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GENUS AND CLASSIFICATION OF RIEMANN SURFACES*)

By

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Introduction

1. Consider two properties for Riemann surfaces R:

(1) *there exists no harmonic Green's function on* R:

(2) *there exists no non-constant harmonic function on* R *with finite Dirichlet integral taken over* R.

It is well known that (1) implies (2) but (2) does not imply (1). On the other hand, for finite Riemann surfaces**), the conditions (1) and (2) are equivalent. Hence the Riemann surfaces satisfying (2) but not (1) must be of infinite genus. In this aspect, there naturally arises the question that, under what condition on genus, Riemann surfaces with the property (2) satisfy automatically the condition (1).

The main purpose of this paper is to give a quantative condition on the distribution of genus which assures the implication from (2) to (1). The condition to be given is satisfied for finite Riemann surfaces. So our result which will be stated below may be regarded as an extension of the fact that (1) and (2) are equivalent for finite Riemann surfaces.

2. Before stating our main result, we need some preliminary definitions. Let R be a Riemann surface. We denote by $[C_1, C_2]$ a pair of mutually disjoint simple closed curves C_1 and C_2 on R satisfying the following two conditions:

(3) C_1 *and* C_2 *are dividing cycles of* R, *i.e. the open set* $R-C_i$ *consists of two components* $(i=1, 2)$;

(4) *the union of* C_1 *and* C_2 *is the boundary of a relatively compact domain* (C_1, C_2) *of* R *such that* (C_1, C_2) *is of genus one.*

We say that two such pairs $[C_1, C_2]$ and $[C_1', C_2']$ are equivalent if there exists such a third pair $[C_1'', C_2'']$ that

$$(C_1, C_2)\frown(C_1', C_2')\supset(C_1'', C_2'').$$

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**) We shall say that R is a finite Riemann surface if R is of finite genus.

This relation is actually an equivalence relation and so the totality of such pairs $[C_1, C_2]$ is divided into equivalence classes. We call each equivalence class H a *handle* of R. Clearly the totality of handles of R is at most countably infinite.

Let G be a subdomain of R and H be a handle of G. Then by an obvious identification, H may be considered to be a handle of R.

An annulus A in R is said to be associated with a handle H of R, $A \in H$ in notation, if there exists a representative $[C_1, C_2]$ of H with $\bar{A} \subset (C_1, C_2)$ satisfying

(5) *each component of the relative boundary of* A *is not a dividing cycle of domain* (C_1, C_2);

(6) *each boundary component of the relative boundary of* A *is not homotopic to any component of an arbitrary level curve of the harmonic function in* (C_1, C_2) *with boundary value* 1 *on* C_1 *and* 2 *on* C_2.

Roughly speaking, conditions (5) and (6) may be summerized as follows: each boundary component of the relative boundary of A rounds the "hole" of (C_1, C_2).

Now consider a Riemann surface R in which there exists a sequence (A_n) of annuli in R satisfying the following conditions:

(7) $A_n \in H_n$, *where* (H_n) *is the totality of handles in* R:

(8) $A_n \cap A_m = \emptyset$ *(empty set) if* $n \neq m$;

(9) $\sum_n 1/\mathrm{mod}\, A_n < \infty$,

where mod A *is the modulus of the annulus* A. For convinience, we shall say that such an R is *almost finite Riemann surface* or that R is *of almost finite genus.* Then our result to be proved is stated as follows:

Theorem 1. *For almost finite Riemann surfaces* R, *the following four conditions are mutually equivalent*:

(a) *there exists no harmonic Green's function on* R;

(b) *there exists no non-constant positive harmonic function on* R;

(c) *there exists no non-constant bounded harmonic function on* R;

(d) *there exists no non-constant harmonic function witn finite Dirichlet integral on* R.

3. Finite Riemann surfaces are clearly of almost finite genus. In order to show that our theorem is not a formal extension of that for finite Riemann surfaces, we must show the existence of an almost finite Riemann surface R with (1) (or without (1)) which is not of finite genus. For the aim, consider the Riemann sphere S; $|z| \leq \infty$. Let (a_n) and (b_n) be two sequences defined by

$$a_n = 3n - 2 + 2\exp(-n^2)$$

and

$$b_n = 3n+1,$$

where $n=1, 2, \cdots$. Let S_0 be a subdomain of S obtained from S by cutting along all intervals $[a_n, b_n]$. Patch two such copies crosswise along $[a_n, b_n]$, $n=1, 2, \cdots$. Then we get a two sheeted covering surface R of S. Clearly R thus obtained is of infinite genus and each interval $[b_n, a_{n+1}]$ corresponds to a handle H_n in one to one and onto manner by an obvious correspondence. Let A_n be the annulus in R which is two sheeted covering surface of the annulus

$$B_n = (z \text{ in } S;\ \exp(-(n+1)^2) < |z-a_{n+1}| < 1)$$

in S. Then clearly $A_n \in H_n$ and $A_n \cap A_m = \emptyset$ $(n \neq m)$. Moreover

$$\operatorname{mod} A_n = \operatorname{mod} B_n/2 = n^2/2.$$

So the sequence (A_n) satisfies the conditions (7), (8) and (9). Thns we have seen that R is an almost finite surface. Evidently the harmonic measure of the ideal boundary of R vanishes and so R satisfies the condition (1). If we remove the compact set with positive capacity from R, then we get non-trivial almost finite surface which does not satisfy (1).

Here we remark the following. Let $a'_n=2n$ and $b'_n=2n+1$ $(n=0, 1, 2, \cdots)$ and construct the two sheeted covering surface R' of S by the similar manner as above. Clearly R and R' are homeomorphic but R' is not of almost finite genus. Hence the almost finite .property is not topologically invaliant. But clearly this notion is quasiconformally invariant.

4. For the proof of our theorem, we use the theory of Royden's compactification ([5]). In Chapter I, we discuss the Royden's compactification of Riemann surfaces with finite genus. In Chapter II, the Royden's compactification of subdomains will be discussed. In Chapter III we shall prove the following theorem which contains the essential part of the proof of Theorem 1:

Theorem 2. *Any point in the Royden's boundary of an almost finite Riemann surface possesses the canonical measure zero.*

This theorem is equivalent to the following assertion: any almost finite Riemann surface does not belong to the Constantinescu-Cornea's class U_{HD} ([2]). In appendix, we shall prove the following Lusin-Privaloff type theorem:

Theorem 3. *Let E be a subset of Royden's boundary of a Riemann* *surface with canonical measure positive and f be a meromorphic function defined on a subdomain whose closure in Royden's compactification is a neighborhood of E. Suppose that f has continuous boundaary value zero at each point of E. Then f vanishes identically.*

I. Royden's compactification of finite Riemann surfaces.

5. Let R be a Riemann surface. We denote its Royden's algebra, Royden's compactification, Royden's boundary, harmonic boundary and canonical measure by $M(R)$, R^*, Γ, Δ and μ respectively ([5])*. For simplicity, we suppose that μ is defined for Borel subsets of Γ by defining the measure of $\Gamma-\Delta$ is zero.

If R is a finite Riemann surface, there exists a compact Riemann surface $\tilde{R}$ such that R is a subdomain of $\tilde{R}$. Let $\bar{R}$ be the closure of R in $\tilde{R}$ and $\gamma=\bar{R}-R$. We shall study the relation between R^* and $\bar{R}$.

Proposition 1. *There exists a unique continuous mapping π of R^* onto $\bar{R}$ fixing R elementwise and such that $\pi^{-1}(R)=R$.*

Proof. The unicity of such a π is obvious. Hence we have only to show the existence. Let A be the totality of functions in $M(R)$ which are considered to be continuous functions on $\bar{R}$. Then A contains sufficiently many functions on $\bar{R}$, since the restriction of a function in $C^\infty(\tilde{R})$ is contained in A. Let S be the space of all characters on A, where a character q on A means an algebraic homomorphism $f\to f(q)$ of A onto the complex number field. The topology in S is defined by the weak* topology, i.e. a directed net (q_λ) in S converges to q in S if and only if $(f(q_\lambda))$ converges to $f(q)$ for any f in A. Then S is a compact Hausdorff space containing R as its open and dense subset (cf. Lemma I. 1, P. 162 in [4]).

First we show that $S=\bar{R}$. It is clear that $\bar{R}\subset S$. Take an arbitrary q in S and set

$$A_q=(f \text{ in } A\,;\ f(q)=0)\,.$$

Then for some z_0 in R, $f(z_0)=0$ for all f in A_q. If this is not so, then we can find an f_z in A_q such that $f_z\geq 0$ on $\bar{R}$ and $f_z(z)=1$ for each z in $\bar{R}$, since A_q is an ideal in A. Using the compactness of $\bar{R}$ and the

*) $M(R)$ is the totality of bounded a.c.T functions on R with finite Dirichlet integral. R^* is the smallest compact Hausdorff space containing R as its open and dense subspace such that any function in $M(R)$ is continuously extended to R^*. $M_\Delta(R)$ is the BD-closure of $M_0(R)$, the totality of functions in $M(R)$ with compact support and $\Gamma=R^*-R$ and $\Delta=\{p\in R\,;\ f(p)=0$ for any f in $M_\Delta(R)\}$. μ is nothing but the harmonic measure.

$\pi^{-1}(\zeta)\cap\Delta\neq\emptyset$ if ζ is in γ_1.

Next suppose that ζ is in $\bar{\gamma}_1-\gamma_1$. We can find a sequence (ζ_n) in γ_1 converging to ζ and $\zeta_n\neq\zeta_m$ $(n\neq m)$. Choose a point p_n in $\Delta\cap\pi^{-1}(\zeta_n)$, which is non-void as we saw above. By the compactness of Δ, there exists a point p in Δ such that p is an accumulation point of the set (p_n). Then by the continuity of π, $\pi(p)$ is an accumulation points of the set (ζ_n). Hence from the nature of (ζ_n), $\pi(p)=\zeta$. Thus $\pi^{-1}(\zeta)\cap\Delta\neq\emptyset$.

Conversely, assume that $\pi^{-1}(\zeta)$ contains a harmonic boundary point p. Assume that ζ is in $\gamma-\bar{\gamma}_1$. As Δ is non-void, so R is hyperbolic and hence there exists a point ζ_1 in γ_1. From above, the set $\pi^{-1}(\zeta_1)$ contains a point p_1 in Δ. Let V be a neighborhood of ζ in $\tilde{R}$ such that $\bar{V}\cap\bar{\gamma}_1=\emptyset$. Then the set $U=\pi^{-1}(V)$ is a neighborhood of p in R^* and U does not contain p_1. Find a function f on Δ such that $0\leq f\leq 1$ on Δ and f vanishes identically in U and $f(p_1)=1$. Then by using notations in [5], the function

$$u(z)=\int_{\Delta}K(z,q)f(q)\,d\mu(q)$$

is a harmonic function on R and continuous on R^* with $0<u(z)<1$ on R. As V contains no point in γ_1, so u is continued harmonically to V and so there exists a positive constant d such that $u(z)>d$ on V. Then by the definition of π, $u\geq d$ on $U\cap R$ and so $u\geq d>0$ on U. Since $u=f$ on $\Delta\cap U$, this is clearly a contradiction. Thus we have proved that ζ is in $\bar{\gamma}_1$ if $\pi^{-1}(\zeta)$ contains a harmonic boundary point.

6. For a moment, R is assumed to be an arbitrary open Riemann surface. Let $f(p)$ be a real valued bounded function defined on Γ. Consider the totality $U^f_{R^*}$ of continuous superharmonic functions $u(z)$ defined on R such that for any point p in Γ

$$\underline{\lim}_{R\ni z\to p}u(z)\geq f(p)$$

in R^*. We define two functions $\bar{H}^f_{R^*}$ and $\underline{H}^f_{R^*}$ by

$$\bar{H}^f_{R^*}(z)=\inf\,(u(z)\,;\,u\in U^f_{R^*})\quad\text{and}\quad \underline{H}^f_{R^*}(z)=-\bar{H}^{-f}_{R^*}(z)$$

respectively. These two functions are harmonic on R and $\bar{H}^f_{R^*}\geq\underline{H}^f_{R^*}$ on R, which are proved by the usual manner. If they are identical on R, then we denote by $H^f_{R^*}$ the common function and f is said to be *resoltive* (with respect to Royden's compactification). A point p in Γ is said to be a *regular point for Dirichlet problem* (with respect to Royden's compacification) if there exists at least one non-constant resoltive function on Γ and for any resoltive function f continuous at p $\lim_{R\ni z\to p}H^f_{R^*}(z)=f(p)$ in R^*.

$$f(p)=\begin{cases} g(p)\sin(\log(\log|z(p)|^{-1}) & \text{in } U; \\ 0 & \text{in } \tilde{R}-U \end{cases}$$

satisfies the above condition. Thus each fiber $\pi^{-1}(\zeta)$, $\zeta\in\gamma$, contains point set whose cardinal number is at least the cardinal number of continuum.

Proposition 3. *For any ζ in γ, the fiber $\pi^{-1}(\zeta)$ always contains a non-harmonic boundary point i.e. $\pi^{-1}(\zeta)\cap(\Gamma-\Delta)\neq\emptyset$.*

Proof. Let (U_n) be a neighborhood system of ζ in $\tilde{R}$ such that $U_n\supset\bar{U}_{n+1}$. Set $D_n=U_n\cap R$. Then by the definition of π, $\pi^{-1}(\zeta)=\bigcap_n\bar{D}_n$, where $\bar{D}_n$ is the closure of D_n in R^*. Hence by Theorem 3 in [6], $\pi^{-1}(\zeta)\cap(\Gamma-\Delta)\neq\emptyset$. Q.E.D.

The boundary γ is divided into two parts γ_0 and γ_1, where γ_1 (resp. γ_0) is the totality of regular (resp. not regular) points for Dirichlet problem with respect to the domain R considered in $\tilde{R}$. We denote by $\bar{\gamma}_1$ the closure of γ_1 in $\tilde{R}$. Then we have

Proposition 4. *The fiber $\pi^{-1}(\zeta)$ contains a harmonic boundary point if and only if the point ζ is contained in $\bar{\gamma}_1$.*

Proof. First suppose that ζ is contained in γ_1. Then there exists a bounded harmonic function $h(z)$ on R such that $h>0$ on R and

$$\lim_{R\ni z\to\zeta}h(z)=0$$

in $\bar{R}$ and for any η in γ with $\eta\neq\zeta$,

$$\lim_{R\ni z\to\eta}h(z)>0$$

in R. Let $E=\pi(\Delta)$, which is compact in γ, since π is continuous and Δ is compact in R^*. We have to show that E contains ζ. Contrry to our assertion, assume that E does not contain ζ. For p in Δ, put

$$\underline{h}(p)=\sup_{(U)}\inf_{U\cap R}h(z),$$

where (U) is a neighborhood system of p in R^*. Then $\underline{h}$ is lower semi-continuous on Δ. By the defintion of π, it is clear that

$$\underline{h}(p)\geq\lim_{R\ni z\to\pi(p)}h(z)>0\,.$$

Since $\pi(p)$ is in $E\subset\gamma-(\zeta)$, the last inequality of the above is assured. Hence there exissts a positive constant d such that

$$\underline{h}(p)>d$$

on Δ. Hence by the maximum principle (Theorem 1.2, P. 190 in [5]), $h(z)>d$ on R. This contradicts the definition of h. Thus we have proved

Considering $f-f(p)$ instead of f, we may assume $f(p)=0$. Contrary to the assertion, assume that 0 is not in $C_R(f, \zeta)$. Then we can find a neighborhood U of ζ in $\bar{R}$ such that

$$|f(z)|>d>0$$

on $U\cap R$. Clearly we can find a function g in $M(R)$ such that

$$g(z) = 1/f(z)$$

on $U\cap R$ and so $g(z)f(z)=1$ there. By the continuity of π, $\pi^{-1}(U\cap\bar{R})$ is a neighborhood of p in R^*. As R is dense in $\bar{R}$ and R^* respectively, so $\pi^{-1}(U\cap\bar{R})\cap R=\pi^{-1}(U\cap R)=U\cap R$ is dense in $\pi^{-1}(U\cap\bar{R})\cap R^*=\pi^{-1}(U\cap\bar{R})$. Hence $f(z)g(z)=1$ on $U\cap R$ implies $f(q)g(q)=1$ on $\pi^{-1}(U\cap\bar{R})$. In particular, $f(p)g(p)=1$. This is clearly a contradiction, since $f(p)=0$.

Conversely assume that a is in $C_R(f, \zeta)$. We have to show the existence of a point p in $\pi^{-1}(\zeta)$ such that $f(p)=a$. To this end, we may assume $a=0$ by considering $f-a$ instead of f. Assume that f does not vanish on $\pi^{-1}(\zeta)$. As $\pi^{-1}(\zeta)$ is compact and f is continuons on this set, there exists a positive number d such that

$$|f(q)|>d$$

on $\pi^{-1}(\zeta)$. On the other hand, as 0 is in $C_R(f, \zeta)$, so we can find a sequence (z_n) in R such that $\lim_n z_n=\zeta$ in $\bar{R}$ and

$$|f(z_n)|<d\,.$$

Let r be an accumulation point of the set (z_n) in R^*. Clearly r is in Γ. Let (z_λ) be a directed net converging to r in R^* whose terms are choosen from the set (z_n). Then by the continuity of the projection π and the fact that r is in Γ and $\pi^{-1}(R)=R$, (z_λ) must converge to ζ in $\bar{R}$ and $\pi(r)=\zeta$. As $|f(z_\lambda)|<d$, so

$$|f(r)| = \lim_\lambda |f(z_\lambda)| \leq d\,.$$

This shows that r is not in $\pi^{-1}(\zeta)$. This is a contradion, since $\pi(r)=\zeta$.

Q.E.D.

It is clear that there exists a function f in $M(R)$ such that the interior cluster set of f at ζ in γ is the closed interval $[-1, 1]$. For example, let U and V be coordinate neighborhoods $(|z|<1/2)$ and $(|z|<1/4)$ in R respectively and g be in $C^\infty(\tilde{R})$ whose carrier is contained in U and $g(z)=1$ on V. Tnen the restriction to R of the function $f(p)$ defined by continuity of each f_z on $\bar{R}$, we can find a system of points $z_1, z_2, \cdots, z_n$ in $\bar{R}$ such that

$$g(z) = \sum_{k=1}^{n} f_{z_k}(z) > 1/2$$

on $\bar{R}$. As g is in A_q and $1/g$ is in A, so the constant function $1=(1/g)g$ is contained in A_q, which is absurd. So we have proved the existence of a point z_0 in $\bar{R}$ with the property mentioned before. As $f-f(q)$ belongs to A_q, so we get

$$f(q) = f(z_0)$$

for any function f in A. This proves that $S=\bar{R}$.

Take a point p in R^*. Then $f \to f(p)$ defines a character $\pi(p)$ on A. This gives rise to a mapping of R^* into $S=\bar{R}$. Moreover π is onto. In fact, for any point z_0 in $\bar{R}$, consider the set

$$M_{z_0} = (f \text{ in } M(R)\,;\ \lim_{R \ni z \to z_0} f(z) = 0 \text{ in } R)\,.$$

This is a proper ideal of $M(R)$. Since $M(R)$ is normed so as to be a Banach algebra, there exists a character p on $M(R)$ vanishing on M_{z_0} by Mazur-Gelfand's theorem that a normed field is the complex number field. This p can be considered to be a point in R^* (cf. Lemma I. 2, P. 163 in [4]). If f belongs to A, $f-f(z_0)$ is contained in M_{z_0} and so

$$f(p) = f(z_0)$$

for any f in A. This shows that $\pi(p)=z_0$ or π is onto. Again by

$$f(p) = f(\pi(p))$$

for any f in A and for any p in R^*, we can conclude π is a continuous mapping of R^* onto $\bar{R}$ fixing R elementwise and $\pi^{-1}(R)=R$. Q.E.D.

We shall quote π as *projection* of R^* onto $\bar{R}$. We also call the set $\pi^{-1}(z)$ the fiber in R^* over a point z in $\bar{R}$. The fiber $\pi^{-1}(z)$ is one point (z) if z is in R but $\pi^{-1}(z)$ contains infinite points if z is in $\gamma=\bar{R}-R$. This is shown by using the following

Proposition 2. *For any function f in $M(R)$ and ζ in γ.*

$$(f(p)\,;\ p \text{ is in } \pi^{-1}(\zeta)) = C_R(f, \zeta)\,,$$

where the right hand side of the above is the interior cluster set of f at ζ, i.e. the totality of a such that there exists a sequence (z_n) in R with $\lim_n z_n=\zeta$ and $\lim_n f(z_n)=a$.

Proof. First we show that $f(p)$ is in $C_R(f, \zeta)$ for any p in $\pi^{-1}(\zeta)$.

Lemma 1. *If R is not of null boundary (i.e. R does not satisfy the condition (1)), then any bounded Borel function f defined on Γ is resoltive and*

$$H_{R^*}^{f}(z) = \int_{\Gamma} K(z, q) f(q) d\mu(q)$$

on R and the totality of regular points in Γ coincides with the harmonic boundary Δ.[1]

Proof. Let R do not satisfy (1). Then by Royden's theorem, $\Delta \neq \emptyset$ (cf. Lemma 1.4, P. 185 in [5]). We first prove that any continuous function f on Γ is resoltive and $H_{R^*}^{f}(z) = v(z)$, where $v(z) = \int_{\Gamma} K(z, q) f(q) d\mu(q)$. We know that $v(z)$ is harmonic on R and continuous on R^* and $v(p) = f(p)$ on Δ (Theorem 2.2 and 2.3 in [5]).

Given an arbitrary positive number t, we can find a compact set K in $\Gamma - \Delta$ such that

$$\min_{\Delta} f(p) - t < f(q),\ v(q) < \max_{\Delta} f(p) + t$$

for any point q in $\Gamma - K$, since f and v are continuous on Γ and $f = v$ on Δ. Let W be an open neighborhood of K in R^* such that $\bar{W} \cap \Delta = \emptyset$ and the relative boundary of $R \cap W$ consists of a piecewise analytic Jordan curves which do not accumulate in R.

Let (R_n) be a normal exhaustion of R. Let the sequence (e_n) of functions e_n on R^* be defined as follows. First choose a real-valued continuous function b on R^* such that $b = -1$ on Δ and $b = 2$ on $\bar{W} - R_n$. As $M(R)$ is uniformly dense in the totality of continuous functions on R^* (cf. P. 185 in [5]), so there exists a real function c in $M(R)$ such that $|b(p) - c(p)| < 1/2$. As the totality of real function in $M(R)$ forms a vector lattice (Lemma 1.7, P. 187 in [5]), so the function

$$d(p) = \max(\min(1, c(p)), 0)$$

is in $M(R)$ and $d(p) = 0$ on Δ and $d(p) = 1$ on $\bar{W} - R_n$. Let $(d_m(p))_{m>n}$ be defined as follows. $d_m(p) = d(p)$ on $(\bar{W} - R_n) \cup (R^* - W - \bar{R}_m)$ and d_m be the solution of Dirichlet problem in $R_m - (\bar{W} - R_n)$ with boundary value $d(p)$. Then by Dirichlet principle and the maximum principle,

$$0 \leq d_m(p) \leq d(p)$$

on R^* and

$$D(d_m) \leq D(d).$$

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1) If R is of null boundary, then there exists no non-constant positive superharmonic function on R (Ohtsuka's theorem). From this, it follows that any point in Γ is not regular.

Hence by choosing a suitable subsequence of (d_m), we may assume that (d_m) converges to a function e_n in BD-topology and since $d-d_m$ is contained in $M_0(R)$, $d-e_n$ is in $M_\Delta(R)$, which shows

$$e_n(p) = d(p) = 0$$

on Δ and

$$e_n(p) = d(p) = 1$$

on $W-R_n$ (cf. Chapter I in [5]). Then e_n is a superharmonic function on R. It is clear that (e_n) forms a decreasing sequence and so its limiting function e is bounded and harmonic on R and e is non-negative and vanishes continuously at Δ. So by the maximum principle (Theorem 1. 2, P. 190 in [5]), e is identically zero on R, i.e.

$$\lim_n e_n(x) = 0$$

on R. Choose an arbitrary function $u(z)$ in $U_{R^*}^f$. Let $a=\sup_\Gamma f(p)$ and $a'=\sup_\Gamma v(p)$. Then for any p in Γ, we have

$$f(p)-t \leq v(p)+ae_n(p)$$

and

$$v(p)-t \leq f(p)+a'e_n(p) \leq \lim_{R\ni z\to p}(u(z)+a'e_n(z)) .$$

From the first inequality, we have

$$\bar{H}_{R^*}^f(z) \leq v(z)+ae_n(z)+t$$

on R and from the second, by the usual minimum principle, $v(z)-t$ is less than $u(z)+a'e_n(z)$ or

$$v(z)-t \leq \bar{H}_{R^*}^f(z)+a'e_n(z) .$$

Hence we have

$$|\bar{H}_{R^*}^f(z)-v(z)| < (a+a')e_n(z)+t$$

on R. First making n tend to infinity and then making t tend to zero, we finally get

$$\bar{H}_{R^*}^f(z) = \int_\Gamma K(z, q)f(q)\,d\mu(q)$$

on R. On the oter hand,

$$\underline{H}_{R^*}^f(z) = -\bar{H}_{R^*}^{-f}(z) = -\int_\Gamma K(z, q)(-f(q))\,d\mu(q) = \bar{H}_{R^*}^f(z) .$$

Hence f is resoltive and $H_{R^*}^f(z)=\int_\Gamma K(z, q)f(q)\,d\mu(q)$ for any continuous real function on Γ. From this, by the usual manner, we can show the validity of the same fact for any bounded Borel function f on Γ.

Let p be in Γ and f be a resoltive function continuous at p. We show that $H_{R^*}^f(z)$ tends to $f(p)$ as z in R tends to p in R^*. For the aim, we may assume $f(p)=0$. We can find open neighborhoods U and V of p in R^* such that U contains the closure of V and

$$|f(q)| < 1/n$$

on U. Set $m=\sup_\Gamma |f(q)|$. Let g be a continuous function on Γ such that $m \geq g \geq 1/n$ on Γ and $g=1/n$ on V and $g=m$ on $\Gamma - U$. Then clearly $g \geq f \geq -g$ on Γ and so

$$H_{R^*}^g \geq H_{R^*}^f \geq -H_{R^*}^g .$$

On the other hand,

$$H_{R^*}^g(z) = \int_\Gamma K(z, q) g(q) d\mu(q)$$

and so

$$\lim_{R \ni x \to p} H_{R^*}^g(z) = g(p) = 1/n .$$

Thus we have

$$1/n \geq \overline{\lim}_{R \ni z \to p} H_{R^*}^f(z) \geq \underline{\lim}_{R \ni z \to p} H_{R^*}^f(z) \geq -1/n .$$

As n is arbitrary, so we get $\lim_{R \ni z \to p} H_{R^*}^f(z) = 0 = f(p)$. Hence p is a regular point.

Let p be in $\Gamma - \Delta$. There exists a continuous real function f on Γ such that $f=0$ on Δ and $f(p)=1$. Then

$$H_{R^*}^f(z) = \int_\Gamma K(z, q) f(q) d\mu(q) = 0$$

on R. This shows that p is not regular. Q.E.D.

7. Again we suppose that R is a finite Riemann surface embedded in a compact surface $\tilde{R}$. Consider a bounded real function f defined on $\gamma = \tilde{R} - R$. We denote by U_R^f the totality of continuous superharmonic functions u such that for any ζ in γ, $\underline{\lim}_{R \ni z \to \zeta} u(z) \geq f(\zeta)$ in $\bar{R}$. We also denote by $\bar{H}_R^f(z) = \inf (u(z) ; u \in U_R^f)$ and $\underline{H}_R^f = -\bar{H}_R^{-f}$. These are harmonic and $\bar{H}_R^f \geq \underline{H}_R^f$ on R. If they are identical, we denote the common function by H_R^f and f is said to be resoltive in the usual sense. It is well known that any Borel function on γ is resoltive. As before, we denote by γ_1 the totality of regular points for Dirichlet problem on γ in the usual sense.

Let f be a continuous real function defined on γ which contains a regular point. Then $f\circ\pi$ is a continuous function on Γ and $\Delta \neq \emptyset$ from Proposition 4. If v is in U_R^f, then by the continuity of π,

$$\underline{\lim}_{R\ni z\to p} v(z) \text{ (in } R^*) \geq \underline{\lim}_{R\ni z\to \pi(p)} v(z) \text{ (in } \bar{R})$$
$$\geq f(\pi(p)) = (f\circ\pi)(p)$$

for any point p in Γ. Hence v is in $U_{R^*}^{f\circ\pi}$, i.e.

$$U_{R^*}^{f\circ\pi} \supset U_R^f .$$

From this we get

$$\bar{H}_{R^*}^{f\circ\pi} \leq \bar{H}_R^f .$$

As this is true for $-f$, so $\bar{H}_{R^*}^{-f\circ\pi} \leq \bar{H}_R^{-f}$ or $-\bar{H}_{R^*}^{-f\circ\pi} \geq -\bar{H}_R^{-f}$. Hence

$$\underline{H}_{R^*}^{f\circ\pi} \leq \underline{H}_R^f .$$

By the fact that f and $f\circ\pi$ are resoltive in the usual sense and in the sense of Royden's compactification, we can conclude that

$$H_{R^*}^{f\circ\pi}(z) = H_R^f(z) \tag{10}$$

holds on R for any continuous function f on γ. From this fact, we get the following proposition which plays one of the central rôle in this paper.

Proposition 5. *Let E be a compact set in γ. Then the canonical measure of $\pi^{-1}(E)$ is zero if and only if the relative harmonic measure of E with respect to R is zero.*

Proof. Let (U_n) be a decreasing sequence of open sets in γ containing E such that $\bigcap_n U_n = E$. Then $(\pi^{-1}(U_n))$ is a decreasing sequence of open sets in Γ containing $\pi^{-1}(E)$ suce that $\bigcap_n \pi^{-1}(U_n) = \pi^{-1}(E)$.

Let ω and μ be the relative harmonic measure on γ with respect to z in R and the canonical measure on Γ with respect to z in R respectively. First assume that γ contains a regular point. Let f_n be a continuous function on γ such that $0 \leq f_n \leq 1$ on γ and $f_n = 1$ on U_{n+1} and $f_n = 0$ on γ outside U_n. Then clearly

$$\omega(U_n) \geq H_R^{f_n}(z) \geq \omega(U_{n+1}) \quad \text{and}$$
$$\mu(\pi^{-1}(U_n)) \geq H_{R^*}^{f_n\circ\pi}(z) \geq \mu(\pi^{-1}(U_{n+1}))$$

and from this with (10), $\omega(U_n) \geq \mu(\pi^{-1}(U_{n+1}))$ and $\mu(\pi^{-1}(U_n)) \geq \omega(U_{n+1})$. By the monotone continuity of ω and μ, we have

$$\omega(E) = \mu(\pi^{-1}(E)) .$$

From this our assertion follows. If γ contains no regular point, then $\Delta=\emptyset$ (Proposition 4) and our assertion is evident. Q.E.D.

II. Royden's compactification of subdomains.

7. Let G be a non-compact subdomain of an arbitrary open Riemann surface R. We denote by $B=B_G$ the set $\bar{G}\cap\Gamma-\overline{\partial G}$ in R^*, where ∂G is the relative boundary of G with respect to R.

Proposition 6. *The set $G\cup B_G$ is an open set in R^*.*

Proof. We have to show that for any point p in $G\cup B$, we can find an open set U in R^* such that $p\in U\subset G\cup B$. This is trivial for p in G. Hence we suppose that p is contained in B. There exists a continuous real function $a(q)$ on R^* such that $a(p)=2$ and $a(q)=-1$ on V, where V is an open neighborhood of $\overline{\partial G}$ in R^* such that p is not in V. This is possible, since (p) and $\overline{\partial G}$ are disjoint compact set in R^*. As $M(R)$ is uniformly dense in the totality of continuous functions on R^*, so we can find a real function $b(q)$ in $M(R)$ such that $|b(q)-a(q)|<1/2$ on R^*. Since $M(R)$ forms a vector lattice, the function $c(q)$ defined by

$$c(q) = \max(\min(1, b(q)), 0)$$

is in $M(R)$ and $c(p)=1$ and $c(q)=0$ on V. As the point p is an accumulation point of G, we can find a directed net (p_λ) in G such that $\lim p_\lambda=p$ and so

$$\lim_\lambda c(p_\lambda) = c(p) = 1 .$$

Now define a function $d(z)$ on R by

$$d(z) = \begin{cases} c(z), & \text{on } G; \\ 0, & \text{on } R-G. \end{cases}$$

As $c(z)$ venishes on a neighborhood of ∂G, so $d(z)$ is a bounded a.c.T function. Moreover $D_R(d)=D_G(c)\leq D_R(c)<\infty$, which shows that d is in $M(R)$ and so it is extended continuously to R^*. Consider the set

$$U = (q\in R^*;\ d(q)>0) .$$

This is clearly an open set in R^*. Let r be a pointin $R^*-G\cup B$. If r belongs to R, then r is in $R-G$ and $d(r)=0$. If r is in Γ, then r is in $\overline{R-G}$ or in $\bar{G}$. In the former case, there exists a directed net (r_λ) in $R-G$ with $\lim_\lambda r_\lambda=r$. Then $d(r)=\lim_\lambda d(r_\lambda)=0$. In the latter case, since r is not in B, r belongs to $\overline{\partial G}$. Hence there exists a directed net (q_λ)

in ∂G with $\lim_\lambda q_\lambda = r$ and so $d(r) = \lim_\lambda d(q_\lambda) = 0$. Thus $d(r) = 0$ for any r in $R^* - G \cup B$. This shows that

$$U \subset G \cup B.$$

Moreover, as (p_λ) is in G, so we have

$$d(p) = \lim_\lambda d(p_\lambda) = \lim_\lambda c(p_\lambda) = 1,$$

which shows that p belongs to U. Q.E,D.

Now we shall investigate the relation between $\bar{G}$ and G^* which is the Royden's compactification of G. Corresponding to Proposition 1, we first prove

Proposition 7. *There exists a unique continous mapping ρ of G^* onto $\bar{G}$ fixing G elementwise such that $\rho^{-1}(G) = G$ and ρ is a homeomorphism between $G \cup \rho^{-1}(B_G)$ and $G \cup B_G$.*

Proof. The unicity of such a ρ is obvious. To show the existence of such a ρ, consider the totality A of functions in $M(G)$ which are continuous on $\bar{G}$. Then A separates points in $\bar{G}$, since the restriction of functions to $\bar{G}$ in $M(R)$ belong to A. As in the proof of Proposition 1, we can show that the totality of characters on A with the weak* topology coincides with $\bar{G}$. A point p in G^* can be considered to be a character on $M(G)$ and its restriction $\rho(p)$ on A is a character on A and so $\rho(p)$ is a point in $\bar{G}$. By the similar manner as in the proof of Proposition 1, we can prove that ρ is a continuous mapping of G^* onto $\bar{G}$ fixing G elementwise and $\rho^{-1}(G) = G$.

Next we shall prove that ρ is univalent on $\rho^{-1}(B)$. For the aim, we have to show that $\rho(p) = \rho(p')$ implies $p = p'$ for p and p' in the set $\rho^{-1}(B)$. As $q = \rho(p) = \rho(p')$ belongs to B, so we can find open neighborhoods U and V of q and $\overline{\partial G}$ respectively such that $\bar{U} \cap \bar{V} = \emptyset$. There exists a continuous real function $a(s)$ on R^* such that $a(q) = 2$ on U and $a(s) = -1$ on V. As $M(R)$ is uniformly dense in the totality of continuous functions on R^*, so we can find a real function $b(s)$ in $M(R)$ such that $|a(s) - b(s)| < 1/2$ on R^*. Since $M(R)$ forms a vector lattice, the function $c(s) = \max(\min(1, b(s)), 0)$ belongs to $M(R)$ and $c(s) = 1$ on U and $c(s) = 0$ on V.

Let f be an arbitrary function in $M(G)$. Then the function $g(z)$ defined on R by

$$g(z) = \begin{cases} f(z)c(z), & \text{on } G; \\ c(z), & \text{on } R - G \end{cases}$$

belongs to $M(R)$. In fact, $c(z)$ vanishes on the open set $V \cap R$ containing

∂G and so g is bounded a.c.T function on R and

$$D_R(g) \leq (\sup_R |c(z)|) D_G(f) + D_R(c) < \infty .$$

Hence the restriction of g on $\bar{G}$ is continuous on $\bar{G}$ and so belongs to A. Thus $g(p)=g(\rho(p))$ and $g(p')=g(\rho(p))$ and so

$$g(p) = g(p') .$$

On the other hand, it is clear that $g(r)=f(r)c(r)$ on G^* and since c is in A, $c(p)=c(\rho(p))=c(q)=1$ and $c(p')=c(\rho(p'))=c(q)=1$. Hence $g(p)=f(p)$ and $g(p')=f(p')$. Thus

$$f(p) = f(p')$$

for all f in $M(G)$. This shows that $p=p'$.

Finally we show that ρ is a homeomorphism between $G \cup \rho^{-1}(B)$ and $G \cup B$. For this aim, it suffices to show that $\lim_\lambda \rho^{-1}(p_\lambda)=\rho^{-1}(q)$ if (p_λ) is a directed net in $G \cup B$ converging to a point q in B in $\bar{G}$. For this q, let U, V and c be defined by the same manner as above. Let f be an arbitrary function in $M(G)$. We have to prove that

$$(*) \qquad \lim_\lambda f(\rho^{-1}(p_\lambda)) = f(\rho^{-1}(q)) .$$

For this f, define g as above. Since g and c are in A, $\lim_\lambda p_\lambda = q$ implies

$$(**) \qquad \lim_\lambda g(p_\lambda) = g(q)$$

and there exists a λ_0 such that $\lambda_0 \leq \lambda$ implies $p_\lambda \in U$ and so $c(p_\lambda)=c(\rho^{-1}(p_\lambda)) =1 (\lambda \geq \lambda_0)$ and $c(q)=c(\rho^{-1}(q))=1$. On the other hand, since g is in A, $g(\rho^{-1}(p_\lambda))=g(p_\lambda)$ and $g(\rho^{-1}(q))=g(q)$. As $g(\rho^{-1}(p_\lambda)))=c(\rho^{-1}(p_\lambda))f(\rho^{-1}(p_\lambda)) =f(\rho^{-1}(p_\lambda))$ and similarly $g(\rho^{-1}(q))=f(\rho^{-1}(q))$ for $\lambda \geq \lambda_0$, so we get $(*)$ from $(**)$. Q.E.D.

Next suppose, for simplicity, that ∂G consists of at most a countable number of disjoint piecewise analytic curves with no end point in R and not accumulating in R. We denote $\Gamma_G = G^* - G$ and by μ_G the canonical measure on Γ_G. Corresponding to Proposition 5, we prove

Proposition 8. *Let E be a compact set E in B_G. The canonical measure $\mu(E)$ of E is positive if and only if the canonical measure $\mu_G(\rho^{-1}(E))$ of $\rho^{-1}(E)$ is positive.*

Proof. Since E and $\overline{\partial G}$ are disjoint compact sets in $\bar{G}-G$, both $\rho^{-1}(E)$ and $\rho^{-1}(\overline{\partial G})$ are disjoint compact sets in Γ_G. As μ and μ_G are regular measures, so we can find, using Proposition 6, sequences (U_n) and (U'_n) of open subsets U_n in R^* and U'_n in G^* such that $G \cup B \supset U_n \supset \bar{U}_{n+1}$

$\supset E$ and $G \cup \rho^{-1}(B) \supset U'_n \supset \bar{U}'_{n+1} \supset \rho^{-1}(E)$ with

$$\mu(E) = \lim_n \mu(U_n \cap \Gamma) \text{ and } \mu_G(\rho^{-1}(E)) = \lim_n \mu_G(U'_n \cap \Gamma_G)$$

respectively. Since ρ is homeomorphic on $G \cup \rho^{-1}(B)$, the set $V_n = U_n \cap \rho(U'_n)$ is an open set in R^* such that $G \cup B \supset V_n \supset \bar{V}_{n+1} \supset E$ and $\rho^{-1}(V_n)$ is an open set in G^* such that $G \cup \rho^{-1}(B) \supset \rho^{-1}(V_n) \supset \overline{\rho^{-1}(V_{n+1})} \supset \rho^{-1}(E)$ with the property

$$\mu(E) = \lim_n \mu(V_n \cap \Gamma) \quad \text{and} \quad \mu_G(\rho^{-1}(E)) = \lim_n \mu_G(\rho^{-1}(V_n) \cap \Gamma_G)$$

respectively. Since $M(R)$ is a vector lattice and uniformly dense in the totality of continuous functions on R^*, we can find a sequence (f_n) of real functions f_n in $M(R)$ such that $0 \leq f_n \leq 1$ and $f_n = 1$ on V_{n+1} and $f_n = 0$ outside V_n in R^*. Moreover we can choose (f_n) so as to satisfy

$$f_n \geq f_{n+1}$$

on R^*. Then f_n vanishes on ∂G and by the property of ρ, we can consider that f_n is in $M(G)$ such that $f_n = 1$ on $\rho^{-1}(V_{n+1})$ and $f_n = 0$ outside $\rho^{-1}(V_n)$ in G^*. Let $u_n(z)$ and $v_n(z)$ be defined by

$$u_n(z) = \int_\Gamma K(z, p) f_n(p) \, d\mu(p)$$

and

$$v_n(z) = \int_{\Gamma_G} K_G(z, p) f_n(p) \, d\mu_G(p)$$

on R and G respectively, where $K_G(z, p)$ is the harmonic kernel belonging to μ_G (cf. P. 149 in [5]).

Let (R_m) be a normal exhaustion of R. Let $v_{n,m}$ be a continuous function on R^* defined by

$$v_{n,m} = \begin{cases} \text{harmonic}, & \text{on } R_m \cap G; \\ f_n, & \text{on } R^* - R_m \cap G. \end{cases}$$

Using Dirichlet principle and the maximum principle, we may assume, by choosing a suitable subsequence, that $(v_{n,m})$ converges in BD-topology to a function v'_n on R and of course on G. By the property of ρ, the function $v_{n,m} - f_n$ vanishes on Γ_G and so $v_{n,m} - f_n$ belongs to $M_\Delta(G)$ and hence $v'_n - f_n$ is in $M_\Delta(G)$ (cf. PP. 187-190 in [5]). From this $v'_n = f_n = v_n$ on Δ_G and so $v'_n = v_n$ on G. Moreover $v_{n,m} - f_n$ belongs to $M_0(R)$ and so $v'_n - f_n = v_n - f_n$ belongs to $M_\Delta(R)$, where v_n is extended to R by $v_n = 0$ on $R - G$. Hence

$$v_n(p) = f_n(p)$$

on Δ_R. Let $u_{n,m}$ be a continuous function on R^* defined by

$$u_{n,m} = \begin{cases} \text{harmonic}, & \text{on } R_m; \\ v_n, & \text{on } R^* - R_m. \end{cases}$$

By the same manner as above, we can prove that $(u_{n,m})$ may be considered to converge to u_n. By the construction of (f_n) and by the maximum principle, we get

$$u_n \geq u_{n+1} \text{ on } R,\ v_n \geq v_{n+1} \text{ on } G \text{ and } u_n \geq v_n \text{ on } G.$$

If the center of μ and μ_G are z_0 in G, then $u_n(z_0) \geq v_n(z_0)$ implies $\mu(V_n) \geq \mu_G(\rho^{-1}(V_{n+1}))$ and by regularity

$$\mu(E) \geq \mu_G(\rho^{-1}(E)).$$

Hence we have proved that $\mu_G(\rho^{-1}(E)) > 0$ implies $\mu(E) > 0$.

Next assume that $\mu(E) > 0$. Coutrary to our assertion, assume that $\mu_G(\rho^{-1}(E)) = 0$. Then $\lim_n v_n(z_0) = \mu_G(\rho^{-1}(E)) = 0$. As the continuous function on R^* which is harmonic in R_m and equals to $v_1 - v_n$ outside R_m in R^* is just $u_{1,m} - u_{n,m}$ and $u_{1,m} - u_{n,m} \geq v_1 - v_n$, so by the maxlmum principle, we see that

$$u_{1,m} - u_{n,m} \leq u_{1,m+1} - u_{n,m+1}$$

and $\lim_m (u_1, m - u_{n,m}) = u_1 - u_n$. Hence $u_1 - u_n \geq u_{1,m} - u_{n,m}$. As (v_n) converges to 0 uniformly on ∂R_m, so $\lim_n u_{n,m} = 0$ uniformly on R_m. Here notice that $u_{n,m} = v_n$ on ∂R_m. From this

$$\lim_n (u_1 - u_n) \geq u_{1,m}$$

on R_m. Thus by making m tend to infinity, we get

$$\lim_n (u_1 - u_n) \geq u_1.$$

Since $u_n \geq 0$, we finally get

$$\lim_n u_n = 0.$$

In particular, $\lim_n u_n(z_0) = 0$ together with $u_n(z_0) \geq \mu(V_{n+1}) \geq \mu(E)$ implies $\mu(E) = 0$, which is a contradiction. Thus $\mu(E) > 0$ implies $\mu_G(\rho^{-1}(E)) > 0$.

Q.E.D.

III. Proofs of Theorems 1 and 2.

8. Let R be an arbitrary open Riemann surface and p be a point in R^*. We say that U is a *normal neighborhood* of p in R^* if U is an open neighborhood of p in R^* such that $R \cap U$ is a subdomain of R

whose relative boundary consists of at most a countable number of analytic Jordan curves with no end point in R and not accumulating in R.

Proposition 9. *Let p_0 be a point in Γ with positive canonical measure and U be an arbitrary neighborhood of p_0 in R^*. Then there exists a normal neighborhood V of p_0 such that V is contained in U.*[*)]

Proof. Choose open neighborhoods U_1, U_2, U_3 and U_4 of p_0 in R^* such that $U_i \supset \bar{U}_{i+1}$ $(i=0,1,2,3)$, where $U_0=U$. Let $((T_m^{(n)})_{m=1}^{\infty})_{n=1}^{\infty})$ be the family of triangulation of R such that $(T_m^{(n)})$ is the barycentric subdivision of $(T_m^{(n+1)})$. Let $(T_{m_k}^{(n)})$ be the greatest subfamily of $(T_m^{(n)})$ such that $\bar{T}_{m_k}^{(n)} \subset R \cap U_1$ and $\bar{T}_{m_k}^{(n)} \cap (R \cap U_2) \neq \emptyset$. Then clearly the set

$$W_1' = \overline{\bigcup_n (\bigcup_k T_{m_k}^{(n)})}$$

is contained in U and contains $\bar{U}_2$. Then the set

$$W_1 = W_1' - \overline{\partial W_1'}$$

is an open neighborhood of p_0 (Proposition 6) such that $\bar{U}_2 \subset W_1 \subset \bar{W}_1 \subset U$ and the relative boundary ∂W_1 of W_1 consists of regular points for Dirichlet problem. Similar construction for the pair U_3 and U_4 gives an open neighborhood W_2 of p_0 such that $\bar{U}_4 \subset W_2 \subset \bar{W}_2 \subset U_2$ and every point in ∂W_2 is regular for Dirichlet problem.

Let (R_n) be a normal exhaustion of R. We define the harmonic function $w_n(z)$ on $R_n \cap (W_1 - \bar{W}_2)$ with boundary value $\varphi(z)$ on $\partial(R_n \cap (W_1 - \bar{W}_2))$, where

$$\varphi(z) = \begin{cases} 0, & \text{on } \partial(R_n \cap (W_1 - \bar{W}_2)) - \partial W_2 ; \\ 1, & \text{elsewhere on } \partial(R_n \cap (W_1 - \bar{W}_2)) . \end{cases}$$

Then (w_n) forms a non-decreasing sequence and there exists the limit function $w(z)$ on $R \cap (W_1 - \bar{W}_2)$ of (w_n). Clearly w is harmonic on $R \cap (W_1 - \bar{W}_2)$ and $0 \leq w \leq 1$ and $w=0$ on $\partial(W_1 - \bar{W}_2) - \partial W_2$ and $w=1$ elsewhere on $\partial(W_1 - \bar{W}_2)$. We set $w=1$ on $\bar{W}_2 \cap R$. Then w is continuous on $\bar{W}_1 \cap R$. Let t be in the open interval $(0,1)$ such that the level curve $(z; w(z)=t)$ contains no multiple point. Put

$$W' = \overline{(z \in R \cap \bar{W}_1 ;\ w(z) > t)} .$$

Then the set

$$W = W' - \overline{\partial W'}$$

*) If p_0 is of canonical measure zero in Γ, then this assertion does not hold in general.

is an open neighborhood (Proposition 6) of p_0 in R^* contained in U with its closure and the relative boundary ∂W of W consists of at most a countable number of analytic Jordan curves with no end point in R and not accumulating in R.

Let V_n be an open neighborhood of p_0 in R^* such that $W \supset \bar{V}_n \supset \bar{V}_{n+1}$ and $\lim_n \mu(V_n \cap \Gamma) = \mu(p_0)$. Since $M(R)$ is a vector lattice and uniformly dense in the totality of continuous functions on R^*, we can find a real function f_n in $M(R)$ such that $0 \leq f_n \leq 1$ and $f_n = 1$ on V_{n+1} and $f_n = 0$ on $R^* - V_n$. Moreover we can assume that

$$f_n(p) \geq f_{n+1}(p)$$

on R^*. Let (R_m) be a normal exhaustion of R and the continuous function $v_{n,m}$ be defined on R^* by

$$v_{n,m}(p) = \begin{cases} \text{harmonic}, & \text{on } R_m \cap W; \\ f_n(p), & \text{on } R^* - R_m \cap W. \end{cases}$$

By the maximum principle and Dirichlet principle, we see that $D(v_{n,m}) \leq D(f_n)$ and $0 \leq v_{n,m} \leq \sup_R f_n$ on R^*. Hence by choosing a suitable subsequence, we may assume that the sequence $(v_{n,m})$ converges in BD-topology to a function v_n on R. As $v_{n,m} - f_n$ belongs to $M_0(R)$, so $v_n - f_n$ is in $M_\Delta(R)$ and so

$$v_n(p) = f_n(p)$$

on Δ. Moreover v_n is harmonic on $R \cap W$ and vanishes on $R - R \cap W$. By the maximum principle (Lemma 2.1, P. 201 in [5]), $v_n \geq v_{n+1}$ on R^*. Next define the continuous function $u_{n,m}$ on R^* by

$$u_{n,m}(q) = \begin{cases} \text{harmonic}, & \text{on } R_m; \\ v_n(q), & \text{on } R^* - R_m. \end{cases}$$

By the same way as above, we may assume that the sequence $(u_{n,m})$ converges in BD-topology to a harmonic function u_n in $M(R)$ such that $u_n - v_n$ belongs to $M_\Delta(R)$. Hence

$$u_n(p) = v_n(p) = f_n(p)$$

on Δ. Again by the maximum principle (Lemma 2.1, ibid), $u_n \geq v_n$ and $u_n \geq u_{n+1}$. As (u_n) ahd (v_n) form decreasing sequences, so there exist a harmonic function $u(z)$ on R and a continuous function $v(z)$ on R such that $u = \lim_n u_n$ and $v = \lim_n v_n$ on R respectively. The function v is harmonic on $R \cap W$ and vanishes on $R - R \cap W$. We assert that

$$v(z) > 0$$

on at least one component of $R \cap W$. Contrary to our assertion, assume that $v(z)=0$ on R. Then $\lim_n v_n(z)=0$ on R. As the continuous function on R^* which is harmonic in R_m and equals $v_1 - v_n$ on $R^* - R_m$ is just $u_{1,m} - u_{n,m}$ and $u_{1,m} - u_{n,m} \geq v_1 - v_n$, we see that

$$u_{1,m} - u_{n,m} \leq u_{1,m+1} - u_{n,m+1}$$

by the usual maximum principle and $\lim_m (u_{1,m} - u_{n,m}) = u_1 - u_n$. Hence

$$u_{1,m} - u_{n,m} \leq u_1 - u_n .$$

As (v_n) converges to zero uniformly on ∂R_m and $u_{n,m} = v_n$ on ∂R_m, so $\lim_n u_{n,m} = 0$ uniformly on R_m. From this

$$\lim_n (u_1 - u_n) \geq u_{1,m}$$

on R_m. Thus by making n tend to infinity, we get

$$\lim_n (u_1 - u_n) \geq u_1$$

on R. Since $u_n \geq 0$ on R, we see that

$$u(z) = \lim_n u_n(z) = 0$$

on R. As $u_n = f_n$ on Δ, so we have $u_n(z) = \int_\Gamma K(z, q) f_n(q)\, d\mu(q)$ on R. Hence if the center of μ is z_0 in R, we get

$$\mu(p_0) \leq \mu(V_{n+1}) \leq u_n(z_0) .$$

But this is impossible since $(u_n(z_0))$ converges to zero and $\mu(p_0) > 0$. Hence there exists a component V' of $W \cap R$ such that $v(z) > 0$ on V'. Then the required V is obtained by choosing

$$V = \bar{V}' - \overline{\partial V'} .$$

In fact, V is an open set in R^* (Proposition 6) such that ∂V consists of at most a countable number of analytic Jordan curves with no end point in R and not accumulating in R. To conclude the proof, we have to show that p_0 is contained in V. Let $f(q) = \lim_n f_n(q)$ on Γ. Then clearly

$$u(z) = \int_\Gamma K(z, q) f(q)\, d\mu(q)$$

on R. Since $(q \in \Gamma ; f(q) \neq 0) - (p_0)$ has canonical measure zero, we may rewrite the above expression as

$$u(z) = \int_{(p_0)} K(z, q)\, d\mu(q) .$$

This have continuous boundary value zero at any point in $\Delta-(p_0)$ (Theorem 2.3, P. 199 in [5]). As $u(z)\geq v(z)\geq 0$ on R, the same is true for $v(z)$. Now assume that p_0 is not in V. Then for any q in the set $\partial V\cup(\bar{V}\cap\Delta)$, we have

$$\lim_{R\ni z\to q} v(z) = 0 .$$

Hence by the maximum principle (Lemma 2.1, ibid), we have $v(z)=0$ on V. This is a contradiction and so p_0 belongs to $\bar{V}$. If p_0 is in $\overline{\partial V}$, then p_0 belongs to $\overline{\partial W}=\overline{\partial W'}$. This shows that p_0 is not in W. This contradiction shows that p_0 belongs to $V=\bar{V}-\overline{\partial V}$. Q.E.D.

9. Proof of Theorem 2. Let R be an almost finite Riemann surface. Contrary to our assertion, assume that Royden's boundary Γ of R contains a point p_0 with positive canonical measure, i.e. $\mu(p_0)>0$. From this we shall derive a contradiction. If R is of finite genus, then R is embedded in a compact surface $\tilde{R}$. Hence there exists a projection π of $R^*=R\cup\Gamma$ onto $R\cup\gamma$ in the sence of Proposition 1. Set $\zeta_0=\pi(p_0)$. Then ζ_0 belongs to γ and clearly its relative harmonic measure with respect to R considered in $\tilde{R}$ is zero. Thus by Proposition 5, $\mu(\pi^{-1}(\zeta_0))=0$. Since the fiber $\pi^{-1}(\zeta_0)$ contains the point p_0, $\mu(\pi^{-1}(\zeta_0))\geq\mu(p_0)>0$. This is a contradiction. Thus we have only to consider the case where R is not of finite genus.

Let $(H_n)_{n=1}^{\infty}$ be the totality of handles in R. By the definition that R is of almost infinite genus, there exists a sequence (A_n) of annuli A_n in R with conditions (7), (8) and (9). We divide A_n into two annuli $A_{n,1}$ and $A_{n,2}$ by a closed analytic Jordan curve j_n in A_n such that

$$\text{mod}\, A_{n,1} = \text{mod}\, A_{n,2} = \text{mod}\, A_n/2 .$$

Define the continuous function $w_n(p)$ on R^* by

$$w_n(p) = \begin{cases} \text{harmonic}, & \text{on } A_n-j_n ; \\ 1, & \text{on } j_n ; \\ 0, & \text{on } R^*-A_n . \end{cases}$$

We also define $g_m(p)$ and $g(p)$ by

$$g_m(p) = \textstyle\sum_{n=1}^{m} w_n(p)$$

and

$$g(p) = \textstyle\sum_{n=1}^{\infty} w_n(p)$$

on R^* respectively. Then it is clear that g_m is in $M_0(R)$ and $|g_m|\leq 1$ on R and $D_R(g-g_m) = \sum_{n=m+1}^{\infty} D_R(w_n) = \sum_{n=m+1}^{\infty} (D_{A_{n,1}}(w_n)+D_{A_{n,2}}(w_n)) =$

$2\pi \sum_{n=m+1}^{\infty} (1/\text{mod } A_{n,1} + 1/\text{mod } A_{n,2}) = 4\pi \sum_{n=m+1}^{\infty} 1/\text{mod } A_n$. Hence by the condition (9), we have

$$\lim_m D_R(g-g_m) = 0 .$$

This shows that (g_m) converges in BD-topology to g and so g belongs to $M_\Delta(R)$. Thus g vanishes on the harmonic boundary Δ of R and, in particular, $g(p_0)=0$, since $\mu(\Gamma-\Delta)=0$ and $\mu(p_0)>0$. By the continuity of g on R^*,

$$U = (p \in R^* ;\ g(p) < 1/2)$$

is an open neighborhood of p_0 in R^*. By Proposition 9, we can find a normal neighborhood V of p_0 in R^* contained in U. Moreover we may assume that $\bar{V} \subset U$. By the definition of U, $\bar{V}$ contains no point in the set $\bigcup_n j_n$.

Next we shall prove that $V \cap R$ is a planer Riemann surface, i.e. $V \cap R$ possesses no handle. In fact, if there exists a handle H' in $V \cap R$, then we can find a pair $[C_1, C_2]$ of closed Jordan curves C_1 and C_2 with two properties (3) and (4) with respect to R. If we consider the pair $[C_1, C_2]$ in R, then it belongs to a handle in R, say H_n. Hence there exists a pair $[C_1', C_2']$ such that A_n is contained in the domain (C_1', C_2') and $[C_1, C_2]$ is equivalent to $[C_1', C_2']$. From this, j_n must meet (C_1, C_2), since j_n is homotopic to each component of ∂A_m in R. This shows that the function g takes the value 1 on (C_1, C_2) and so on $V \cap R$. But this cannot occur, since $V \cap R \subset U$, on which $g < 1/2$.

Thus $V \cap R$ is conformally equivalent to a plane domain G. Let $\tilde{G}$ be the Riemann sphere and $\bar{G}$ be the closure of G in $\tilde{G}$ and $\gamma = \bar{G} - G$. Then there exists a projection π of G^* onto $\bar{G}$ in the sense of Proposition 1. As any point ζ in γ is of relative harmonic measure zero with respect to G considereed in $\tilde{G}$, so the canonical measure of $\pi^{-1}(\zeta)$ is zero with respect to G^*. Then G carries no $\underline{HD}$-minimal function (Theorem 3.6, P. 216 in [5]) and since this property is clearly conformally invariant, $V \cap R$ carries no $\underline{HD}$-minimal function. Thus any point in the Royden's boundary $\Gamma_{V \cap R} = (V \cap R)^* - (V \cap R)$ of $V \cap R$ is of canonical measure zero (Theorem 3.6, ibid).

On the other hand, there exists a projection ρ of $(V \cap R)^*$ onto $\overline{V \cap R}$ in R^* in the sense of Proposition 7. Clearly p_0 is contained in $B_{V \cap R} = \overline{V \cap R} - \overline{\partial(V \cap R)}$ in R^* and since $q_0 = \rho^{-1}(p_0)$ is one point in $(V \cap R)^*$ (Proposition 7), $\mu_{V \cap R}(q_0) > 0$ follows from $\mu_R(p_0) > 0$ by using Proposition 8, i.e. $\Gamma_{V \cap R}$ possesses a point with positive canonical measure. This contradicts the above fact that $\mu_{V \cap R}(p) = 0$ for all p in $\Gamma_{V \cap R}$. Q.E.D.

Constantinescu-Cornea's class U_{HD} of open Riemann surface R is defined by the property that $R \notin O_G$ and R carries an HD- or $\underline{HD}$-minimal function. Since any HD- or $\underline{HD}$-minimal function $u(z)$ on R is of the form

$$u(z) = c \int_{(p)} K(z, q)\, d\mu(q)\,,$$

where c is a positive constant and p is a point in Γ with positive canonical measure (Theorem 3. 6, P. 126 in [5]), the class U_{HD} consists of all open Riemann surfaces R whose Royden's ideal boundary contains at least one point with positive canonical measure. Hence Theorem 2 may be restated as follows:

Theorem 2′. *Any almost finite Riemann surface does not belong to the Constantinescu-Cornea's class* U_{HD}.

10. Proof of Theorem 1. Since the general implication scheme $(a)\rightarrow(b)\rightarrow(c)\rightarrow(d)$ is well known, we have to show that if R is an almost finite surface belonging to O_{HD} (i.e. R satisfies (d)), then R belongs to O_G. Assume that R belongs to $O_{HD}-G_G$. Then by Royden's theorem (c.f. Lemma 1. 4, P. 185 in [5]), Δ consists of only one point and since $\mu(\Delta)=1$, Γ possesses a point with positive canonical measure. This contradicts the assertion of Theorem 2. Q.E.D.

11. Finally we give a remark on the bahaviour of quasiconformal mapping on the Royden's boundary of a Riemann surface. Let T be a quasiconformal mapping of a Riemann surface R_1 onto another surface R_2. This T can be extended so as to be a topological mapping of R_1^* onto R_2^* such that $T(\Delta_1)=\Delta_2$ (Theorem 5, P. 218 in [3]). Concerning this, there naturally arises a question whether T is absolutely continuous on Δ_1 with respect to canonical measures or not. If this is affirmative, then we can conclude that U_{HD}-property is quasiconformally invariant. But the former question is negatively answered. This follows at once from an example of Beurling-Ahlfors.

Let $R_1=R_2=(z\,;\,|z|<1)$. Beurling and Ahlfors gave an example of quasiconformal mapping T of R_1 onto R_2 and a compact set E_1 in $\gamma_1=\bar{R}_1-R_1=(z\,;\,|z|=1)$ with positive linear measure such that $E_2=T(E_1)$ is of linear measure zero (cf. [1]). Here notice that any quasiconformal mapping of R_1 onto R_2 can be extended so as to be a topological mapping of $\bar{R}_1$ onto $\bar{R}_2$.

Let π_i be the projection of R_i^* onto $\bar{R}_i$ in the sense of Proposition 1. We set

$$E_i^* = \pi_i^{-1}(E_i)\,.$$

This set is compact in Γ_i and by Propostion 5,

$$\mu_1(E_1^*) > 0 \quad \text{and} \quad \mu_2(E_2^*) = 0 .$$

Consider T as the topological mapping of R_1^* onto R_2^* (resp. $\bar{R}_1$ onto $\bar{R}_2$) and denote it by T^* (resp. $\bar{T}$). Then for any point p in R_1^*

$$\bar{T}(\pi_1(p)) = \pi_2(T^*(p)) .$$

In fact, this is true for any p in R_1. Let p be in Γ_1. We can find a directed net (p_λ) in R_1 snch that $\lim_\lambda p_\lambda = p$. Then by the continuity of π_1 and T^*, $\lim_\lambda \pi_1(p) = \pi_1(p)$ and $\lim_\lambda T^*(p_\lambda) = T^*(p)$ in $\bar{R}_1$ and R_2^* respectively. Hence $\bar{T}(\pi_1(p_\lambda)) = \pi_2(T^*(p_\lambda))$ implies the desired conclusion. From this we see that

$$T^*(E_1^*) = E_2^* .$$

Thus T^* carries a set with positive canonical measure onto a set with canonical measure zero.

Appendix

12. Proof of Theorem 3. Let G be a subdomain of R whose closure $\bar{G}$ in R^* is a neighborhood of E and $f(z)$ be a meromorphic function in G possessing continuous boundary value zero at each point of E in R^*. We have to show that $f(z) \equiv 0$ on G.

First we show that we can reduce the proof to the case where $G = R$. To show this, we first remark that we can assume E is a compact subset of Δ and the relative boundary ∂G of G consists of at most a countable number of piecewise analytic Jordan curves without end point in R and not accumulating in R. In fact, since μ is a regular Borel measure on Γ with $\mu(\Gamma - \Delta) = 0$, we may clearly assume that E is compact and contained in Δ. To verify the second assertion, we choose an open set U in R^* such that $E \subset U \subset \bar{U} \subset$ (the open kernel of $\bar{G}$ in R^*). Since $R \cap U$ is an open set in R, we can decompose $R \cap U$ into at most a countable number of connected components $U_k : R \cap U = \bigcup_{k=1}^N U_k$ $(N \leq \infty)$. We can choose points z_k in U_k and arcs a_k connecting z_0 and z_k in G such that $(a_k)_k$ does not accumulate in ∂G. We then set $U' = (\bigcup_{k=1}^N a_k) \cup U$. Let $(T_n^{(m)})_n$ be triangulations of R whose each triangle have piecewise analytic contour such that $(T_n^{(m+1)})_n$ is the barycentric subdivision of $(T_n^{(m)})_n$. Consider the totality $(T_k)_k$ of triangles T_k in $(T_n^{(m+1)})_{n,m}$ such that $\bar{T}_k \subset G$ and $\bar{T}_k \cap U' \neq \emptyset$. Then the set $G' =$ (the open kernel in R of $\overline{\bigcup_k T_k}$) is a subdomain of G with piecewise analytic boundary curves not ending and not accumulating in R and $\bar{G}' \supset \overline{R \cap U} = \bar{U} \supset E$ shows that $\bar{G}'$ is a neighborhood of E in R^*. Hence we have only to replace G by G'.

Let ρ be the projection of G^* (Royden's compactification of G) onto $\bar{G}$ (the closure of G in R^*) in the sense of Proposition 7 and $E^*=\rho^{-1}(E)$. Since E is contained in $B_G=\bar{G}\cap\Gamma-\overline{\partial G}$ and $\mu(E)>0$, we can conclude that $\mu_G(E^*)>0$ by Proposition 8. By the continuity of ρ, f is a meromorphic function on G possessing continuous boundary value zero at each point of E^* in G^*. Hence we can reduce the proof of Theorem 3 to the case where $G=R$.

Contrary to our assertion, assume that $f(z)\not\equiv 0$ on R. Let

$$F=(z\in R\,;\,|f(z)|<1)\,.$$

Then F is an open set in R and $\bar{F}$ is a neighborhood of E in R^*, since f has continuous boundary value zero at each point of E. Let

$$F=\bigcup_{k=1}^{N}F_k \qquad (N\leq\infty)$$

be the decomposition of F into connected components F_k and set

$$E_k=E\cap\bar{F}_k\,.$$

Let ρ_k be the projection of F_k^* (Royden's compactification of F_k) onto $\bar{F}_k$ (the closure of F_k in R^*) in the sense of Proposition 7 and set $E_k^*=\rho_k^{-1}(E_k)$, First we assert that

$$(*) \qquad \mu_{F_k}(E_k^*)>0 \quad \text{for at least one } k\,.$$

If this is not the case, $\mu_{F_k}(E_k^*)=0$ for all k. Since E_k is contained in $B_{F_k}=\Gamma\cap\bar{F}_k-\overline{\partial F_k}$, we conclude that $\mu(E_k)=0$ for all k by Proposition 8. In this case we must have $N=\infty$. In fact, if $N<\infty$, then from $\bar{F}=\overline{\bigcup_{k=1}^{N}F_k}=\bigcup_{k=1}^{N}\bar{F}_k$, we have $E=\bigcup_{k=1}^{N}E_k$ and so we get the following contradiction: $0<\mu(E)\leq\sum_k\mu(E_k)=0$.

Since μ is a regular Borel measure, we can find an open set U in R^* such that $E\subset U\subset\bar{U}\subset$ (the open kernel of $\bar{F}$) and

$$\mu(U\cap\Gamma-E)<\mu(E)/2\,.$$

As $M(R)$ is dense in the totality of continuous functions on R^* in the sense of uniform convergence and $M(R)$ forms a vector lattice, so we can find a function f_∞ in $M(R)$ with $0\leq f_\infty\leq 1$ on R such that

$$f_\infty=\begin{cases}1\,, & \text{on } E\,;\\ 0\,, & \text{on } R^*-U\,.\end{cases}$$

For n, $1\leq n<\infty$, we define functions f_n on R by

$$f_n=\begin{cases}f_\infty\,, & \text{on } \bigcup_{k=1}^{n}F_k\,;\\ 0\,, & \text{elsewhere on } R\,.\end{cases}$$

Then clearly f_n is bounded and a.c.T. on R and since

$$D_R(f_\infty) = \sum_{k=1}^{\infty} D_{F_k}(f_\infty) < \infty ,$$

we get

$$D_R(f_n - f_\infty) = \sum_{k=n+1}^{\infty} D_{F_k}(f_\infty) \to 0$$

as n tends infinity and, in particular, f_n is in $M(R)$. Noticing the relation $(\overline{\bigcup_{k=1}^{n} F_k}) = \bigcup_{k=1}^{n} \bar{F}_k$ and that f_n is continuous on R^*, we see that

$$f_n = \begin{cases} 1, & \text{on} \quad \bigcup_{k=1}^{n} E_k ; \\ 0, & \text{on} \quad (R^* - U) \cup (U - \bigcup_{k=1}^{n} \bar{F}_k) . \end{cases}$$

By the Royden's decomposition (c.f. Theorem 1. 1 (harmonic decomposition), P. 188 in [5]) and the definition of μ (c.f. P. 194 in [5]),

$$f_n = u_n + g_n \qquad (1 \leq n \leq \infty) .$$

where g_n vanishes on Δ and

$$u_n = \int_{\Gamma} K(z, p) f_n(p) d\mu(p)$$

and

$$D(u_n) \leq D(f_n) \qquad (1 \leq n \leq \infty) .$$

From the integral representation of u_n, we see that the sequence $(u_n)_n$ is monotone non-decreasing and dominated by u_∞. Hence there exists a harmonic function u on R such that $u = \lim_n u_n$ on R and $u \leq u_\infty$. As the harmonic decomposition of $f_n - f_\infty$ is as follows:

$$f_n - f_\infty = (u_n - u_\infty) + (g_n - g_\infty) ,$$

so we have

$$D(u_n - u_\infty) \leq D(f_n - f_\infty) .$$

By Fatou's lemma,

$$D(u - u_\infty) \leq \underline{\lim}_n D(u_n - u_\infty) \leq \lim_n D(f_n - f_\infty) = 0 .$$

Thus $u - u_\infty = c$ is a non-negative constant. Choose a point p_0 in E_1 which is contained in Δ. Then $u_n(p_0) = u_n(p_0) + g_n(p_0) = f_n(p_0) = 1$ $(1 \leq n \leq \infty)$. Hence $\lim_{R \ni p \to p_0} (u_n(p) - u_\infty(p)) = 0$. Combining this wish $0 \leq c = u_\infty - u \leq u_\infty - u_n$ on R, we get $c = 0$ or $u = u_\infty$. Thus

$$u_\infty = \lim_n u_n$$

on R. Let z_0 be the center of μ. Then for $n < \infty$,

$$\begin{aligned} u_n(z_0) &= \int_\Gamma K(z_0, p) f_n(p)\, d\mu(p) \\ &= \int_H K(z_0, p)\, d\mu(p) \qquad (H = (\textstyle\bigcup_{k=1}^n \bar{F}_k) \cap \Gamma \cap U) \\ &= \mu(H) = \mu(H \cap E) + \mu(H - (H \cap E)) \\ &= \mu((\textstyle\bigcup_{k=1}^n \bar{F}_k) \cap \Gamma \cap E) + \mu((\textstyle\bigcup_{k=1}^n \bar{F}_k) \cap (\Gamma \cap U - E)) \\ &\leq \textstyle\sum_{k=1}^n \mu(E_k) + \mu(\Gamma \cap U - E) \leq \mu(E)/2 . \end{aligned}$$

Thus $u_\infty(z_0) \leq \mu(E)/2$. But this is impossible, since

$$u_\infty(z_0) = \int_\Gamma K(z_0, p) f_\infty(p)\, d\mu(p) \geq \int_E K(z_0, p)\, d\mu(p) = \mu(E) .$$

Thus we have proved (*).

Now we close our proof by showing the following:

$$(\ast\!\!\ast) \text{ for each } k , \qquad \mu_{F_k}(E_k^*) = 0 .$$

If we can show this, then the impossibility of validity of both of (*) and ($\ast\!\ast$) implies that our assumption $f(z) \not\equiv 0$ on R is false and we have $f(z) \equiv 0$ on R. To show ($\ast\!\ast$), contrary to the assertion, assume that $\mu_{F_k}(E_k^*) > 0$ for some k. Let z_0^* be in F_k such that $f(z_0^*) \neq 0$. We may assume that the center of μ_{F_k} is z_0^* (c.f. Corollary to Theorem 2.1, P. 196 in [5]). As $f(z)$ has continuous boundary value zero at each point of E_k^*, so we can find an open set V_n in F_k^* such that $V_n \supset E_k^*$ and

$$|f(z)| < e^{-n} \qquad \text{on} \quad V_n \cap F_k .$$

We can find a coutinuous function k_n on F_k^* such that $0 \leq k_n \leq 1$ on F_k^* and

$$k_n = \begin{cases} 1 , & \text{on} \quad E_k^* \\ 0 , & \text{on} \quad F_k^* - V_n . \end{cases}$$

Let $w_n(z) = \int_{\Gamma_{F_k}} K_{F_k}(z, p) k_n(p)\, d\mu_{F_k}(p)$. Then w_n is harmonic on F_k and continuous on F_k^* and $0 \leq w_n \leq 1$ on F_k^* and $w_n = 0$ on $\Delta_{F_k} - V_n$. Let

$$w(z) = -\log |f(z)| .$$

Then $w(z)$ is positive superharmonic on F_k and $w(z) \geq n$ on $V_n \cap F_k$. From these, we conclude that

$$w(z)/n \geq w_n(z)$$

on F_k. In fact, if this is not so, then there exists a negative number $s < 0$ such that a component Z of $(z \in F_k ; w(z)/n - w_n(z) < s)$ is a nonempty Jordan subdomain of F_k. Clearly $s+w_n(z)-w(z)/n$ is a non-constant HB-function on Z vanishing on the relative boundary ∂Z of Z with respect to F_k, or $Z \notin SO_{HB}$. On the other hand, $\bar{Z} \cap \Delta_{F_k}=\emptyset$ shows that $Z \in SO_{HB}$ (c.f. Lemma 2. 2, P. 202 in [5]). This is a contradiction. Thus

$$w(z_0^*)/n \geq w_n(z_0^*) = \int_{\Gamma_{F_k}} K_{F_k}(z_0^*, p)\, k_n(p)\, d\mu_{F_k}(p)$$
$$\geq \int_{E_k^*} K_{F_k}(z_0^*, p)\, d\mu_{F_k}(p) = \mu_{F_k}(E_k^*)\,.$$

Since $\mu_{F_k}(E_k^*)>0$, this is a contradiction and so we get (⁑). Q.E.D.

From this Lusin-Privaloff type theorem, we can conclude at once that if $R \in U_{HD}$, then there exists no non-constant meromorphic function on R continuous on R^* (or continuous near $\underline{HD}$-minimal point), in particular, $U_{HD} \subset O_{AD}$ (Constantinescu-Cornea's generalization [2] of Kuramochi's theorem).

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