE 2. In the above, one has proved that $\mathcal{H}_\alpha$ *is closed in the* H_α*-norm.* We will give another proof of this fact. Let $\{u_n\}$ be a sequence in $\mathcal{H}_\alpha$ and $u=\mathrm{H}_\alpha\text{-}\lim u_n$. Note that u is the pointwise limit of u_n. Since $\mathrm{H}_\alpha(u_n-u, u-u_n) \geqq \langle u-u_n, u-u_n\rangle_\alpha$, it follows that $u_n(x)\to u(x)$ and $Au_n(x)\to Au(x)$ for every x (see the paragraph preceding Proposition 3.1). Hence one has

$$(\alpha u-Au)(x) = \lim_{n\to\infty}(\alpha u-Au_n)(x) = 0,$$

proving that $u\in\mathcal{H}_\alpha$.

4. Resolvents associated with the Dirichlet norm and their lateral conditions

Let $\mathcal{X}^\bullet$ be the space of all L^2-functions with finite Dirichlet norm.

Proposition 4.1. *The pair* $(\mathcal{X}^\bullet, \langle\ ,\ \rangle)$ *is an* L^2*-Dirichlet space.*

It is enough to show that $\mathcal{X}^\bullet$ is complete in the norm $\langle\ ,\ \rangle_\alpha^{1/2}$. Let $\{u_n\}$ be a Cauchy sequence in the norm $\langle\ ,\ \rangle_\alpha^{1/2}$. Let u be the L^2-limit of $\{u_n\}$ and therefore, in our special context, the pointwise limit of $\{u_n\}$. By the Fatou lemma,

$$\langle u-u_n, u-u_n\rangle \leqq \lim_{m\to\infty}\langle u_m-u_n, u_m-u_n\rangle,$$

which proves that $u\in\mathcal{X}^\bullet$ and $\langle u-u_n, u-u_n\rangle_\alpha\to 0$.

Let $\{G_\alpha^\bullet(x, B), \alpha>0\}$ be the μ-symmetric resolvent associated with $(\mathcal{X}^\bullet, \langle\ ,\ \rangle)$.

Theorem 2. *The condition*

(4.1)[9] $\quad u\in\mathcal{X}^{\bullet}$ *and* $\langle u, v\rangle+(Au, v)=0$ *for every* $v\in\mathcal{X}^{\bullet}$

is a lateral condition for the Kolmogorov equation

$$(\alpha-A)u=f, \qquad f\in C_0 \tag{4.2}$$

and the corresponding A-resolvent is given by $\{G_\alpha^{\bullet}(x, B), \alpha>0\}$. *More precisely, for every f in* C_0 *(or more generally, in* $\mathcal{L}^2$*),* $u=G_\alpha^{\bullet}f$ *is the unique solution of* (4.2) *and* (4.1).

Recall that $u=G_\alpha^{\bullet}f$ is the unique solution of

$$u\in\mathcal{X}^{\bullet} \quad \text{and} \quad \langle u, v\rangle_\alpha=(f, v) \qquad \text{for every} \quad v\in\mathcal{X}^{\bullet}, \tag{4.3}$$

so that one has only to show that a function u satisfies (4.2) and (4.1) iff it satisfies (4.3). The necessity is evident. For the converse, note that the equation in (4.1) is always satisfied for every $u\in\mathcal{X}^{\bullet}$ and $v\in C_0$ (see Proposition 3.1 and footnote 8). If u satisfies (4.3), therefore, then

$$((\alpha-A)u, v)=(f, v) \qquad \text{for every} \quad v\in C_0, \tag{4.4}$$

which implies (4.2). But it follows from $u\in\mathcal{L}^2$ and (4.2) that $Au\in\mathcal{L}^2$, so that (4.4) is valid for every $v\in\mathcal{L}^2$. Hence one has

$$(Au, v)=(f, v)-\alpha(u, v)=-\langle u, v\rangle, \qquad v\in\mathcal{X}^{\bullet},$$

proving (4.1).

Proposition 4.2. *If* μ *is a finite measure, then* $\{G_\alpha^{\bullet}(x, B), \alpha>0\}$ *is a Markov resolvent,* i.e.,

$$\alpha G_\alpha^{\bullet}(x, E)=1 \qquad \textit{every } \alpha>0 \textit{ and every } x\in E. \tag{4.5}$$

Since $1\in\mathcal{X}^{\bullet}$ and since

$$\langle 1, v\rangle_\alpha=\langle 1, v\rangle+\alpha(1, v)=(\alpha 1, v),$$

one has $1=G_\alpha^{\bullet}(\alpha 1)=\alpha G_\alpha^{\bullet}(\cdot, E)$.

Since $\{G_\alpha^{\bullet}(x, B), \alpha>0\}$ is a μ-symmetric A-resolvent, Theorem 1 applies to obtain

Proposition 4.3. *Let* $\mathcal{H}_\alpha^{\bullet}$ *be the space of all* α*-harmonic functions with finite Dirichlet norm. The kernel*

$$R_\alpha^{\bullet}(x, y)=G_\alpha^{\bullet}(x, y)-G_\alpha^{0}(x, y) \tag{4.6}$$

is characterized by $R_\alpha^{\bullet}f\in\mathcal{H}_\alpha^{\bullet}$ *for each* $f\in C_0$ *and by the functional equation*

$$\langle R_\alpha^{\bullet}, f, v\rangle=(f, v) \qquad \textit{for every} \quad v\in\mathcal{H}_\alpha^{\bullet}. \tag{4.7}$$

9) This form of the lateral condition for $\{G^{\bullet}_\alpha(x, B), \alpha>0\}$ was informed us by H. Kunita,

This proposition shows that the resolvent $\{G_\alpha^\bullet(x, B),\ \alpha>0\}$ is an exact counterpart in Markov chains of what Fukushima [9] called the resolvent of the r.b. Brownian motion on an arbitrary bounded domain. In accordance with Fukushima's terminology, $\{G_\alpha^\bullet(x, B),\ \alpha>0\}$ will be called the *resolvent of the reflecting barrier Markov chain associated with* (μ, A).[10] In the next section we will give another justification of this nomenclature, discussing the boundary condition of the r.b. (in the present sense) birth and death process. The resolvent $G_\alpha^\bullet$ coincides with the minimal resolvent G_α^0 if $\mathcal{H}_\alpha^\bullet=\{0\}$. The condition "$\mathcal{H}_\alpha^\bullet=\{0\}$", however, may be considered the case when there is no boundary point admitting the "reflection"; the resolvent $G_\alpha^\bullet$ still determines the r.b. Markov chain (though in the trivial sense). This fact also will be illustrated for birth and death processes in the next section.

The following generalization of Theorem 2 is now obvious.

Theorem 3. *Let* $\mathcal{X}\supset C_0$ *and let* $(\mathcal{X}, \langle\ ,\ \rangle)$ *be an* L^2*-Dirichlet subspace of* $(\mathcal{X}^\bullet, \langle\ ,\ \rangle)$. *Then the condition*

$$(4.8)\qquad u\in\mathcal{X}\quad \textit{and}\quad \langle u, v\rangle+(Au, v)=0\qquad \textit{for every } v\in\mathcal{X}$$

is a lateral condition for the Kolmogorov equation (4.2) *and the corresponding A-resolvent is the* μ*-symmetric resolvent associated with* $(\mathcal{X}, \langle\ ,\ \rangle)$.

Let $\bar{C}_0$ be the closure of C_0 with respect to the $\langle\ ,\ \rangle_\alpha$-norm, $\alpha>0$, in $\mathcal{X}^\bullet$. Obviously $\bar{C}_0$ is independent of $\alpha>0$.

Theorem 4. *The pair* $(\bar{C}_0, \langle\ ,\ \rangle)$ *is an* L^2*-Dirichlet space. The condition*

$$(4.9)\qquad u\in\bar{C}_0$$

is a lateral condition for the Kolmogorov equation (4.2) *and the corresponding A-resolvent is the minimal A-resolvent* $\{G_\alpha^\bullet(x, B),\ \alpha>0\}$. *This is equivalent to saying that*

$$(4.10)\qquad (\bar{C}_0, \langle\ ,\ \rangle)=(\mathcal{X}^{(0)}, \mathrm{H}^{(0)})\,.$$

Note that the relation

$$\langle u, v\rangle+(Au, v)=0\,,\qquad v\in\bar{C}_0$$

is automatically satisfied if $u\in\mathcal{X}^\bullet$ and $Au\in\mathcal{L}^2$ and hence that, assuming that $(\bar{C}_0, \langle\ ,\ \rangle)$ is an L^2-Dirichlet space, (4.9) is nothing but (4.8) with $\mathcal{X}$ replaced by $\bar{C}_0$. Then it is obvious from Theorem 3 that the proof of the theorem is reduced to showing (4.10).

The relation (4.10) follows immediately from Proposition 3.3 and the note

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10) This actually depends only on A (see Proposition 6.2).

for Proposition 3.4. We will give, however, an alternative proof of the theorem without using Proposition 3.3. This also turns out to give an alternative proof of Proposition 3.3.

One proves first that $(\bar{C}_0, \langle\ ,\ \rangle)$ is an L^2-Dirichlet space. Given a function u, let Tu denote the function $(u\vee 0)\wedge 1$. It is enough to show that $u\in\bar{C}_0$ implies $Tu\in\bar{C}_0$; the other conditions for L^2-Dirichlet space are satisfied obviously. Let $u_n\in C_0$ and $u=\langle\ ,\ \rangle_\alpha\text{-lim}\,u_n$. Then u is the pointwise limit of $\{u_n\}$. Since $|Tu_n|\leqq|u_n|$ and $|Tu_n-Tu|\leqq|u_n-u|$, it follows that $Tu_n\in C_0$, $Tu_n\to Tu$ both in $\mathcal{L}^2$ and in the pointwise sense. Let $J(x, y)=0$ if $x=y$, and $=1$ if $x\neq y$. Since $|u_n(x)-u_n(y)|^2$ is uniformly integrable relative to the measure $J(x, y)a(x, y)\mu(x)\mu(y)$ and since

$$(4.11)\qquad \begin{aligned}&|Tu(x)-Tu_n(x)-(Tu(y)-Tu_n(y))|^2\\ &\leqq 2\{|u(x)-u(y)|^2+|u_n(x)-u_n(y)|^2\},\end{aligned}$$

the left side of (4.11) is uniformly integrable relative to $J(x, y)a(x, y)\mu(x)\mu(y)$ and converges to zero for every $(x, y)\in E\times E$. Therefore $Tu_n\to Tu$ in the Dirichlet norm. One has proved that $Tu=\langle\ ,\ \rangle_\alpha\text{-lim}\,Tu_n\in\bar{C}_0$.

In the same way as in the proof of Proposition 3.4 (but without using Proposition 3.3) it is easy to show that, if $u\in C_0$, then $u\in\mathcal{X}^{(0)}$ and $H^{(0)}(u, u)=\langle u, u\rangle$. It remains to prove that $\mathcal{X}^{(0)}\subset\bar{C}_0$. Let $\{\tilde{G}_\alpha(x, B), \alpha>0\}$ be the μ-symmetric A-resolvent associated with $(\bar{C}_0, \langle\ ,\ \rangle)$. Since $\{G^0_\alpha(x, B), \alpha>0\}$ is the minimal A-resolvent, one has $\tilde{G}_\alpha(x, y)\geqq G^0_\alpha(x, y)\geqq 0$ for every $(\alpha, x, y)\in(0, \infty)\times E\times E$. If $u\in\mathcal{X}^{(0)}$, then $u^+=u\vee 0$ and $u^-=(-u)\vee 0$ are in $\mathcal{X}^{(0)}$ and $u=u^+-u^-$. One has

$$\lim_{\beta\to\infty}\beta(u^\pm-\beta\tilde{G}_\alpha u^\pm, u^\pm)\leqq\lim_{\beta\to\infty}\beta(u^\pm-\beta G^0_\beta u^\pm, u^\pm)=H^{(0)}(u^\pm, u^\pm)<\infty,$$

which implies that $u^\pm\in\bar{C}_0$ by Proposition 2.2.

NOTE for Theorem 3. In the case of Brownian motions, Fukushima [10] has given an example of $(\mathcal{X}, \langle\ ,\ \rangle)$, which is *nontrivial* in the sense that $C_0\subsetneqq\mathcal{X}\subsetneqq\mathcal{X}^\bullet$.

5. Case of birth and death processes

The Kolmogorov-Feller kernel for the birth and death process, satisfying condition (1.5), is given by the infinite matrix

$$(5.1)\qquad A=\begin{pmatrix}-\beta_0 & \beta_0 & 0 & 0 & \cdots\\ \delta_1 & -(\delta_1+\beta_1) & \beta_1 & 0 & \cdots\\ 0 & \delta_2 & -(\delta_2+\beta_2) & \beta_2 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{pmatrix}$$

with every $\beta_i>0$ and $\delta_i>0$. The integration problem of the Kolmogorov equation for such process has been completely analyzed by Feller [7]. He gave the most general lateral condition in terms of the boundary condition. We will construct the resolvent of the reflecting barrier (in the sense of Section 4) birth and death process and find its boundary condition. Moreover we will construct Feller's *elastic barrier* solution by the method of L^2-Dirichlet space.

We follow Feller's notation and terminology with some obvious exceptions[11]. The *natural scale* $\{x_n\}$ and the *canonical measure* μ are defined, respectively, by

$$(5.2)\qquad \begin{cases} x_0=0 \\ x_n=\dfrac{1}{\beta_0}+\cdots+\dfrac{\delta_1\delta_2\cdots\delta_{n-1}}{\beta_0\beta_1\cdots\beta_{n-1}} & (n=1,2,\cdots) \\ x_\infty=\lim\limits_{n\to\infty}x_n \end{cases}$$

and by

$$(5.3)\qquad \mu(0)=1\,,\quad \mu(n)=\frac{\beta_0\beta_1\cdots\beta_{n-1}}{\delta_1\delta_2\cdots\delta_n}\qquad (n=1,2,\cdots)\,.$$

Then the pair (μ, A) satisfies condition (S) of Section 3; the infinite matrix of the symmetric density $a(x, y)$ is of the form

$$\begin{pmatrix} -\beta_0 & \delta_1 & 0 & 0 & \cdots \\ \delta_1 & -\dfrac{\delta_1(\delta_1+\beta_1)}{\beta_0} & \dfrac{\delta_1\delta_2}{\beta_0} & 0 & \cdots \\ 0 & \dfrac{\delta_1\delta_2}{\beta_0} & -\dfrac{\delta_1\delta_2(\delta_2+\beta_2)}{\beta_0\beta_1} & \dfrac{\delta_1\delta_2\delta_3}{\beta_0\beta_1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The state space $E=\{0, 1, 2, \cdots\}$ is identified with the set $\{x_0, x_1, x_2, \cdots\}$ by the correspondence $n\leftrightarrow x_n$. For a function f over E, one denotes $f(x_n)$ for $f(n)$. The right difference ratio f^+ of f and the difference ratio $D_\mu f$ with respect to μ are respectively defined by

$$(5.4)\qquad f^+(x_n)=\frac{f(x_{n+1})-f(x_n)}{x_{n+1}-x_n}\qquad (n=0,1,2,\cdots)$$

and by

$$(5.5)\qquad D_\mu f(x_0)=f(x_0)\,,\quad D_\mu f(x)=\frac{f(x_n)-f(x_{n-1})}{\mu(n)}\qquad (n=1,2,\cdots)\,.$$

Write

11) For example, we use $\mu(n)$ for μ_n, Af for Ωf, $G_\alpha f$ [resp. $G_\alpha{}^0 f$] for $F_\lambda{}^*$ [resp. F_λ] etc.

(5.6) $$f(x_\infty) = \lim_{i\to\infty} f(x_i)\,,$$

(5.7) $$f^-(x_\infty) = \lim_{i\to\infty} f^+(x_i)\,,$$

if the limits exist. Then one has

(5.8) $$Af(x_n) = D_\mu f^+(x_n) \qquad (n = 0, 1, 2, \cdots)\,,$$

so that the Kolmogorov equation is of the form

(5.9) $$\alpha u - D_\mu u^+ = f, \qquad f \in C_0\,.$$

We will make use of some elementary results among those by Feller. Some of them will be stated just below, and some others, in the place where they are needed.

The boundary point x_∞ is classified into the four types called *regular*, *exit*, *entrance* and *natural* [7; Section 6]. One need only the definition of the regular boundary: x_∞ is *regular* if $x_\infty < \infty$ and if μ is a finite measure. For each $\alpha > 0$, there is one and only one solution h_α[12] of

(5.10) $$\alpha h_\alpha - D_\mu h_\alpha^+ = 0\,, \quad h_\alpha(x_0) = 1\,.$$

Therefore, any α-harmonic function is a constant multiple of h_α. Both h_α and h_α^+ are strictly increasing, so that $h_\alpha(x_\infty)$ and $h_\alpha^-(x_\infty)$ are well defined. The boundary x_∞ is regular iff both $h_\alpha(x_\infty)$ and $h_\alpha^-(x_\infty)$ are finite (see [7; Theorem 7.1] for the proof of the assertions on h_α).

Noting that, for any function u over E,

(5.11) $$\lim_{n\to\infty} \sum_{j=1}^{n} D_\mu(u^2)^+(x_j)\mu(j) = \lim_{n\to\infty} [u(x_{n+1}) + u(x_n)]\, u^+(x_n)$$

and that the conclusions of Proposition 3.5 are valid for *every* α-harmonc function if one allows the both sides of (3.17) to be infinite, one has immediately the following proposition.

Proposition 5.1. *If u is α-harmonic, both $u(x_\infty)$ and $u^-(x_\infty)$ exist and*

(5.12) $$\langle u, u\rangle_\alpha = u(x_\infty)u^-(x_\infty) \leqq \infty\,.$$

Therefore $\mathcal{H}_\alpha^\bullet = \{0\}$ if x is not regular, while $\mathcal{H}_\alpha^\bullet = \{ch_\alpha,\ -\infty < c < \infty\}$ if x is regular.

Moreover one has

Proposition 5.2. *Let x_∞ be regular. If u is a solution of the Kolmogorov equation* (5.9), *then $u \in \mathfrak{X}^\bullet \cap \mathcal{B}$ and*

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12) This function is denoted by u in Feller's notation.

$$(5.13)\qquad \langle u, u\rangle+(Au, u)=\frac{1}{2}\lim_{n\to\infty}[u(x_{n+1})+u(x_n)]u^{+}(x_n)\,.$$

Since $u-G_\alpha^0 f=ch_\alpha\in\mathcal{H}_\alpha^{\bullet}$, then $u\in\mathcal{X}^{\bullet}\cap\mathcal{B}$. Formula (5.13) follows from Proposition 3.6 and (5.11).

Proposition 5.1 implies the first assertion of the following theorem.

Theorem 5. (i) *If x_∞ is not regular, one has*

$$G_\alpha^{\bullet}(i, j)=G_\alpha^0(i, j)\,,\quad \alpha>0\,,\qquad i, j=0, 1, 2, \cdots\,.$$

(ii) *Let x_∞ be regular. Then the lateral condition* (4.1) *for the Kolmogorov equation* (5.9) *of the birth and death process is equivalent to Feller's reflecting barrier boundary condition*

$$(5.14)\qquad u^{-}(x_\infty)=0\,.$$

The kernel $\{R_\alpha^{\bullet}(i, j),\ \alpha>0\}$ is given by

$$(5.15)\qquad \frac{R_\alpha^{\bullet}(i, j)}{\mu(j)}=\frac{h_\alpha(x_i)h_\alpha(x_j)}{h_\alpha(x_\infty)h_\alpha^{-}(x_\infty)}\,.$$

One first proves (5.15). Let $R_\alpha^{\bullet}f=ch_\alpha$ $(f\in C_0)$ and $v=dh_\alpha$ with constants c, d. By (4.7) one has

$$c\cdot d\langle h_\alpha, h_\alpha\rangle_\alpha=d(f, h_\alpha)\,,$$

so that $c=(f, h_\alpha)/\langle h_\alpha, h_\alpha\rangle_\alpha$ and hence by (5.12),

$$R_\alpha f=\frac{h_\alpha\cdot(f, h_\alpha)}{h_\alpha(x_\infty)h_\alpha^{-}(x_\infty)}\,,$$

proving (5.15).

The former half of (ii) is obvious by comparing formula (5.15) with [7; Theorem 11.1]. However, one prefers to prove that, if u is a solution of the Kolmogorov equation (5.9), the function u satisfies condition (5.14) iff it satisfies (4.1). Suppose that u satisfies (4.1), i.e., $u=G_\alpha^{\bullet}f$. One may assume that $f\geqq 0$. Let $R_\alpha^{\bullet}f=ch_\alpha$. Note that $c>0$ unless f is identically zero, for $\langle R_\alpha^{\bullet}f, h_\alpha\rangle_\alpha=(f, h_\alpha)>0$. Then, therefore, $u=G_\alpha^0 f+R_\alpha^{\bullet}f$ is bounded below by a positive constant. Taking u for v in (4.1) and using (5.13), one has (5.14). Conversely, suppose that u satisfies (5.14). By Proposition 5.2, $u\in\mathcal{X}^{\bullet}\cap\mathcal{B}$. It is enough to show that, for every $g\in C_0$,

$$(5.16)\qquad \langle u, G_\alpha^{\bullet}g\rangle+(Au, G_\alpha^{\bullet}g)=0\,,$$

because $\{G_\alpha^{\bullet}g\,;\,g\in C_0\}$ is dense in $\mathcal{X}^{\bullet}$ with respect to the norm $\langle\ ,\ \rangle^{1/2}$. Since $u^{-}(x_\infty)=(G_\alpha^{\bullet}g)^{-}(x_\infty)=0$, one has by (5.13)

$$I_1 = \langle u+G^{\bullet}_{\alpha}g,\ u+G^{\bullet}_{\alpha}g\rangle+(A[u+G^{\bullet}_{\alpha}g],\ u+G^{\bullet}_{\alpha}g) = 0\,,$$
$$I_2 = \langle u,\ u\rangle+(Au,\ u) = 0\,.$$

On the other hand one has by (4.1)

$$I_3 = \langle G^{\bullet}_{\alpha}g,\ u\rangle+(A\ G^{\bullet}_{\alpha}g,\ u) = 0\,,$$
$$I_4 = \langle G^{\bullet}_{\alpha}g,\ G^{\bullet}_{\alpha}g\rangle+(A\ G^{\bullet}_{\alpha}g,\ G^{\bullet}_{\alpha}g) = 0\,.$$

Hence

$$\langle u,\ G^{\bullet}_{\alpha}g\rangle+(Au,\ G^{\bullet}_{\alpha}g) = I_1-I_2-I_3-I_4 = 0\,.$$

We next proceed to discuss Feller's elastic barrier solution. Let x_∞ be regular and let $u\in\mathfrak{X}^{\bullet}$. Consider the orthogonal decomposition of u (see Theorem 1);

$$u = u_1+u_2\,,\quad u_1\in\mathfrak{X}^{(0)},\quad u_2\in\mathcal{H}^{\bullet}_{\alpha}\,. \tag{5.17}$$

Proposition 5.3. (i) *$u_2(x_\infty)$ is a continuous linear functional of $u\in\mathfrak{X}^{\bullet}$ with respect to the $\langle\ ,\ \rangle_\alpha$-norm.* (ii) *$u_2(x_\infty)$ is independent of $\alpha>0$. This common value is called the boundary value of u and it is denoted by $u(x_\infty)$.*

Let $u_n\in\mathfrak{X}^{\bullet}$ and let $\langle u_n,\ u_n\rangle_\alpha\to 0$. The α-harmonic component of u_n is expressed as $u_{n2}=c_nh_\alpha$. Since $\langle u_{n2},\ u_{n2}\rangle_\alpha=c_n^2\langle h_\alpha,\ h_\alpha\rangle\to 0$, one has $c_n\to 0$, so that $u_{n2}(x_\infty)=c_nh_\alpha(x_\infty)\to 0$. One has proved assertion (i).

For assertion (ii) one need one more result due to Feller [7; Lemma 9.1]: if x_∞ is regular (or exit), then

$$G^0_\alpha f(x_i)\to 0\quad\text{as}\quad i\to\infty \tag{5.18}$$

for every $f\in\mathcal{L}^2\cap\mathcal{B}$ and every $\alpha>0$. Let $u\in\mathfrak{X}^{\bullet}$ and let

$$\begin{aligned} u &= u_1+u_2\,,\quad u_1\in\mathfrak{X}^{(0)},\quad u_2\in\mathcal{H}^{\bullet}_{\alpha}\,,\\ &= v_1+v_2\,,\quad v_1\in\mathfrak{X}^{(0)},\quad v_2\in\mathcal{H}^{\bullet}_{\beta}\,. \end{aligned}$$

Choose $f_n\in\mathcal{L}^2\cap\mathcal{B}$ such that $u_n=G^{\bullet}_{\alpha}f_n\to u$ in the $\langle\ ,\ \rangle_\alpha$-norm. Then $u_n\to u$ in the $\langle\ ,\ \rangle_\beta$-norm and $u_n=G^{\bullet}_{\beta}g_n$ with $g_n=f_n+(\beta-\alpha)G^{\bullet}_{\alpha}f_n\in\mathcal{L}^2\cap\mathcal{B}$. Since $G^0_\alpha f_n(x_\infty)=G^0_\beta g_n(x_\infty)=0$ by (5.18), one has $R^{\bullet}_{\alpha}f_n(x_\infty)=R^{\bullet}_{\alpha}g_n(x)$. By assertion (i), $u_2(x_\infty)=\lim_{n\to\infty}R^{\bullet}_{\alpha}f_n(x_\infty)=\lim_{n\to\infty}R^{\bullet}_{\beta}g_n(x_\infty)=v_2(x_\infty)$, proving (ii).
One can now introduce a bilinear form on $\mathfrak{X}^{\bullet}$ by

$$\mathrm{N}(u,\ v) = u(x_\infty)v(x_\infty)\,. \tag{5.19}$$

For a positive constant a, define

$$\mathrm{H}(u,\ v) = \langle u,\ v\rangle+a\mathrm{N}(u,\ v)\,,\qquad u,\ v\in\mathfrak{X}^{\bullet}. \tag{5.20}$$

By Proposition 5.3 it follows that the pair $(\mathfrak{X}^{\bullet},\ \mathrm{H})$ is an L^2-Dirichlet space. The following theorem is easily proved in the same way as Theorem 2 and Theorem 5.

Theorem 6. *The condition*

$$u\in\mathcal{X}^{\bullet} \quad \textit{and} \quad \mathrm{H}(u, v)+(Au, v) = 0 \quad \textit{for every} \quad v\in\mathcal{X}^{\bullet} \tag{5.21}$$

is a lateral condition for the Kolmogorov equation (5.9). *The corresponding A-resolvent is the μ-symmetric resolvent $\{G_\alpha(x, B), \alpha>0\}$ associated with* ($\mathcal{X}$, H). *The lateral condition* (5.21) *is equivalent to Feller's elastic barrier boundary condition*

$$u^-(x_\infty)+au(x_\infty) = 0\,. \tag{5.22}$$

The kernel $R_\alpha(i, j)$ $(=G_\alpha(i, j)-G_\alpha^0(i, j))$ is given by

$$\frac{R_\alpha(i, j)}{\mu(j)} = \frac{h_\alpha(x_i)h_\alpha(x_j)}{h_\alpha(x_\infty)\{h_\alpha^-(x_\infty)+ah_\alpha(x_\infty)\}}\,. \tag{5.23}$$

NOTE. The kernel $G_\alpha(i, j)$ in Theorem 6 reduces to $G_\alpha^0(i, j)$ for $a=\infty$ and to $G_\alpha^{\bullet}(i, j)$ for $a=0$.

6. Concluding remarks and open problems

In this section we will give some complementary results and propose some open problems.

(a) Given the Kolmogorov-Feller kernel $A(x, y)$, one can ask how many measures satisfy condition (S) of Section 3. Since the resolvent $\{G_\alpha^{\bullet}(x, B), \alpha>0\}$ of the r.b. Markov chain depends at least *formally* on the measure μ as well as A, one can also ask if it *really* depends on μ or not. We will answer these problems in the sequel.

Let μ be an arbitrary (but fixed) measure satisfying condition (S).

Proposition 6.1. *There is a unique decomposition of E into the disjoint subsets $\{E_k\}$ such that a measure ν satisfies condition* (S) *iff*

$$\nu(B) = c_i\mu(B)\,, \qquad c_i > 0\,, \tag{6.1}$$

whenever B is a subset of E_i.

The decomposition of E is carried out in a way similar to the decomposition of the state space into the minimal closed *recurrent* subsets in Markov chain theory (see [8]). Let $\Pi(x, y)$ be the stochastic matrix defined in (1.7). For two states x, y in E, denote $x\rightsquigarrow y$ if $\Pi^n(x, y)>0$ for some $n\geqq 1$. From condition (S) it follows that $x\rightsquigarrow y$ implies $y\rightsquigarrow x$. Then one can easily verify that the relation "$\rightsquigarrow$" is an equivalence relation. Denote the equivalence classes by E_i, $i=1, 2,\cdots$. It is shown that each E_i is a minimal closed set with respect to Π, i.e., that $\Pi(x, E-E_i)=0$ for every $x\in E_i$ and for any proper subset B of E_i, $\Pi(x, E-B)>0$ for some $x\in B$.

Let ν be a measure satisfying condition (S) and $b(x, y)$, the symmetric density $A(x, y)/\nu(y)$. If both x and y are in E_i, there is a chain $\{x_1, x_2, \cdots, x_{n-1}\}$ such that $\Pi(x, x_1)>0$, $\Pi(x_i, x_{i+1})>0$ and $\Pi(x_{n-1}, y)>0$. Since $A(x, x_1)>0$, one has $a(x, x_1)>0$ and $b(x, x_1)>0$. Hence

$$\frac{\nu(x)}{\mu(x)} = \frac{a(x_1, x)}{b(x_1, x)} = \frac{a(x, x_1)}{b(x, x_1)} = \frac{\nu(x_1)}{\mu(x_1)}$$

In the same way one has $\nu(x)/\mu(x)=\nu(x_1)/\mu(x_1)=\cdots=\nu(x_{n-1})/\mu((x_{n-1})=\nu(y)/\mu(y)$. The converse is easy, because $A(x, y)=0$ if $x\in E_i$, $y\in E_j$, $(i\neq j)$.

Proposition 6.2. *The resolvent $\{G^{\bullet}_{\alpha}(x, B), \alpha>0\}$ of Section 4 depends only on A.*

Let ν be a measure satisfying condition (S) and ν_i [resp. μ_i], the restriction of ν [resp. μ] to the set E_i. Let $A_i(x, y)$ be the restriction of $A(x, y)$ to $E_i\times E_i$. The kernel $A_i(x, y)$ is a Kolmogorov-Feller kernel satisfying condition (1.5) on E_i. The pairs (μ_i, A_i) and (ν_i, A_i) satisfy condition (S) on E_i. Since $\nu_i=c_i\mu_i$, it follows that the pairs (μ_i, A_i) and (ν_i, A_i) generate the same r.b. Markov chain resolvent $\{{}^iG^{\bullet}_{\alpha}(x, B), \alpha>0\}$ over E_i. Define

(6.2) $$\begin{aligned} G^{\bullet}_{\alpha}(x, y) &= {}^iG^{\bullet}_{\alpha}(x, y) \qquad \text{for} \quad x, y\in E_i,\ i=1, 2, \cdots, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

One wants to prove that the above $\{G^{\bullet}_{\alpha}(x, B), \alpha>0\}$ is the resolvent of the r.b. Markov chain associated with (μ, A) as well as (ν, A). Let I_{E_i} be the indicator of the set E_i and f_i, the restriction of a function f (over E) to E_i. By (6.2),

(6.3) $$I_{E_i}G^{\bullet}_{\alpha}f = I_{E_i}{}^iG^{\bullet}_{\alpha}f_i\,.$$

From the formula

(6.4) $$\langle u, v\rangle_{\alpha} = \textstyle\sum_i \langle I_{E_i}u, I_{E_i}v\rangle_{\alpha}\,,$$

it follows that if $u\in\mathcal{X}^{\bullet}$, then $I_{E_i}u\in\mathcal{X}^{\bullet}_i$, where $\mathcal{X}^{\bullet}_i$ is the L²-Dirichlet space on E_i associated with the Dirichlet norm $\langle\ ,\ \rangle_i$ with respect to (μ_i, A_i). Since $\langle {}^iG^{\bullet}_{\alpha}f_i, v_i\rangle_{i,\alpha}=\langle I_{E_i}{}^iG^{\bullet}_{\alpha}f_i, I_{E_i}v_i\rangle_{\alpha}=(I_{E_i}f_i, I_{E_i}v_i)$ for every $v_i\in\mathcal{X}^{\bullet}_i$, one has by (6.3) and (6.4)

(6.5) $$\begin{aligned} \langle G^{\bullet}_{\alpha}f, v\rangle_{\alpha} &= \textstyle\sum_i \langle I_{E_i}{}^iG^{\bullet}_{\alpha}f_i, I_{E_i}v\rangle_{\alpha} \\ &= \textstyle\sum_i (I_{E_i}f_i, I_{E_i}v) \\ &= (f, v)\,, \qquad v\in\mathcal{X}^{\bullet}. \end{aligned}$$

One has proved that the $\{G^{\bullet}_{\alpha}(x, B), \alpha>0\}$ in (6.2) is the r.b. Markov chain resolvent associated with (μ, A). The same argument applies to the pair (ν, A).

(b) Let $\{G_{\alpha}(x, B), \alpha>0\}$ be any μ-symmetric resolvent. For $f\in C_0^+$,

$(\alpha G_\alpha f, 1)=(f, \alpha G_\alpha 1)\leqq(f, 1)$, so that the measure μ is *excessive relative to any μ-symmetric resolvent*, i.e.,

$$\alpha \textstyle\sum_x \mu(x)G_\alpha(x, B) = \mu(\alpha G_\alpha)(B)\leqq\mu(B) \tag{6.6}$$

for every subset B of E. In the same way, by hypothesis (S) one has

$$\textstyle\sum_x \mu(x) \sum_{y\in B} A(x, y) = \mu A(B) = 0\,, \qquad B\subset E\,. \tag{6.7}$$

Note that the condition

$$\mu A(B)\leqq 0\,, \qquad B\subset E \tag{6.8}$$

is equivalent to the condition that μ is excessive relative to the minimal A-resolvent $\{G^n_\alpha(x, B), \alpha>0\}$. Condition (6.8) will be a standard which should be imposed on the measure μ in the generalization to the unsymmetric case (see paragraph (c)).

(c) Let $A(x, y)$ be a Kolmogorov-Feller kernel satisfying condition (1.5) and μ, a measure (strictly positive and finite at every state) satisfying condition (6.8). We will ask, *without the symmetric hypothesis* (S), if one can construct something like the r.b. Markov chain associated with (μ, A), developing the method of Section 4. Replacing hypothesis (S) by other kind of hypotheses, Kunita (unpublished, see Theorem 7) has obtained a partial answer to the above question. In this paragraph we will give a slightly more general formulation than Kunita's and propose an open problem, together with Kunita's result.

We will use the same notation as in the symmetric case. Let $a(x, y)$ be the density $A(x, y)/\mu(y)$ and $A^*(x, y)$, the co-kernel of $A(x, y)$ with respect to μ:

$$A^*(x, y) = a(y, x)\mu(y)\,. \tag{6.9}$$

Kernel $A^*(x, y)$ is also a Kolmogorov-Feller kernel by condition (6.8), but it does not satisfy condition (1.5) unless $\mu A(B)=0$. In the same way as in Section 3, introduce the Dirichlet form with respect to (μ, A) by

$$\begin{aligned}\langle u, v\rangle &= \frac{1}{2} \textstyle\sum_{x,y} (u(x)-u(y))(v(x)-v(y))a(x, y)\mu(x)\mu(y) \\ &\quad+\frac{1}{2} \textstyle\sum_x u(x)v(x)(-A^*1)(x)\mu(x)\end{aligned} \tag{6.10}$$

and $\langle u, v\rangle_\alpha$, by

$$\langle u, v\rangle_\alpha = \langle u, v\rangle+\alpha(u, v)\,.$$

As in Section 4 one denotes by $\mathcal{X}^\bullet$ the space of all L^2-functions with finite Dirichlet norm. For each $\alpha>0$, $\mathcal{X}^\bullet$ forms a Hilbert space with respect to the norm $\langle\ ,\ \rangle_\alpha^{1/2}$. As in Section 3, if $u\in\mathcal{X}^\bullet$, both Au and A^*u are finite everywhere.

The basic hypothesis on the pair (μ, A) is this:
(B) A^*-A is a bounded operator from $\mathcal{X}^\bullet$ into $\mathcal{L}^2$.
The above hypothesis is valid, for example, in the symmetric case or in the case when $a(x, y)-a(y, x)$ is square-integrable with respect to the product measure $\mu\times\mu$. We do not know if there is a pair (μ, A) which does not satisfy hypothesis (B).

Problem 1. *Under hypothesis* (B), *the condition*

$$u\in\mathcal{X}^\bullet, \langle u, v\rangle+\frac{1}{2}([A+A^*]u, v)=0 \qquad \text{for every} \quad v\in\mathcal{X}^\bullet \tag{6.11}$$

is a lateral condition for the Kolmogorov equation

$$(\alpha-A)u=f, \qquad f\in C_0\,. \tag{6.12}$$

That is, there is a unique solution u of (6.12) *and* (6.11) *for each $\alpha>0$ and the family of kernels $\{G_\alpha^\bullet(x, B), \alpha>0\}$ defined by*

$$u=G_\alpha^\bullet f \tag{6.13}$$

is a resolvent.

Theorem 7. (H. Kunita) *The assertion of Problem* 1 *is true if μ is a finite measure satisfying condition* (6.7): $\mu A=0$. *In this case, $\{G_\alpha^\bullet(x, B), \alpha>0\}$ is a Markov resolvent.*

We now introduce the outline of Kunita's proof. Define a bilinear form on $\mathcal{X}^\bullet$ by

$$\mathrm{D}_\alpha(u, v)=\langle u, v\rangle_\alpha+\frac{1}{2}([A^*-A]u, bv)\,. \tag{6.14}$$

In the same way as in Theorem 2, u satisfies (6.11) and (6.12) iff it satisfies the functional equation

$$u\in\mathcal{X}^\bullet \quad \text{and} \quad \mathrm{D}_\alpha(u, v)=(f, v) \qquad \text{for every} \quad v\in\mathcal{X}^\bullet\,. \tag{6.15}$$

From hypothesis (B) and $\mathrm{D}_\alpha(u, u)=\langle u, u\rangle_\alpha$ it follows that (6.15) has a unique solution. It is not difficult to see that the kernels $\{G_\alpha^\bullet(x, B), \alpha>0\}$ defined by (6.13) satisfy the resolvent equation (2.3). One next proves the positivity of $G_\alpha^\bullet$. A direct computation shows that, for every $u\in\mathcal{X}^\bullet$,

$$\mathrm{D}_\alpha(u^+, u^-)\leqq 0 \quad \text{with} \quad u^+=u\vee 0\,, \quad u^-=(-u)\vee 0\,. \tag{6.16}$$

On the other hand, if $f\geqq 0$ and $u=G_\alpha^\bullet f$, then $\mathrm{D}_\alpha(u, u^-)=(f, u^-)\geqq 0$. Therefore, by (6.16)

$$\langle u^-, u^- \rangle_\alpha = \mathrm{D}_\alpha(u^-, u^-) = \mathrm{D}_\alpha(u^+, u^-) - \mathrm{D}_\alpha(u, u^-) \leqq 0\,,$$

proving $u^-=0$, i.e., $u=G_\alpha^\bullet f \geqq 0$.

The proof of $\alpha G_\alpha^\bullet 1 = 1$ is the same as in Proposition 4.2, since $A1 = A^*1 = 0$ under the hypothesis of the theorem.

In proving the general case of Problem 1, the crucial point is to verify the sub-Markov property of $\alpha G_\alpha^\bullet$; $\alpha G_\alpha^\bullet 1 \leqq 1$. Kunita's proof remains valid for the other parts.

(d) It will be interesting to find the boundary condition which the resolvent of the r.b. Markov chain should satisfy at the ideal boundary of Feller, Martin or other types. The case of the birth and death process, in which the ideal boundary consists of one point x_∞, was discussed in Section 5. We propose two problems related to the boundary condition.

Let us introduce the following hypothesis:

(F) There are at most finitely many linearly independent, bounded α-harmonic functions.

Under this hypothesis, Feller [6] and Dynkin [4] have obtained the most general lateral condition for the Kolmogorov equation (1.10) in terms of the boundary condition at the exit[13] boundary (consisting of finitely many points by hypothesis (F)).

Problem 2. *Find the Feller-Dynkin boundary condition for the resolvent of the r.b. Markov chain.*

The second problem is concerned with a natural ideal boundary induced by the r.b. Markov chain itself. Let $\{P^\bullet(t, x, B),\ t \geqq 0\}$ be the transition function corresponding to $\{G_\alpha^\bullet(x, B),\ \alpha > 0\}$ (see Section 1) and $\tilde{x}_t$, the Markov chain with $P^\bullet(t, x, B)$ as its transition function. By [15; Theorem 1], there is a *nice*[14] standard modification x_t of $\tilde{x}_t$, taking values in an extended state space $\bar{E}$. We will say that the process x_t is *the reflecting barrier Markov chain associated with* A. Space $\bar{E}$ is compact and the original state space E is dense in $\bar{E}$, so that the set $\bar{E}-E$ is an ideal boundary. The kernels $G_\alpha^\bullet(x, y)$ and $P^\bullet(t, x, y)$ are extended to $\bar{E} \times \bar{E}$. If $f \in \bar{C}$ (=the space of all continuous functions over $\bar{E}$), $G_\alpha^\bullet f$ is continuous on $\bar{E}$.

Problem 3. *Find the boundary condition of* $u(x) = G_\alpha^\bullet f(x)$, $f \in \bar{C}$, $x \in \bar{E}$, *at the boundary* $\bar{E}-E$.

(e) Similarly to the latter half of the preceding paragraph, Fukushima [9; Theorem 2 and p. 213] has defined the r.b. Brownian motion on a bounded

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13) The word "exit" is taken in the sense of Feller [6].

14) The strong Markov property, the right continuity of sample paths and so on.

domain as a nice Markov process defined over an extended space[15] and with the transition function of his Theorem 1, (v). He has proved that the almost all sample functions of the r.b. Brownian motion are continuous at the boundary (as well as in the interior of the original domain). One can expect the following result.

Problem 4. *The almost all sample functions of the r.b. Markov chain* (*see paragraph* (d)) *are continuous at the boundary* $\bar{E}-E$.

It seems to us that Fukushima's proof depends heavily on the fact that his resolvent is a Markov resolvent. In fact, if our $\{G_\alpha^\bullet(x, B), \alpha>0\}$ is a Markov resolvent, Fukushima's proof applies to the r.b. Markov chain. The complete proof is given in [19][16]. In the case of *sub-Markov* resolvents[17], Problem 4 remains still open.

(f) One can ask how the results of this paper can be extended to the Markov process over a general (but nice topological) state space. So far one has two such examples; the Brownian motion [9] (where A is the Laplacian multiplied by $\frac{1}{2}$) and the stable process [5] (where A is a certain integro-differential operator), both defined on a domain of Euclidean space. Note that, in these cases as well as in the Markov chain case, the operator A is the characteristic operator [3] of its associated minimal process. One possible formulation in the general case is this.

Problem 5. *Let x_t^0 be a standard* (*or Hunt*) *process over a locally compact, separable Hausdorff space E and A, the characteristic operator of x_t^0. Let μ be a Randon measure on E, strictly positive for every open set. Suppose that the resolvent $\{G_\alpha^0(x, B), \alpha>0\}$ of the process x_t^0 is μ-symmetric. Then find the Dirichlet norm with respect to the pair* (μ, A) *and construct its associated r.b. Markov process.*

Appendix. The resolvent of the reflecting barrier Brownian motion

We will now outline how the resolvent of the r.b. Brownian motion by Fukushima can be constructed by the same method as in Section 4. This method is, mostly, nothing but the Hilbert space method for elliptic partial differential equations (PDE). At least, the crucial parts in the proofs depend on those results in PDE. All the results on PDE which will be used with no reference in the sequel are proved, for example, in [17].

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15) His compactification is slightly different from ours in paragraph (d). But the arguments of [9; Sections 3 and 4] are much simplified under our compactification (applied to the case of Brownian motion), all the results there remaining valid [20].

16) One needs only establish the analogue in the r.b. Markov chain of [9; Lemma 4.4]; the other parts are the same as in the r.b. Brownian motion (see footnote 15).

17) This situation can arise even in the case of Brownian motion, if the original domain is allowed to be unbounded. Fukushima considers only bounded domains.

We will often use the same notation as in the previous sections in a different context. Let E be a (possibly unbounded) domain of Euclidean N-space and Δ, the Laplacian $\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$. The measure μ is the Lebesgue measure on E. Space $\mathcal{L}^2$ is the L^2-space with respect to the Lebesgue measure dx. Space C [resp. C_0] is the space of all real-valued, bounded continuous functions [resp. with compact support]. Space C^∞ [resp. C_0^∞] is the subspace of C [resp. C_0] formed by C^∞-functions[18].

One is concerned with the Laplace transform of the heat equation (1.1):

$$\left(\alpha - \frac{1}{2}\Delta\right)u = f, \qquad f \in C_0^\infty . \tag{A.1}$$

The words *symmetric* and *Brownian resolvent* stand for "μ-symmetric" and "$\frac{1}{2}\Delta$-resolvent", respectively. For a locally integrable function u, let $\frac{\partial u}{\partial x_i}$ denote the *derivative in the sense of Schwartz distribution* and $\langle\ ,\ \rangle$, *the classical Dirichlet form* (up to constant $\frac{1}{2}$):

$$\langle u, v\rangle = \frac{1}{2}\int_E \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx . \tag{A.2}$$

Let $\mathcal{X}^\bullet$ be the space of all L^2-functions u such that $\frac{\partial u}{\partial x_i} \in \mathcal{L}^2$, $i=1, 2, \cdots, N$.

Proposition A.1. *The pair* $(\mathcal{X}^\bullet, \langle\ ,\ \rangle)$ *is an* L^2*-Dirichlet space.*

It is a standard fact in PDE that $\mathcal{X}^\bullet$ forms a Hilbert space with respect to the $\langle\ ,\ \rangle_\alpha$-norm. It is enough to prove assertion (c) of Proposition 2.1. Let $u \in \mathcal{X}^\bullet$ and v, a normal contraction of u. In general, let $\left[\frac{\partial y}{\partial x_1}\right]$ denote the Radon-Nikodym derivative of u relative to the measure dx_1 for each fixed $(x_2, \cdots, x_N)$. It is known that $\left[\frac{\partial u}{\partial x_1}\right]$ exists for almost all $(x_2, \cdots, x_N)$ and it is locally integrable iff $\frac{\partial u}{\partial x_1}$ is a locally integrable function and that, in this case, $\frac{\partial u}{\partial x_1} = \left[\frac{\partial u}{\partial x_1}\right]$ almost everywhere. This result obviously implies that $v \in \mathcal{X}^\bullet$ and then $\langle u, u\rangle \geqq \langle v, v\rangle$.

Proposition A.2. *Let* $f \in \mathcal{L}^2$. *A function u is a solution of the equation in* $\mathcal{X}^\bullet$

$$\langle u, v\rangle_\alpha = (f, v) \qquad \textit{for every} \quad v \in \mathcal{X}^\bullet, \tag{A.3}$$

iff it satisfies the distribution equation

18) Functions having continuous partial derivatives of all order.

(A.4) $$\left(\alpha-\frac{1}{2}\Delta\right)u=f$$

and

(A.5) $$u\in\mathcal{X}^{\bullet},\ \langle u, v\rangle+\left(\frac{1}{2}\Delta u, v\right)=0 \qquad \text{for every} \quad v\in\mathcal{X}^{\bullet}.$$

The "if" part is obvious. The "only if" part also goes along the same line as in Theorem 2 of Section 4 as follows. Note that, for each $u\in\mathcal{X}^{\bullet}$,

(A.6) $$\langle u, v\rangle+\left(u, \frac{1}{2}\Delta v\right)=0 \qquad \text{for every} \quad v\in C_0^{\infty}.$$

Therefore, if u is a solution of (A. 3), one has

$$(\alpha u, v)-\left(u, \frac{1}{2}\Delta v\right)=(f, v)\,, \qquad v\in C_0^{\infty},$$

proving that u is a solution of the distribution equation (A. 4). Hence it follows that the function $\frac{1}{2}\Delta u$, with the derivative taken in the distribution sense, is in $\mathcal{L}^2$. Therefore one has

$$(\alpha u, v)-\left(\frac{1}{2}\Delta u, v\right)=(f, v)\,, \qquad v\in\mathcal{L}^2\,,$$

which, together with (A. 3), proves (A. 5).

As noted in Section 2, equation (A. 3) determines only an L^2-resolvent in general. In the present case, however, this result can be strengthened as follows.

Proposition A.3. *There is a unique resolvent* $\{G_\alpha^{\bullet}(x, B),\ \alpha>0\}$ *which is associated with* $(\mathcal{X}^{\bullet}, \langle\ ,\ \rangle)$ *and maps* C_0 *into* C: $G_\alpha^{\bullet}f\in C$ *or every* $\alpha>0$ *and every* $f\in C_0$.

Let $f\in\mathcal{L}^2\cap C^{\infty}$. Since the solution of (A. 3) satisfies (A. 4), it has a C^{∞}-version[19], denoted by $G_\alpha^{\bullet}f$. In particular, $G_\alpha^{\bullet}f$ is continuous, so that the arbitrariness of measure zero which is involved in the L^2-resolvent associated with $(\mathcal{X}^{\bullet}, \langle\ ,\ \rangle)$ is eliminated in the version $G_\alpha^{\bullet}f$. Hence, whenever $f\in\mathcal{L}^2\cap C^{\infty}$ and $0\leqq f\leqq 1$,

(A.7) $$0\leqq G_\alpha^{\bullet}f(x)\leqq 1 \qquad \text{for every} \quad x\in E\,,$$

and, for every $f\in\mathcal{L}^2\cap C^{\infty}$ and every $x\in E$,

(A.8) $$G_\alpha^{\bullet}f(x)-G_\beta^{\bullet}f(x)+(\alpha-\beta)G_\alpha^{\bullet}G_\beta^{\bullet}f(x)=0\,, \qquad \alpha, \beta>0\,.$$

19) If two functions f and g equal almost everywhere, g[resp. f] is called a version of f [resp. g]. Then the stated result is known as the lemma of Weyl.

From (A. 7) and (A. 8) it follows that there is a unique resolvent $\{G^{\bullet}_{\alpha}(x, B), \alpha>0\}$ such that $G^{\bullet}_{\alpha}f(x)=\int_E G^{\bullet}_{\alpha}(x, dy)f(y)$, $f\in\mathcal{L}^2\cap C^{\infty}$. Since C_0^{∞}, a subset of $\mathcal{L}^2\cap C^{\infty}$, is dense in C_0 with uniform norm, the function $\int_E G^{\bullet}_{\alpha}(x, dy)f(y)$, $f\in C_0$, is continuous.

One next proves the analogue or Theorem 4 (Section 4) in the two propositions.

Proposition A.4. *Let $\bar{C}_0^{\infty}$ be the closure of C_0^{∞} with respect to the $\langle\ ,\ \rangle_{\alpha}$-norm, $\alpha>0$. The pair $(\bar{C}_0^{\infty}, \langle\ ,\ \rangle)$ is an* L^2-*Dirichlet space.*

It is evident that $\bar{C}_0^{\infty}$ is independent of $\alpha>0$.

One uses the same notation as in the proof of Theorem 4. Let $u\in\bar{C}_0^{\infty}$ and let $u_n\in C_0^{\infty}$ such that $u=\langle\ ,\ \rangle_{\alpha}\text{-}\lim u_n$. One may assume that u_n converges almost everywhere to u[20]. Then $Tu_n\to Tu$ both almost everywhere and in $\mathcal{L}^2$. Also since $Tu_n\in C_0\cap\mathcal{X}^{\bullet}$, it can be approximated in the $\langle\ ,\ \rangle_{\alpha}$-norm by a sequence of regularizations of Tu_n belonging to C_0^{∞}, which proves that $Tu_n\in\bar{C}_0^{\infty}$. The rest of the proof is the same as in Theorem 4; it is enough to replace the partial derivatives for the difference and, the Lebesgue measure dx for $J(x, y)a(x, y)\mu(x)\mu(y)$.

It is well known (for example, [11], [13]) that the minimal Brownian resolvent $\{G^0_{\alpha}(x, B), \alpha>0\}$ exists and it is symmetric. Moreover G^0_{α} maps C_0 (actually, the space $\mathcal{B}$ of all bounded measurable functions) into C.

Proposition A.5. *The condition*

$$u\in C_0^{\infty}\cap\bar{C}^{\infty} \tag{A.9}$$

is a lateral condition for the equation (A. 1) *and the corresponding Brownian resolvent, say $\{\tilde{G}_{\alpha}(x, B), \alpha>0\}$, is nothing but the minimal Brownian resolvent.*

Since Proposition A. 2 and A. 3 are valid for any L^2-Dirichlet subspace, containing C_0^{∞}, of $(\mathcal{X}^{\bullet}, \langle\ ,\ \rangle)$, then the former half is obvious. Also since both $\tilde{G}_{\alpha}$ and G^0_{α} map C_0 into C, it is enough to prove that they define the same L^2-resolvent, i.e.,

$$(\bar{C}_0^{\infty}, \langle\ ,\ \rangle) = (\mathcal{X}^{(0)}, H^{(0)})\,. \tag{A.10}$$

One first proves that, if $u\in C_0^{\infty}$, then $u\in\mathcal{X}^{(0)}$ and

$$H^{(0)}(u, u) = \langle u, u\rangle\,. \tag{A.11}$$

The proof is similar to that of Proposition 3.2. Indeed, one can easily verify that $\beta(u-\beta G^0_{\beta}u)$ converges to $-\frac{1}{2}\Delta u$ in the distribution sense, so that

20) Rewrite by u_n the a.e. convergent subsequence, if necessary.

$$\mathrm{H}^{(0)}(u, u) = \lim_{\beta\to\infty} \beta(u-\beta G^0_\beta u, u) = \left(-u, \frac{1}{2}\Delta u\right) = \langle u, u\rangle .$$

Then, in the same way as in Proposition 3.4, it follows that, if $u\in\bar{C}_0^\infty$, $u\in\mathcal{X}^{(0)}$ and (A. 11) remains valid. The proof of $\mathcal{X}^{(0)}\subset\bar{C}_0^\infty$ is the same as in the final argument of Theorem 4.

Let $\mathcal{H}^\bullet_\alpha$ be the space of all α-harmonic functions[21] in $\mathcal{X}^\bullet$. By (A.10) one has the same orthogonal decomposition as in Theorem 1 (or rather in Proposition 4.3). The kernel

$$R^\bullet_\alpha(x, B) = G^\bullet_\alpha(x, B) - G^0_\alpha(x, B) \tag{A.12}$$

is characterized by $R^\bullet_\alpha f\in\mathcal{H}^\bullet_\alpha$, $f\in C_0^\infty$, and

$$\langle R^\bullet_\alpha f, v\rangle_\alpha = (f, v) \qquad \text{for every} \quad v\in\mathcal{H}^\bullet_\alpha . \tag{A.13}$$

Proposition A.6 *There is a symmetric density* $r^\bullet_\alpha(x, y) = R^\bullet_\alpha(x, dy)/dy$ *such that, for each* y, $r^\bullet_\alpha(x, y)$ *is* α*-harmonic.*

Kernel $r^\bullet_\alpha(x, y)$ is denoted by $R_\alpha(x, y)$ in [9]. Note that, for each fixed y, $v(y)$ is a continuous linear functional on $v\in\mathcal{H}^\bullet_\alpha$ [9; Lemma 2.2]. Let $r^\bullet_\alpha(x, y)$ the reproducing kernel in $\mathcal{H}^\bullet_\alpha$:

$$\langle r^\bullet_\alpha(\cdot, y), v\rangle_\alpha = v(y) , \qquad v\in\mathcal{H}^\bullet_\alpha . \tag{A.14}$$

Using [9; Lemma 2.1] in the same way as in the proof of [9; Lemma 2.7], one has, for each x,

$$\int_E \frac{\partial r^\bullet_\alpha(x, y)}{\partial x_i} f(y)dy = \frac{\partial}{\partial x_i}\left(\int_E r^\bullet_\alpha(x, y) f(y)dy\right),$$

which implies that

$$\begin{aligned}(f, v) &= \int_E f(y)\langle r^\bullet_\alpha(\cdot, y), v\rangle_\alpha dy \\ &= \left\langle \int_E f(y) r^\bullet_\alpha(\cdot, y)dy, v\right\rangle_\alpha .\end{aligned}$$

Hence, $R^\bullet_\alpha f(x) = \int_E f(y) r^\bullet_\alpha(x, y)dy$ for every $f\in C_0^\infty$ and every x in E.

We will now summarize the main results of this section.

Theorem 8. *The condition*

$$u\in\mathcal{X}^\bullet\cap C^\infty, \quad \langle u, v\rangle + \left(\frac{1}{2}\Delta u, v\right) = 0 \quad \textit{for every} \quad v\in\mathcal{X}^\bullet \tag{A.15}$$

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21) The *genuine* solutions of $\left(\alpha - \frac{1}{2}\Delta\right)u = 0$.

is a lateral condition for the equation (A.1). *The corresponding Brownian resolvent* $\{G_\alpha^\bullet(x, B), \alpha > 0\}$ *is symmetric and it generates the* L^2*-resolvent associated with* $(\mathcal{X}^\bullet, \langle\ ,\ \rangle)$. *For each* $\alpha > 0$, $G_\alpha^\bullet$ *maps* C_0 *into* C. *Moreover, if* E *is bounded,* $\{G_\alpha^\bullet(x, B), \alpha > 0\}$ *is a Markov resolvent. The harmonic part* $R_\alpha^\bullet(x, B)$ *has the symmetric density* $r_\alpha^\bullet(x, y)$ *which is* α*-harmonic in each variable.*

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