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ON THE ABSTRACT EVOLUTION EQUATION

Dedicated to Professor K. Shoda on his 60th birthday

By

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§ **0. Introduction.** The present paper is concerned with the abstract evolution equation

$$du/dt+A(t)u = f(t), \qquad 0\leq t\leq T, \tag{0.1}$$

in a Banach space X. $u=u(t)$ and $f(t)$ are functions on $[0, T]$ to X and $A(t)$ is a function on $[0, T]$ to the set of unbounded operators acting in X.

We have already published a number of papers on the integration of this equation based on the theory of semi-groups of operators; in particular the reader is referred to Kato [3] for a survey of recent results, including those obtained by other authors. In most (but not all) of these papers of ours, $-A(t)$ are assumed to be infinitesimal generators of *analytic semi-groups* $\exp(-sA(t))$ of bounded linear operators on X; this is equivalent to assuming that the resolvent $(\lambda I+A(t))^{-1}$ of $-A(t)$ covers a closed sector of the form $|\arg\lambda|\leq\frac{\pi}{2}+\theta$, $\theta>0$, and satisfies the inequality

$$\|(\lambda I+A(t))^{-1}\|\leq M/|\lambda| . \tag{0.2}$$

Regarding the dependence of $A(t)$ on t, it has so far been necessary to assume that the domain $D(A(t))$ of $A(t)$ or at least the domain $D(A(t)^h)$ of a certain fractional power $A(t)^h$ of $A(t)$ is independent of t, with other anxiliary assumptions such as the Hölder continuity of $A(t)^hA(0)^{-h}$ (see [3]).

The main object of the present article is to eliminate such an assumption on the domain of $A(t)$ or of $A(t)^h$. We shall prove the existence and the uniqueness of the solution of (0.1) or what comes essentially to the same thing, of the *evolution operator* $U(t, s)$ associated with (0.1); in addition to the condition that $-A(t)$ be the infinitesimal generator of an analytic semi-group, our principal assumption will be that an inequality of the form

$$\left\| \frac{d}{dt}(\lambda I + A(t))^{-1} \right\| \leq N/|\lambda|^{1-\rho} \tag{0. 3}$$

is valid with a constant ρ such that $0 \leq \rho < 1$. Of course (0. 3) implies that $A(t)^{-1}$ be differentiable in t, but it does *not* imply that $D(A(t)^h)$ be constant for any $h > 0$. In this respect (0. 3) is weaker than the assumptions used in our previous papers and is believed to be an essential improvement.

The condition (0. 3) is not very easy to verify in a given problem. We have given a criterion for the validity of (0. 3) (Theorem 2. 1). Also we have a rather general case in which (0. 3) is satisfied (see § 7) ; it is interesting to note that this case is a generalization of a case dealt with in detail by Lions in his recent book [6].

Actually we find it difficult to construct a *strict solution* of (0. 1) under the assumptions stated above alone : we had to assume further the Hölder continuity in norm of the derivative $dA(t)^{-1}/dt$. It must be admitted that this is a rather strong assumption. It is possible, however, to construct a solution (and the associated evolution opepator) which satisfies (0. 1) in a weak sense and yet is determined by the initial value $u(0)$, without assuming this Hölder continuity of $dA(t)^{-1}/dt$.

§ 1. Analytic semi-group and its infinitesimal generator. For the sake of convenience, we state some results from the theory of analytic semi-groups which will be used in the sequel.

Let A be a linear operator from a complex Banach space X into itself. Let us assume

(A) A is a densely defined, closed linear operator. The resolvent set $\rho(-A)$ of $-A$ contains a closed sector $\Sigma : |\arg \lambda| \leq \pi/2 + \theta$, $0 < \theta < \pi/2$. The resolvent of $-A$ satisfies

$$\|(\lambda I + A)^{-1}\| \leq M/|\lambda| \qquad \text{for} \quad \lambda \in \Sigma\,, \tag{1. 1}$$

where M is a constant independent of λ.

Note that the assumption $0 \in \rho(-A)$ is contained in (A).

Under the assumption (A), $-A$ generates a semi-group $\exp(-tA)$ by means of the formula

$$\exp(-tA) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I + A)^{-1} d\lambda\,, \tag{1. 2}$$

where Γ is a smooth contour running in Σ from $\infty e^{-i(\pi/2+\theta)}$ to $\infty e^{i(\pi/2+\theta)}$. $\exp(-tA)$ is analytic in the sector $|\arg t| \leq \theta$, $t \neq 0$. For any real α

with $0<\alpha<1$ we define

$$A^{-\alpha}=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\lambda^{-\alpha}(\lambda I+A)^{-1}d\lambda\,. \tag{1.3}$$

$A^{-\alpha}$ has an inverse and we define A^{α} by

$$A^{\alpha}=(A^{-\alpha})^{-1}\,. \tag{1.4}$$

For $\alpha=n+\alpha'$ with some natural number n and some α' satisfying $0<\alpha'<1$ we define $A^{\alpha}=A^{n}A^{\alpha'}$ and then $A^{-\alpha}=(A^{\alpha})^{-1}$. In this way A^{α} has been defined for any real α.

The fractional power can be defined for more general operators and for the details see Kato [1] and [2]. $A^{-\alpha}$, $\alpha>0$, is bounded; however, A^{α}, $\alpha>0$, is bounded only when A is bounded. For any real α and β, we have

$$A^{\alpha+\beta}=A^{\alpha}A^{\beta} \tag{1.5}$$

and $A^1=A$. For any positive number α_0 there exists a constant M_{α_0} such that for any α with $0\leqq\alpha\leqq\alpha_0$ we have

$$||A^{\alpha}\exp(-tA)||\leqq M_{\alpha_0}|t|^{-\alpha}, \tag{1.6}$$

in the sector $|\arg t|\leqq\theta$. Furthermore to each α in $[0, 1]$ there corresponds a constant C_{α} such that we have

$$||A^{\alpha}(\lambda I+A)^{-1}||\leqq C_{\alpha}/|\lambda|^{1-\alpha} \tag{1.7}$$

for any $\lambda\in\Sigma$. This can be proved by

$$(\lambda I+A)^{-1}=\int e^{\lambda t}\exp(-tA)dt\,, \tag{1.8}$$

where the integral path runs from 0 to ∞ along the upper or the lower boundary of the sector $|\arg t|<\theta$, according to $\operatorname{Im}\lambda<0$ or $\operatorname{Im}\lambda>0$ respectively.

§ 2. **Assumptions and Definitions.** In what follows, we denote by Σ a fixed closed angular domain (as in the previous section): $\Sigma=\{\lambda\,;\,|\lambda|\leqq\pi/2+\theta\}$, $0<\theta<\pi/2$.

We first state the assumption to be made in the theorems.

(E. 1) For each $t\in[0, T]$, $A(t)$ is a densely defined, closed linear operator. The resolvent set $\rho(-A(t))$ of $-A(t)$ contains Σ. The resolvent of $-A(t)$ satisfies

$$||(\lambda I+A(t))^{-1}||\leqq M/|\lambda| \tag{2.1}$$

for any $\lambda\in\Sigma$ and $t\in[0, T]$, where M is a constant independent of λ and t.
(E. 2) $A(t)^{-1}$, which is a bounded operator for each t, is continuously differentiable in $t\in[0, T]$ in the uniform operator topology.
(E. 3) For any $\lambda\in\Sigma$ and $t\in[0, T]$, the following inequality holds:

$$\left\|\frac{\partial}{\partial t}(\lambda I+A(t))^{-1}\right\|\leqq\frac{N}{|\lambda|^{1-\rho}} \tag{2.2}$$

where N and ρ are constants independent of t and λ with $0\leqq\rho<1$.
(E. 4) $dA(t)^{-1}/dt$ is Hölder continuous in $t\in[0, T]$ in the uniform operator topology:

$$\|dA(t)^{-1}/dt-dA(s)^{-1}/ds\|\leqq K|t-s|^{\alpha}, \qquad K>0,\ \alpha>0. \tag{2.3}$$

In what follows, we denote by C constants which depend only on the constants appearing in the above assumptions.

As a sufficient condition for (E. 3), we have

Theorem 2.1. *If there exist positive numbers ρ, ρ_1 and a natural number l satisfying $1=l\rho+\rho_1$, $0\leqq\rho_1<\rho<1$, such that both $A(t)^{-\rho}$ and $A(t)^{-\rho_1}$ are continuously differentiable in t in the strong operator topology, then* (E. 3) *is satisfied.*

Proof. First we remark the relation

$$(\partial/\partial t)(\lambda I+A(t))^{-1}=A(t)(\lambda I+A(t))^{-1}\cdot dA(t)^{-1}/dt\cdot A(t)(\lambda I+A(t))^{-1}. \tag{2.4}$$

Using $A(t)^{-1}=(A(t)^{-\rho})^l A(t)^{-\rho_1}$, we have

$$\begin{aligned}
&(\partial/\partial t)(\lambda I+A(t))^{-1}=\\
&=A(t)(\lambda I+A(t))^{-1}\{\sum_{j=1}^{l}A(t)^{-(j-1)\rho}dA(t)^{-\rho}/dt\cdot A(t)^{-(l-j)\rho-\rho_1}\\
&\quad+A(t)^{\rho_1-1}dA(t)^{-\rho_1}/dt\}A(t)(\lambda I+A(t))^{-1}=\\
&=\sum_{j=1}^{l}A(t)^{1-(j-1)\rho}(\lambda I+A(t))^{-1}dA(t)^{-\rho}/dt\cdot A(t)^{1-(l-j)\rho-\rho_1}(\lambda I+A(t))^{-1}\\
&\quad+A(t)^{\rho_1}(\lambda I+A(t))^{-1}dA(t)^{-\rho_1}/dt\cdot A(t)(\lambda I+A(t))^{-1},
\end{aligned}$$

whence using (1. 7) we readily obtain (2. 2).

REMARK. Let $X=L^p[a, b]$ $(1\leqq p\leqq\infty)$ with the norm $\|u\|=\left[\int_a^b|u(x)|^p dx\right]^{1/p}$, and let $A(t)$ be a multiplication operator defined by

$$(A(t)u)(x)=|x-t|^{-k}u(x)$$

with some constant $k>1$. Then it is easily seen that the assumptions (E. 1)$\sim$(E. 4) are all satisfied. Especially (2. 2) holds good with $\rho=k^{-1}$, but $A(t)^{-\rho'}$ is differentiable in $(0, T)\cap(a, b)$ only when $\rho' k \geqq 1$. Hence if k is sufficiently near 1, the assumption of Theorem 2.1 is not satisfied because then $A(t)^{-\rho_1}$ is not differentiable. Thus, Theorem 2.1 does not give a very satisfactory sufficient condition for the validity of (2.2). Note that the domain of $A(t)^h$ does change with t in $(0, T)\cap(a, b)$ for any $h>0$ in this example.

In what follows the inhomogeneous term $f(t)$ of (0.1) will be assumed to be strongly continuous unless otherwise stated.

Definition 2.1. We call $u(t)$ a *strict solution* of (0.1) in $(s, T]$ if

(1) $u(t)$ is strongly continuous in the closed interval $[s, T]$ and strongly continuously differentiable in the open-closed interval $(s, T]$;

(2) for each $t\in(s, T]$, $u(t)$ belongs to $D(A(t))$;

(3) $u(t)$ satisfies (0.1) in $(s, T]$.

Definition 2.2. We call $u(t)$ a *weak solution* of (0.1) in $(s, T]$ if

(1) $u(t)$ is strongly continuous in $[s, T]$;

(2) we have

$$\int_s^T (u(t), \varphi'(t)-A(t)^*\varphi(t))dt+\int_s^T (f(t), \varphi(t))dt+(u(s), \varphi(s)) = 0 \quad (2.5)$$

where $\varphi(t)$ is any function with values in X^* with the properties

(i) for each t, $\varphi(t)$ belongs to $D(A(t)^*)$,

(ii) $\varphi(t)$, $\varphi'(t)\,(=d\varphi(t)/dt)$ and $A(t)^*\varphi(t)$ are strongly continuous in $[s, T]$,

(iii) $\varphi(T)=0$.

By the assumption (E.1) each $-A(s)$ generates an analptic semi-group $\exp(-tA(s))$. The derivatives of $\exp(-(t-s)A(t))$ satisfy

$$\|(\partial/\partial t)\exp(-(t-s)A(t))\| \leqq C(t-s)^{-1}, \quad (2.6)$$

$$\|(\partial/\partial s)\exp(-(t-s)A(t))\| \leqq C(t-s)^{-1}. \quad (2.7)$$

(2.7) is a direct consequence of (1.6) with $\alpha=1$ and the relation

$$(\partial/\partial s)\exp(-(t-s)A(t)) = A(t)\exp(-(t-s)A(t)).$$

(2.6) follows from (2.7), the relation

$$\frac{\partial}{\partial t}\exp(-(t-s)A(t)) = -\frac{\partial}{\partial s}\exp(-(t-s)A(t))+\frac{1}{2\pi i}\int_\Gamma e^{\lambda(t-s)}\frac{\partial}{\partial t}(\lambda I+A(t))^{-1}d\lambda$$

and the uniform boundedness of $(\partial/\partial t)(\lambda I+A(t))^{-1}$ which is a consequence of (2.4) (for the proof of (2.7), (E.3) is not necessary).

Supposing that the assumptions (E.1)~(E.3) hold, we will prove that there exists a unique weak solution to (0.1), and assuming (E.4) in addition to (E.1)$\sim$(E.3), we will show that the weak solution is actually a strict one.

§3. Existence and uniqueness of weak solutions. Let us construct the evolution operator in the form

$$U(t, s) = \exp(-(t-s)A(t)) + \int_s^t \exp(-(t-\tau)A(t))R(\tau, s)d\tau. \tag{3.1}$$

Calculating formally, we get

$$\frac{\partial}{\partial t}U(t, s) = \frac{\partial}{\partial t}\exp(-(t-s)A(t)) + R(t, s) + \int_s^t \frac{\partial}{\partial t}\exp(-(t-\tau)A(t))R(\tau, s)d\tau,$$

$$A(t)U(t, s) = \frac{\partial}{\partial s}\exp(-(t-s)A(t)) + \int_s^t \frac{\partial}{\partial \tau}\exp(-(t-\tau)A(t))R(\tau, s)d\tau.$$

Putting

$$R_1(t, s) = -(\partial/\partial t + \partial/\partial s)\exp(-(t-s)A(t)), \tag{3.2}$$

we obtain

$$\frac{\partial}{\partial t}U(t, s) + A(t)U(t, s) = -R_1(t, s) + R(t, s) - \int_s^t R_1(t, \tau)R(\tau, s)d\tau.$$

Therefore we will determine $R(t, s)$ as the solution of the integral equation

$$R(t, s) - \int_s^t R_1(t, \tau)R(\tau, s)d\tau = R_1(t, s). \tag{3.3}$$

Lemma 3.1. *$R_1(t, s)$ is continuous in $0 \leqq s < t \leqq T$ in the uniform operator topology and satisfies*

$$\|R_1(t, s)\| \leqq C_1(t-s)^{-\rho}. \tag{3.4}$$

Proof. This follows from the integral representation

$$R_1(t, s) = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda(t-s)}\frac{\partial}{\partial t}(\lambda I + A(t))^{-1}d\lambda \tag{3.5}$$

and (E.3).

By Lemma 3.1 the integral equation (3.3) can be solved by successive approximation:

$$R(t, s) = \sum_{m=1}^{\infty} R_m(t, s), \tag{3.6}$$

$$R_m(t, s) = \int_s^t R_1(t, \sigma)R_{m-1}(\sigma, s)d\sigma, \qquad m = 2, 3, \cdots. \tag{3.7}$$

Lemma 3.2. *$R(t, s)$ is continuous in $0 \leqq s < t \leqq T$ in the uniform operator topology and satisfies*

$$\|R(t, s)\| \leqq C(t-s)^{-\rho}. \tag{3.8}$$

Proof. This follows from

$$\|R_m(t, s)\| \leqq \frac{C_1^m \Gamma(1-\rho)^m (t-s)^{(m-1)(1-\rho)-\rho}}{\Gamma(m(1-\rho))}, \tag{3.9}$$

where C_1 is the same constant as in (3.4), and from the preceding lemma.

Lemma 3.1 and 3.2 show that $U(t, s)$ is well defined by the formula (3.1).

For sufficiently small positive h, we define

$$U_h(t, s) = \exp(-(t-s)A(t)) + \int_s^{t-h} \exp(-(t-\tau)A(t))R(\tau, s)d\tau. \tag{3.10}$$

Then we have

$$\begin{aligned}&(\partial/\partial t)U_h(t, s) + A(t)U_h(t, s)\\ &= -R_1(t, s) + \exp(-hA(t))R(t-h, s) - \int_s^{t-h} R_1(t, \tau)R(\tau, s)d\tau.\end{aligned}$$

The right member is uniformly bounded in h and

$$(\partial/\partial t)U_h(t, s) + A(t)U_h(t, s) \to 0 \tag{3.11}$$

strongly as $h \downarrow 0$. The solution of (0.1) in $(s, T]$ is formally given by

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \sigma)f(\sigma)d\sigma. \tag{3.12}$$

We can, however, only prove that this is a weak solution unless we assume (E.4). Let $\varphi(t)$ be any function satisfying (i), (ii) and (iii) in Definition 2.2. Then,

$$\begin{aligned}&\int_s^T (U(t, s)u(s), \varphi'(t))dt\\ &= \lim_{k\downarrow 0}\lim_{h\downarrow 0}\int_{s+k}^T (U_h(t, s)u(s), \varphi'(t))dt.\end{aligned}$$

Hence we have

$$\begin{aligned}&\int_{s+k}^T (U_h(t, s)u(s), \varphi'(t))dt\\ &= -(U_h(s+k, s)u(s), \varphi(s+k))\end{aligned}$$

$$-\int_{s+k}^{T}(((\partial/\partial t)U_h(t, s)+A(t)U_h(t, s))u(s),\ \varphi(t))dt$$
$$+\int_{s+k}^{T}(U_h(t, s)u(s),\ A(t)^*\varphi(t))dt$$
$$\to -(U(s+k, s)u(s),\ \varphi(s+k))$$
$$+\int_{s+k}^{T}(U(t, s)u(s),\ A(t)^*\varphi(t))dt \qquad (\text{as} \quad h\downarrow 0)$$
$$\to -(u(s),\ \varphi(s))+\int_{s}^{T}(U(t, s)u(s),\ A(t)^*\varphi(t))dt \qquad (\text{as} \quad k\downarrow 0),$$

which implies

$$\int_{s}^{T}(U(t, s)u(s),\ \varphi'(t)-A(t)^*\varphi(t))dt+(u(s),\ \varphi(s)) = 0. \tag{3.13}$$

Similarly,

$$\int_{s}^{T}\left(\int_{s}^{t}U(t, \sigma)f(\sigma)d\sigma,\ \varphi'(t)\right)dt = \int_{s}^{T}\int_{\sigma}^{T}(U(t, \sigma)f(\sigma),\ \varphi'(t))dtd\sigma$$
$$= \lim_{k\downarrow 0}\lim_{s\downarrow 0}\lim_{h\downarrow 0}\int_{s}^{T-k}\int_{\sigma+\delta}^{T}(U_h(t, \sigma)f(\sigma),\ \varphi'(t))dtd\sigma.$$

However

$$\int_{s}^{T-k}\int_{\sigma+s}^{T}(U_h(t, \sigma)f(\sigma),\ \varphi'(t))dtd\sigma = -\int_{s}^{T-k}(U_h(\sigma+\delta, \sigma)f(\sigma),\ \varphi(\sigma+\delta))d\sigma$$
$$-\int_{s}^{T-k}\int_{\sigma+\delta}^{T}(((\partial/\partial t)U_h(t, \sigma)+A(t)U_h(t, \sigma))f(\sigma),\ \varphi(t))dtd\sigma$$
$$+\int_{s}^{T-k}\int_{\sigma+\delta}^{T}(U_h(t, \sigma)f(\sigma),\ A(t)^*\varphi(t))dtd\sigma$$
$$\to -\int_{s}^{T-k}(U(\sigma+\delta, \sigma)f(\sigma),\ \varphi(\sigma+\delta))d\sigma$$
$$+\int_{s}^{T-k}\int_{\sigma+\delta}^{T}(U(t, \sigma)f(\sigma),\ A(t)^*\varphi(t))dtd\sigma \qquad (\text{as} \quad h\downarrow 0)$$
$$\to -\int_{s}^{T-k}(f(\sigma),\ \varphi(\sigma))d\sigma+\int_{s}^{T-k}\int_{\sigma}^{T}(U(t, \sigma)f(\sigma),\ A(t)^*\varphi(t))dtd\sigma$$
$$(\text{as} \quad \delta\downarrow 0)$$
$$\to -\int_{s}^{T}(f(\sigma),\ \varphi(\sigma))d\sigma+\int_{s}^{T}\int_{\sigma}^{T}(U(t, \sigma)f(\sigma),\ A(t)^*\varphi(t))dtd\sigma$$
$$(\text{as} \quad k\downarrow 0),$$

and hence we obtoin

$$\int_{s}^{T}\left(\int_{s}^{t}U(t, \sigma)f(\sigma)d\sigma,\ \varphi'(t)-A(t)^*\varphi(t)\right)dt+\int_{s}^{T}(f(\sigma),\ \varphi(\sigma))d\sigma = 0. \tag{3.14}$$

Adding (3.13) to (3.14), we get (2.5).

In order to prove the uniqueness of the weak solution, we will first construct a bounded-operator-valued function $V(t, s)$ which has the following properties:

i) $V(t, s)$ is continuons in $0 \leqq s \leqq t \leqq T$ in the strong topology;

ii) $V(t, t)=I$ for any $t \in [0, T]$:

iii) for any $u \in D(A(s))$, $\lim_{h \to 0} h^{-1}(V(t, s+h)-V(t, s))u$ exists and is equal to $V(t, s)A(s)u$.

Such a $V(t, s)$ can be constructed by setting

$$V(t, s) = \exp(-(t-s)A(s)) + \int_s^t Q(t, \tau) \exp(-(\tau-s)A(s))d\tau, \tag{3.15}$$

where $Q(t, s)$ is the solution of the integral equation

$$Q(t, s) - \int_s^t Q(t, \tau)Q_1(\tau, s)d\tau = Q_1(t, s) \tag{3.16}$$

with kernel and inhomogeneous term

$$Q_1(t, s) = (\partial/\partial t + \partial/\partial s)\exp(-(t-s)A(s)) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda(t-s)}(\partial/\partial s)(\lambda I + A(s))^{-1}d\lambda.$$

As in Lemma 3.1, it can be shown that $Q_1(t, s)$ is continuous in $0 \leqq s < t \leqq T$ in the uniform operator topology, and satisfies

$$\|Q_1(t, s)\| \leqq C(t-s)^{-\rho}. \tag{3.17}$$

Therefore (3.16) can be solved by successive approximation as before. The solution $Q(t, s)$ is continuous in $0 \leqq s < t \leqq T$ in the uniform operator topology and satisfies

$$\|Q(t, s)\| \leqq C(t-s)^{-\rho}. \tag{3.18}$$

If we set

$$V_h(t, s) = \exp(-(t-s)A(s)) + \int_{s+h}^t Q(t, \tau)\exp(-(\tau-s)A(s))d\tau \tag{3.19}$$

for a sufficiently small positive h, then $(\partial/\partial s)V_h(t, s)$ and $\overline{V_h(t, s)A(s)}$ (=the bouuded extension of $V_h(t, s)A(s)$) are continuous in $s \in [a, t-h]$ in the uniform operator topology. Furthermore $(\partial/\partial s)V_h(t, s) - \overline{V_h(t, s)A(s)}$ tends to 0 strongly as $h \downarrow 0$.

Let $u(t)$ be any weak solution of the homogeneous equation of (0.1) in $(s, T]$. We have then by definition

$$\int_s^T (u(t), \psi'(t) - A(t)^*\psi(t))dt + (u(s), \psi(s)) = 0 \tag{3.20}$$

for any ψ satisfying i), ii) and iii). Let t_0 be an arbitrarily fixed number in $(s, T]$, and let $\varphi(t)$ be a continuously differentiable function with value in X^* and with support in (s, t_0). Then, if h is sufficiently small depending on the support of φ, $\psi_h(t) = V_h(t_0, t)^*\varphi(t)$ has all the properties required of ψ in (3.20). We have also

$$\int_s^{t_0}(V(t_0, t)u(t), \varphi'(t))dt = \lim_{h\downarrow 0}\int_s^{t_0}(V_h(t_0, t)u(t), \varphi'(t))dt$$
$$= \lim_{h\downarrow 0}\int_s^{t_0}(u(t), V_h(t_0, t)^*\varphi'(t))dt = \lim_{h\downarrow 0}\int_s^{t_0}(u(t), \psi_h'(t) - A(t)^*\psi_h(t))dt$$
$$-\lim_{h\downarrow 0}\int_s^{t_0}(u(t), (\partial V_h(t_0, t)^*/\partial t - A(t)^* V_h(t_0, t)^*)\varphi(t))dt .$$

The first term vanishes because $u(t)$ satisfies (3.20) with $\psi = \psi_h$ and because $\psi_h(s) = 0$. Therefore the right member is equal to

$$\lim_{h\downarrow 0}\int_s^{t_0}((\partial V_h(t_0, t)/\partial t - \overline{V_h(t_0, t)A(t)})u(t), \varphi(t))dt = 0 .$$

This implies that the distribution derivative of $V(t_0, t)u(t)$ vanishes, and hence that $V(t_0, t)u(t) \equiv \text{const.}$ in (s, t_0). Letting $t \downarrow s$ and then $t \uparrow t_0$, we get $u(t_0) = V(t_0, s)u(s)$. As t_0 was an arbitrary number in $(s, T]$, it follows that

$$u(t) = V(t, s)u(s) \tag{3.21}$$

for any $t \geqq s$. This shows that the weak solution of (0.1) is uniquely determined by its initial data and the inhomogeneous term. As (2.12) was seen to be a weak solution, we have also proved

$$V(t, s) = U(t, s)\,, \tag{3.22}$$

$$U(t, r)U(r, s) = U(t, s)\,, \qquad s \leqq r \leqq t\,. \tag{3.23}$$

We do not know whether the range of $U(t, s)$, $s < t$, is contained in $D(A(t))$ without assuming (E.4), but the following weaker result is obvious:

$$\left.\begin{array}{l} A(t)^\beta U(t, s) \text{ is bounded if } 0 \leqq \beta < 1 \text{ and } t > s \\ \text{and we have } \|A(t)^\beta U(t, s)\| \leqq C(t-s)^{-\beta} \end{array}\right\}. \tag{3.24}$$

Summing up, we have established

Theorem 3.1. *Under the assumptions* (E.1) ~ (E.3), *the operator-valued function* $U(t, s)$ *determined by* (3.1) *and* (3.3) *satisfies* (3.23), (3.24) *and*

$$U(s, s) = I \qquad \text{for any } \ 0 \leqq s \leqq T\,. \tag{3.25}$$

The function $u(t)$ *determined by* (3.12) *is a unique weak solution of* (0.1) *in* $(s, T]$, $u(t)$ *belongs to* $D(A(t)^{\beta})$ *for each* $t\in(s, T]$ *if* $0\leqq\beta<1$, *and* $A(t)^{\beta}u(t)$ *is strongly continuous in* $t\in(s, T]$.

§4. Existence of the strict solution.

In this section, we will prove that the weak solution whose existence was proved in the previous section is a strict solution under the additional assumption (E.4). We assume (E.1)$\sim$(E.4) throughout this section.

Lemma 4.1. *For* $0\leqq s<\tau<t\leqq T$, *we have*

$$\|R(t, s)-R(\tau, s)\|\leqq C\left\{\frac{t-\tau}{(t-s)(\tau-s)^{\rho}}+\frac{(t-\tau)^{\alpha}}{t-s}+\frac{(t-\tau)^{1-\rho}}{(t-s)^{\rho}}+\frac{(t-\tau)^{\alpha}}{(t-s)^{\rho}}\log\frac{t-s}{t-\tau}\right\}. \tag{4.1}$$

Proof. First we have

$$R_1(t, s)-R_1(\tau, s) = -\frac{1}{2\pi i}\int_{\Gamma}e^{\lambda(t-s)}\left\{\frac{\partial}{\partial t}(\lambda I+A(t))^{-1}-\frac{\partial}{\partial\tau}(\lambda I+A(\tau))^{-1}\right\}d\lambda$$

$$-\frac{1}{2\pi i}\int_{\Gamma}(e^{\lambda(t-s)}-e^{\lambda(\tau-s)})\frac{\partial}{\partial\tau}(\lambda I+A(\tau))^{-1}d\lambda = I+II, \tag{4.2}$$

and by (2.4),

$$\begin{aligned}&(\partial/\partial t)(\lambda I+A(t))^{-1}-(\partial/\partial\tau)(\lambda I+A(\tau))^{-1}\\ &= \{A(t)(\lambda I+A(t))^{-1}-A(\tau)(\lambda I+A(\tau))^{-1}\}\,dA(t)^{-1}/dt\cdot A(t)(\lambda I+A(t))^{-1}\\ &+ A(\tau)(\lambda I+A(\tau))^{-1}\{dA(t)^{-1}/dt-dA(\tau)^{-1}/d\tau\}A(t)(\lambda I+A(t))^{-1}\\ &+ A(\tau)(\lambda I+A(\tau))^{-1}dA(\tau)^{-1}/d\tau\,\{A(t)(\lambda I+A(t))^{-1}-A(\tau)(\lambda I+A(\tau))^{-1}\}\,.\end{aligned}$$

Since

$$\begin{aligned}&A(t)(\lambda I+A(t))^{-1}-A(\tau)(\lambda I+A(\tau))^{-1} = \lambda((\lambda I+A(t))^{-1}-(\lambda I+A(\tau))^{-1})\\ &= \lambda\int_{\tau}^{t}(\partial/\partial\sigma)(\lambda I+A(\sigma))^{-1}d\sigma\,,\end{aligned}$$

we have

$$\begin{aligned}&\|A(t)(\lambda I+A(t))^{-1}-A(\tau)(\lambda I+A(\tau))^{-1}\|\\ &\leqq|\lambda|\int_{\tau}^{t}N|\lambda|^{\rho-1}d\sigma = C(t-\tau)|\lambda|^{\rho}\,.\end{aligned}$$

Making use of this inequality together with (E.4), we get

$$\|(\partial/\partial t)(\lambda I+A(t))^{-1}-(\partial/\partial\tau)(\lambda I+A(\tau))^{-1}\|\leqq C\{(t-\tau)|\lambda|^{\rho}+(t-\tau)^{\alpha}\}\,,$$

and therefore

$$\begin{aligned}\|I\| &\leqq C\int_\Gamma e^{\mathrm{Re}\,\lambda(t-s)}\{(t-\tau)|\lambda|^\rho+(t-\tau)^\alpha\}\,|d\lambda| \\ &\leqq C\left\{\frac{t-\tau}{(t-s)^{1+\rho}}+\frac{(t-\tau)^\alpha}{t-s}\right\}. \end{aligned} \tag{4.3}$$

As for II, we have

$$\begin{aligned} II &= \int_{\tau-s}^{t-s}\frac{\partial}{\partial\sigma}\left\{\frac{1}{2\pi i}\int_\Gamma e^{\lambda\sigma}\frac{\partial}{\partial\tau}(\lambda I+A(\tau))^{-1}d\lambda\right\}d\sigma \\ &= \frac{1}{2\pi i}\int_{\tau-s}^{t-s}\left\{\int_\Gamma \lambda e^{\lambda\sigma}\frac{\partial}{\partial\tau}(\lambda I+A(\tau))^{-1}d\lambda\right\}d\sigma. \end{aligned}$$

Noting

$$\left\|\int_\Gamma \lambda e^{\lambda\sigma}\frac{\partial}{\partial\tau}(\lambda I+A(\tau))^{-1}d\lambda\right\| \leqq C\int_\Gamma e^{\mathrm{Re}\,\lambda\cdot\sigma}|\lambda|^\rho|d\lambda| \leqq \frac{C}{\sigma^{1+\rho}},$$

we get

$$\begin{aligned}\|II\| &\leqq C\int_{\tau-s}^{t-s}\sigma^{-1-\rho}d\sigma = C\{(\tau-s)^{-\rho}-(t-s)^{-\rho}\} = C(\tau-s)^{-\rho}\left\{1-\left(\frac{\tau-s}{t-s}\right)^\rho\right\} \\ &\leqq C(\tau-s)^{-\rho}\left(1-\frac{\tau-s}{t-s}\right) = C\frac{t-\tau}{(t-s)(\tau-s)^\rho}. \end{aligned} \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we obtain

$$\|R_1(t, s)-R_1(\tau, s)\| \leqq C\left\{\frac{t-\tau}{(t-s)(\tau-s)^\rho}+\frac{(t-\tau)^\alpha}{t-s}\right\}.$$

From the identity

$$\begin{aligned} &\int_s^t R_1(t, \sigma)R(\sigma, s)d\sigma-\int_s^\tau R_1(\tau, \sigma)R(\sigma, s)d\sigma \\ &= \int_\tau^t R_1(t, \sigma)R(\sigma, s)d\sigma+\int_s^\tau (R_1(t, \sigma)-R_1(\tau, \sigma))R(\sigma, s)d\sigma, \end{aligned}$$

we get

$$\begin{aligned} &\left\|\int_s^t R_1(t, \sigma)R(\sigma, s)d\sigma-\int_s^\tau R_1(\tau, \sigma)R(\sigma, s)d\sigma\right\| \\ &\leqq C\left\{\frac{(t-\tau)^{1-\rho}}{(t-s)^\rho}+\frac{(t-\tau)(\tau-s)^{1-2\rho}}{t-s}+\frac{(t-\tau)^\alpha}{(t-s)^\rho}\left(\log\frac{t-s}{t-\tau}+1\right)\right\} \end{aligned}$$

(cf. Lemma 1.2 in [7]), which completes the proof of the lemma.

We denote by $W(t, s)$ the second term in $U(t, s)$, i.e.

$$W(t, s) = \int_s^t \exp(-(t-\tau)A(t))R(\tau, s)d\tau, \tag{4.5}$$

and for a sufficiently small positive number h we set

$$W_h(t, s) = \int_s^{t-h} \exp(-(t-\tau)A(t))R(\tau, s)d\tau .$$

Then, we have

$$\frac{\partial}{\partial t}W_h(t,s) = \exp(-hA(t))R(t-h, s) + \int_s^{t-h} \frac{\partial}{\partial t}\exp(-(t-\tau)A(t))(R(\tau, s)-R(t, s))d\tau$$
$$+\int_s^{t-h}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\exp(-(t-\tau)A(t))d\tau\cdot R(t, s)-\exp(-hA(t))R(t, s)$$
$$+\exp(-(t-s)A(t))R(t, s) .$$

Letting $h\downarrow 0$, it follows that

$$\frac{\partial}{\partial t}W(t, s) = \int_s^t \frac{\partial}{\partial t}\exp(-(t-\tau)A(t))(R(\tau, s)-R(t, s))d\tau$$
$$-\int_s^t R_1(t, \tau)d\tau\cdot R(t, s)+\exp(-(t-s)A(t))R(t, s) . \tag{4.6}$$

It follows from (3.4), (3.8) and (4.1) that

$$\left\|\frac{\partial}{\partial t}U(t, s)\right\| \leqq \frac{C}{t-s}, \quad \left\|\frac{\partial}{\partial t}W(t, s)\right\| \leqq C\left\{\frac{1}{(t-s)^{\rho}}+\frac{1}{(t-s)^{1-\alpha}}\right\} . \tag{4.7}$$

As regards $A(t)U(t, s)$, we can show in a similar way that

$$A(t)U(t, s) = A(t)\exp(-(t-s)A(t))$$
$$+\int_s^t A(t)\exp(-(t-\tau)A(t))(R(\tau, s)-R(t, s))d\tau \tag{4.8}$$
$$-R(t, s)+\exp(-(t-s)A(t))R(t, s) .$$

Using (4.8), we can also deduce estimates similar to (4.7) for $\|A(t)U(t, s)\|$ and $\|A(t)W(t, s)\|$. However, if we use (3.11), we can show this and the equality $(\partial/\partial t)U(t, s)+A(t)U(t, s)=0$ at the same time without verifying (4.8). Making use of $U(t, s)=V(t, s)$, it can also be proved that $(\partial/\partial s)U(t, s)$ is bounded when $s<t$ and that we have

$$(\partial/\partial s)U(t, s) = \overline{U(t, s)A(s)} \tag{4.9}$$

as well as

$$\|(\partial/\partial s)U(t, s)\| \leqq C/(t-s) . \tag{4.10}$$

In order to deduce this property, we have only to calculate $(\partial/\partial s)V(t, s)$ just as we did for $(\partial/\partial t)U(t, s)$ using the following lemma which can be proved in just the same way as Lemma 4.1.

Lemma 4.2. *For any $s<\tau<t$, we have*

$$\|Q(t,s)-Q(t,\tau)\| \leqq C\left\{\frac{\tau-s}{(t-s)(t-\tau)^{\rho}}+\frac{(\tau-s)^{\alpha}}{t-s}+\frac{(\tau-s)^{1-\rho}}{(t-s)^{\rho}}+\frac{(\tau-s)^{\alpha}}{(t-s)^{\rho}}\log\frac{t-s}{\tau-s}\right\}.$$

Thus we have established

Theorem 4.1. *Under the assumptions* (E. 1)~(E. 4), *there exists an evolution operator $U(t,s)$ for the equation* (0, 1) *which satisfies*

$$\left\|\frac{\partial}{\partial t}U(t,s)\right\| = \|A(t)U(t,s)\| \leqq \frac{C}{t-s}, \tag{4.11}$$

$$\left\|\frac{\partial}{\partial t}\{U(t,s)-\exp(-(t-s)A(t))\}\right\| \leqq C\left\{\frac{1}{(t-s)^{\rho}}+\frac{1}{(t-s)^{1-\alpha}}\right\}. \tag{4.12}$$

Moreover $U(t,s)$ satisfies (4.9) *and* (4.10), *and*

$$\left\|\frac{\partial}{\partial s}\{U(t,s)-\exp(-(t-s)A(s))\}\right\| \leqq C\left\{\frac{1}{(t-s)^{\rho}}+\frac{1}{(t-s)^{1-\alpha}}\right\} \tag{4.13}$$

as well as (3.23) *and* (3.25).

Theorem 4.2. *If we suppose that $f(t)$ is Hölder continuous in $t\in[s,T]$:*

$$\|f(t)-f(t)\| \leqq F|t-s|^{\gamma}, \qquad F>0,\quad \gamma>0,$$

and that (E. 1)~(E. 4) *hold, then* (3.12) *is the strict solution of* (0. 1).

Proof. We have only to notice that

$$\frac{\partial}{\partial t}\int_s^t \exp(-(t-\sigma)A(t))f(\sigma)d\sigma = \int_s^t \frac{\partial}{\partial t}\exp(-(t-\sigma)A(t))(f(\sigma)-f(t))d\sigma$$
$$-\int_s^t R_1(t,\sigma)f(t)d\sigma+\exp(-(t-s)A(t))f(t),$$

§5. Analyticity of the solution. In this section we assume that $A(t)$ is defined in a convex complex neighborhood Δ of $[0, T]$ and that

(A. 1) for each $t\in\Delta$, $A(t)$ is a densely defined, closed linear operator;
(A. 2) the resolvent set of $-A(t)$ contains Σ for each $t\in\Delta$ and the resolvent of $-A(t)$ satisfies

$$\|(\lambda I+A(t))^{-1}\| \leqq M/|\lambda|, \qquad t\in\Delta,\quad \lambda\in\Sigma,$$

where M is a constant independent of t and λ;
(A. 3) $A(t)^{-1}$ is holomorphic in Δ in the uniform operator topology;
(A. 4) for any $\lambda\in\Sigma$ and $t\in\Delta$, the following inequality holds:

$$\|(\partial/\partial t)(\lambda I+A(t))^{-1}\| \leqq N/|\lambda|^{1-\rho},$$

where N is a constant independent of t and λ.

For any $\lambda \in \Sigma$, $(\lambda I+A(t))^{-1}$ and $(\partial/\partial t)(\lambda I+A(t))^{-1}$ are holomorphic in $t \in \Delta$ with $A(t)^{-1}$. Hence we have

Lemma 5.1. $\exp(-(t-s)A(t))$ *and* $R_1(t, s)$ *are both holomorphic in the domain* $t, s \in \Delta$, $|\arg(t-s)| < \theta$. $\exp(-(t-s)A(t))$ *is uniformly bounded in this domain and* $R_1(t, s)$ *satisfies*

$$\|R_1(t, s)\| \leqq C/|t-s|^{\rho}.$$

Lemma 5.2. *Let* $P(t, s)$ *and* $P'(t, s)$ *be two bounded-operator-valued functions defined for* $t, s \in \Delta$ *and* $|\arg(t-s)| < \theta$. *If they satisfy*

$$\|P(t, s)\| \leqq C_2/|t-s|^{\rho_2}, \quad \|P'(t, s)\| \leqq C_3/|t-s|^{\rho_3},$$
$$-\infty < \rho_2 < 1, \quad -\infty < \rho_3 < 1,$$

and are holomorphic with respect to two complex arguments t *and* s *in the above domain, then*

$$\int_s^t P(t, r)P'(r, s)dr$$

is defined in the same domain as above and holomorphic in t *and* s *there. It satisfies, moreover,*

$$\left\| \int_s^t P(t, r)Q(r, s)dr \right\| \leqq C_2C_3B(1-\rho_2, 1-\rho_3)|t-s|^{1-\rho_2-\rho_3}.$$

The proof is quite similar to that of Lemma 3 in Komatsu [4].

By Lemma 5.1 and 5.2, $R(t, s)$ is holomorphic in the domain mentioned above, and so are $W(t, s)$ and $U(t, s)$. To see this we have only to note that

$$\|R_m(t, s)\| \leqq C_1^m\Gamma(1-\rho)^m|t-s|^{(m-1)(1-\rho)-\rho}/\Gamma(m(1-\rho))$$

and that the uniform limit of a series of holomorphic functions is holomorphic. Thus we have proved

Theorem 5.1. *Under the assumptions* (A. 1)$\sim$(A. 4), *the evolution operator* $U(t, s)$ *constructed in* § 3 *can be extended holomorphically to the domain* $s, t \in \Delta$, $|\arg(t-s)| < \theta$.

We can also prove easily

Theorem 5.2. *Under the assumptions of Theorem* 5.1, *the solution* $u(t)$ *of the inhomogeneous equation* (0.1) *is holomorphic in the domain where* $f(t)$ *is holomorphic.*

§ 6. **Perturbation theory.** We consider a perturbed equation

$$du(t)/dt+A(t)u(t)+B(t)u(t) = f(t)\,. \tag{6.1}$$

(E. 5) For each $t\in[0, T]$, $B(t)$ is a closed linear operator whose domain contains that of $A(t)$. There exist positive constants M' and γ' such that

$$\|B(t)(\lambda I+A(t))^{-1}\| \leqq M'/|\lambda|^{1-\gamma'}\,, \tag{6.2}$$

for each $\lambda\in\Sigma$ and $t\in[0, T]$.
(E. 6) There exist positive constants K_2 and β such that

$$\|B(t)A(t)^{-1}-B(s)A(s)^{-1}\| \leqq K_2|t-s|^{\rho}\,, \tag{6.3}$$

for each $s, t\in[0, T]$,
By (E. 5) we readily obtain

$$\|B(s)\exp(-tA(s))\| \leqq C/t^{\gamma} \tag{6.4}$$

for $t>0$, and hence

$$\|B(t)U_0(t, s)\| \leqq C_3(t-s)^{-\gamma}\,, \tag{6.5}$$

where $U_0(t, s)$ is the evolution operator for (0. 1). The evolution operator $U(t, s)$ of (6. 1) is formally constructed by

$$U(t, s) = \sum_{m=0}^{\infty} U_m(t, s)\,, \tag{6.6}$$

where

$$U_m(t, s) = (-1)^m \int_s^t U_0(t, \sigma)B(\sigma)U_{m-1}(\sigma, s)d\sigma\,, \qquad m = 1, 2, 3, \cdots. \tag{6.7}$$

If C_2 is a constant such that $\|U_0(t, s)\|\leqq C_2$, we easily obtain by induction

$$\|U_m(t, s)\| \leqq C_2C_3^m\Gamma(1-\gamma)^m(t-s)^{m(1-\gamma)}/\Gamma((m+1)(1-\gamma))\,, \tag{6.8}$$

$$\|B(t)U_m(t, s)\| \leqq C_3^{m+1}\Gamma(1-\gamma)^{m+1}(t-s)^{m(1-\gamma)-\gamma}/\Gamma((m+1)(1-\gamma))\,. \tag{6.9}$$

For the formal constrution of $U(t, s)$, it is not necessary to assume (E. 4) and (E. 6), and $U(t, s)$ thus constructed can be used to form a weak solution of (0. 1). In this section, however, we do not consider weak solutions.

Lemma 6. 1. *For $s<\tau<t$, we have*

$$\begin{aligned}\|B(t)U(t, s)-B(\tau)U(\tau, s)\| \leqq C\Big\{&\frac{t-\tau}{(t-s)(t-s)^{\rho'}}+\frac{(t-\tau)^{1-\rho'}}{(t-s)^{\rho'}}\\ &+\frac{(t-\tau)^{\beta}}{t-s}+\frac{(t-\tau)^{\beta}}{(t-s)^{\rho'}}\log\frac{t-s}{t-\tau}+C_8(t-\tau)^{\beta+\delta-1}(t-s)^{2-\rho-\delta-\gamma}\Big\}\,,\end{aligned} \tag{6.10}$$

where $\rho'=\max(\rho, \gamma)$ *and* δ *is any constant with* $0<\delta<1$.

The proof follows from

$$\|B(t)\exp(-(t-s)A(t))-B(\tau)\exp(-(\tau-s)A(\tau))\| \leqq C\left\{\frac{t-\tau}{(t-s)^{1+\rho}}+\frac{(t-\tau)^{\beta}}{t-s}+\frac{t-\tau}{(t-s)(\tau-s)^{\gamma}}\right\}, \tag{6.11}$$

$$\left\|\int_s^{\tau}\{B(t)\exp(-(t-\sigma)A(t))-B(\tau)\exp(-(\tau-\sigma)A(\tau))\}R(\sigma, s)d\sigma\right\|$$
$$\leqq C\left\{\frac{(t-\tau)(\tau-s)^{1-\rho}}{(t-s)^{1+\rho}}+\frac{(t-\tau)^{1-\rho}(\tau-s)}{(t-s)^{1+\rho}}+\frac{(t-\tau)(\tau-s)^{1-\rho-\gamma}}{t-s}\right.$$
$$\left.+\frac{(t-\tau)^{1-\gamma}(\tau-s)}{(t-s)^{1+\rho}}+\frac{(t-\tau)^{\beta}(\tau-s)^{1-\rho}}{t-s}+\frac{(t-\tau)^{\beta}}{(t-s)^{\rho}}\log\frac{t-s}{t-\tau}\right\}, \tag{6.12}$$

$$\left\|\int_{\tau}^{t}B(t)\exp(-(t-\sigma)A(t))R(\sigma, s)d\sigma\right\|\leqq C\frac{(t-\tau)^{1-\gamma}}{(t-s)^{\rho}}. \tag{6.13}$$

(6.11) and (6.13) are easily proved; the proof of (6.12) is tedious but straight forward and may be omitted.

As is easily seen, we have

$$\begin{aligned} U(t, s) &= U_0(t, s)+\sum_{m=1}^{\infty}\int_s^t U_0(t, \sigma)B(\sigma)U_{m-1}(\sigma, s)d\sigma \\ &= U_0(t, s)+\int_s^t U_0(t, \sigma)B(\sigma)U(\sigma, s)d\sigma \\ &= U_0(t, s)+\int_s^t U(t, \sigma)B(\sigma)U_0(\sigma, s)d\sigma, \end{aligned} \tag{6.14}$$

and

$$\frac{\partial}{\partial t}\int_s^t \exp(-(t-\tau)A(t))B(\tau)U(\tau, s)d\tau$$
$$=\int_s^t\frac{\partial}{\partial t}\exp(-(t-\tau)A(t))\{B(\tau)U(\tau, s)-B(t)U(t, s)\}d\tau \tag{6.15}$$
$$+\int_s^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\exp(-(t-\tau)A(t))d\tau\, B(t)U(t, s)+\exp(-(t-s)A(t))B(t)U(t, s).$$

The norm of the first term is dominated by $C_3\{(t-s)^{-\rho'}+(t-s)^{\beta-1}\}$, and hence we have

$$\|(\partial/\partial t)U(t, s)\|\leqq C/(t-s). \tag{6.16}$$

If we define

$$U_h(t, s) = U_0(t, s)+\int_s^{t-h}U_0(t, \sigma)B(\sigma)U(\sigma, s)d\sigma \tag{6.17}$$

we have

$$(\partial/\partial t)U_h(t, s)-(A(t)+B(t))U_h(t, s) = U_0(t, t-h)B(t-h)U(t-h, s)$$
$$-B(t)U_0(t, s)-\int_s^{t-h} B(t)U_0(t, \sigma)B(\sigma)U(\sigma, s)d\sigma \to 0 \tag{6.18}$$

as $h \downarrow 0$. This implies that $U(t, s)$ is the evolution operator with the desired property.

In order to prove the uniquencess of the solution, we will show that

$$\lim_{h\to 0} h^{-1}\{U(t, s+h)u-U(t, s)u\} = U(t, s)(A(s)+B(s))u \tag{6.19}$$

for any $u \in D(A(s))$. For this purpose, we will investigate the limit of each term on the right of

$$h^{-1}\{U(t, s+h)u-U(t, s)u\} = h^{-1}\{U_0(t, s+h)-U_0(t, s)\}u$$
$$+h^{-1}\int_{s+h}^{t} U(t, \sigma)B(\sigma)(U_0(\sigma, s+h)-U_0(\sigma, s))ud\sigma$$
$$-h^{-1}\int_s^{s+h} U(t, \sigma)B(\sigma)U_0(\sigma, s)ud\sigma = I+II+III\,. \tag{6.20}$$

We will begin with the second term.

$$h^{-1}B(\sigma)(U_0(\sigma, s+h)-U_0(\sigma, s))u = h^{-1}B(\sigma)U_0(\sigma, \sigma_1)(U_0(\sigma_1, s+h)-U_0(\sigma_1, s))u$$
$$\to -B(\sigma)U_0(\sigma, \sigma_1)U_0(\sigma_1, s)A(s)u = -B(\sigma)U_0(\sigma, s)A(s)u$$

as $h\to 0$, where σ_1 is an arbitrary number satisfying $s<\sigma_1<\sigma$. On the other hand, we have

$$h^{-1}B(\sigma)(U_0(\sigma, s+h)-U_0(\sigma, s))u = B(\sigma)U_0(\sigma, s+h)h^{-1}(I-U_0(s+h, s))u$$

and

$$h^{-1}\{U_0(s+h, s)-I\}u = h^{-1}\{\exp(-hA(s))-I\}u$$
$$-h^{-1}\int_s^{s+h} Q(s+h, \tau)\exp(-(\tau-s)A(s))d\tau\cdot u\,. \tag{6.21}$$

The norm of the second term on the right in (6.12) is dominated by

$$\| h^{-1}\int_s^{s+h} Q(s+h, \tau)(A(s)^{-1}-A(\tau)^{-1})\exp(-(\tau-s)A(s))A(s)ud\tau\|$$
$$+\|h^{-1}\int_s^{s+h} Q(s+h, \tau)A(\tau)^{-1}\exp(-(\tau-s)A(s))A(s)ud\tau\| \leqq C\|A(s)u\|$$

where we used an easily verifiable inequality $\|Q(s+h, \tau)A(\tau)^{-1}\|\leqq C$. Hence the norm of the integrand in II is not larger than $C(\sigma-s-h)^{-\gamma}\|A(s)u\|$. Thus, II tends to

$$-\int_s^t U(t, \sigma)A(\sigma)U_0(\sigma, s)A(s)ud\sigma$$

as $h \to 0$. Next, let us consider III.

$$\begin{aligned} B(\sigma)U_0(\sigma, s)u - B(s)u &= B(\sigma)\exp(-(\sigma-s)A(\sigma))u - B(s)u \\ &+ B(\sigma)\int_s^\sigma \exp(-(\sigma-\tau)A(\sigma))R(\tau, s)ud\tau \\ &= B(\sigma)\exp(-(\sigma-s)A(\sigma))(A(s)^{-1}-A(\sigma)^{-1})A(s)u \\ &+ B(\sigma)\exp(-(\sigma-s)A(\sigma))A(\sigma)^{-1}A(s)u - B(s)u \\ &+ B(\sigma)\int_s^\sigma \exp(-(\sigma-\tau)A(\sigma))R(\tau, s)(A(s)^{-1}-A(\tau)^{-1})A(s)ud\tau \\ &+ B(\sigma)\int_s^\sigma \exp(-(\sigma-\tau)A(\sigma))R(\tau, s)A(\tau)^{-1}A(s)ud\tau \to 0 \qquad (\text{as} \quad \sigma \downarrow s), \end{aligned}$$

where we used $\|R(\tau, s)A(\tau)^{-1}\| \leqq C$, which is easily proved. Thus, we have proved that

$$III \to -U(t, s)B(s)u$$

as $h \to 0$. I tends to $U_0(t, s)A(s)u$ as was shown in § 3. In this way we obtain (6. 19). Hence, for any strict solution of (6. 1) we have

$$(\partial/\partial\sigma)(U(t, \sigma)u(\sigma)) = U(t, \sigma)du(\sigma)/d\sigma - U(t, \sigma)(A(\sigma)+B(\sigma))u(\sigma) = U(t, \sigma)f(\sigma).$$

which implies the uniqueness in question. Summing up, we have proved

Theorem 6. 1. *Under the assumptions* (E. 1)∼(E. 6), *and evolution operator* $U(t, s)$ *exists for the perturbed equation* (6. 1), *and it satisfies* (6. 16), (6. 19) *and*

$$\|A(t)U(t, s)\| \leqq C/(t-s), \quad \|B(t)U(t, s)\| \leqq C/(t-s)^\gamma, \tag{6. 22}$$

$$U(t, r)U(r, s) = U(t, s), \qquad s \leqq r \leqq t. \tag{6. 23}$$

Let $f(t)$ *be Hölder continuous iu* $(s, T]$. *Then the unique strict solution in* $(s, T]$ *of* (6. 1) *is given by*

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \sigma)f(\sigma)d\sigma. \tag{6. 24}$$

§ 7. Applications

In this section we shall show that our basic assumptions (E. 1) to (E. 4) are satisfied in a rather general case in which $A(t)$ is defined by certain sesquilinear forms in a Hilbert space.

We follow the terminology and notations of Lions [6]. Let H be a

Hilbert space, the inner product and norm in H being denoted by (f, g) and $|f|$. Let K be another Hilbert space such that $K \subset H$ algebraically and topologically, the inner product and norm in K being denoted by $((u, v))$ and $\|u\|$. Thus there is a constant M_0 such that $|u| \leqq M_0 \|u\|$ for $u \in K$. The norms of bounded linear operators on H to itself and of those on K to itself will be denoted by $|\quad|$ and $\|\quad\|$ respectively.

Let $a(t; u, v)$, $0 \leqq t \leqq T$, be a family of continuous sesquilinear forms on $K \times K$, and let $V(t)$ be a family of closed subspaces of K. We now introduce the following assumptions.

(K. 1) For each $t \in [0, T]$, $V(t)$ is dense in H.

(K. 2) There exist two families $P(t)$ and $Q(t)$ of (not necessarily orthogonal) projection operators on K onto $V(t)$, depending on t continuously differentiably for $t \in [0, T]$ in the strong topology of K. ($P(t)$ and $Q(t)$ may or may not be identical).

(K. 3) For any $u, v \in K$, $a(t; u, v)$ is continuously differentiable in $t \in [0, T]$ and the derivative $\dot{a}(t; u, v) = (\partial/\partial t)a(t; u, v)$ satisfies

$$|\dot{a}(t; u, v)| \leqq M_1 \|u\| \, \|v\| \tag{7.1}$$

for any $u, v \in K$ and $0 \leqq t \leqq T$, where M_1 is a constant independent of u, v and t.

(K. 4) There exists a positive constant δ such that for any $t \in [0, T]$ and $u \in V(t)$ we have

$$\operatorname{Re} a(t; u, u) \geqq \delta \|u\|^2 . \tag{7.2}$$

(K. 1)$\sim$(K. 4) are generalizations of the assumptions used in [6], p. 138.

It follows from the assumptions stated above that there exists a constant M_2 such that

$$|a(t; u, v)| \leqq M_2 \|u\| \, \|v\| \quad \text{for any} \quad u, v \in K \quad \text{and} \quad t \in [0, T], \tag{7.3}$$

$$\|P(t)\| \leqq M_2, \quad \|Q(t)\| \leqq M_2, \quad \|\dot{P}(t)\| \leqq M_2, \quad \|\dot{Q}(t)\| \leqq M_2 \quad \text{for any} \quad t \in [0, T], \tag{7.4}$$

where $\dot{P}(t) = dP(t)/dt$ and $\dot{Q}(t) = dQ(t)/dt$. In this section we use the notation C to denote constants which depend only on M_0, M_1, M_2 and δ.

Let us define the operator $A(t)$ for each $t \in [0, T]$ in the following manner:

$$\left.\begin{array}{l} u \in V(t) \text{ belongs to } D(A(t)) \text{ and } A(t)u = f \in H \\ \text{if } a(t; u, v) = (f, v) \text{ for each } v \in V(t) . \end{array}\right\} \tag{7.5}$$

In the terminology of [2], $A(t)$ is the regularly accretive operator associated with the regular sesquilinear form $a(t; u, v)$ with domain

$V(t)$. The following lemmas are direct consequences of this remark (see Theorems 2.1, 2.2 of [2]).

Lemma 7.1. *For each* $t\in[0, T]$, $-A(t)$ *is the infinitesimal generator of an analytic semi-group of bounded linear operators on* H. *The resolvent set* $\rho(-A(t))$ *of* $-A(t)$ *contains some fixed angular domain* $\Sigma=\{\lambda\,;\,\arg|\lambda|\leqq\pi/2+\theta\}$, *where* θ *is an angle with* $0<\theta<\pi/2$. *Furthermore, there exists a constant* C *such that*

$$|(\lambda I+A(t))^{-1}|\leqq C/|\lambda| \tag{7.6}$$

for any $\lambda\in\Sigma$ *and* $t\in[0, T]$.

Lemma 7.2. *Let* v *and* g *be elements in* $V(t)$ *and* H *respectively. Then* $v\in D(A(t)^*)$ *and* $A(t)^*v=g$ *if and only if we have* $a(t\,;u, v)=(u, g)$ *for any* $u\in V(t)$.

Lemma 7.3. *For any* $f, g\in H$ *and* $t, s\in[0, T]$, *we have*

$$||A(t)^{-1}f||\leqq C|f|, \quad ||A(t)^{*-1}g||\leqq C|g|, \tag{7.7}$$

$$||A(t)^{-1}f-A(s)^{-1}f||\leqq C|t-s|\,|f|, \tag{7.8}$$

$$||A(t)^{*^{-1}}g-A(s)^{*^{-1}}g||\leqq C|t-s|\,|g|. \tag{7.9}$$

Proof. Let u be any element in $D(A(t))$. Then by definition we have

$$a(t\,;u, v) = (A(t)u, v)$$

for any $v\in V(t)$. Putting $v=u(\in V(t))$, we have by (7.2)

$$\delta||u||^2\leqq \mathrm{Re}\,a(t\,;u, u) = \mathrm{Re}\,(A(t)u, u)\leqq|A(t)u|\,|u|\leqq M_0|A(t)u|\,||u||,$$

showing that (7.7) holds with $C=M_0/\delta$.

Before proving (7.8) and (7.9) we notice that

$$P(t)A(t)^{-1} = A(t)^{-1} \quad\text{and}\quad Q(t)A(t)^{*^{-1}} = A(t)^{*^{-1}}$$

hold for any $t\in[0, T]$ because $D(A(t))$ and $D(A(t)^*)$ are both subsets of $V(t)$. Let v be an arbitrary element in $V(t)$. Then by the above remark

$$a(t\,;A(t)^{-1}f-P(t)A(s)^{-1}f, v) = a(t\,;A(t)^{-1}f, v)$$
$$-a(t\,;(P(t)-P(s))A(s)^{-1}f, v)-a(t\,;A(s)^{-1}f, v).$$

Noting that $a(t\,;A(t)^{-1}f, v)=(f, v)=a(s\,;A(s)^{-1}f, v)$, we see that the right member is equal to

$$-a(t\,;(P(t)-P(s))A(s)^{-1}f, v)-\{a(t\,;A(s)^{-1}f, v)-a(s\,;A(s)^{-1}f, v)\}.$$

Therefore

$$\begin{aligned}&\operatorname{Re} a(t; A(t)^{-1}f-P(t)A(s)^{-1}f, v)\\ &\leqq M_2\|(P(t)-P(s))A(s)^{-1}f\|\,\|v\|+M_1|t-s|\,\|A(s)^{-1}f\|\,\|v\|\\ &\leqq (M_2^2+M_1)|t-s|\,\|A(s)^{-1}f\|\,\|v\|\leqq C|t-s|\,|f|\,\|v\|.\end{aligned}$$

As v was an arbitrary element in $V(t)$, we can set $v=A(t)^{-1}f-P(t)A(s)^{-1}f$. Then,

$$\delta\|v\|^2\leqq \operatorname{Re} a(t; v, v)\leqq C|t-s|\,|f|\,\|v\|,$$

and hence

$$\|A(t)^{-1}f-P(t)A(s)^{-1}f\|\leqq C|t-s|\,|f|. \tag{7.10}$$

Using (7. 10) together with

$$\begin{aligned}&\|P(t)A(s)^{-1}f-A(s)^{-1}f\| = \|(P(t)-P(s))A(s)^{-1}f\|\\ &\leqq C|t-s|\,|f|\end{aligned}$$

we obtain (7. 8). The proof of (7. 9) is similar.

Theorem 7. 1. *Under the assumptions* (K. 1), (K. 2), (K. 3) *and* (K. 4), *we have*

$$\left|\frac{\partial}{\partial t}(\lambda I+A(t))^{-1}\right|\leqq\frac{C}{|\lambda|^{1/2}} \tag{7.11}$$

for any $\lambda\in\Sigma$ *and* $t\in[0, T]$.

REMARK. The right member of (7. 11) may be replaced by $C/|\lambda|$ if $P(t)=Q(t)=I_K$ (identity operator of K).

Proof. First, let us notice the relation

$$\begin{aligned}&(A(t)^{-1}f-A(s)^{-1}f, g) = -a(t; A(t)^{-1}f, A(s)^{*^{-1}}g)+a(s; A(t)^{-1}f, A(s)^{*^{-1}}g)\\ &+((P(t)-P(s))A(t)^{-1}f, g)+(f, (Q(t)-Q(s))A(s)^{*^{-1}}g) \qquad (7.12)\\ &-a(t; A(t)^{-1}f, (Q(t)-Q(s))A(s)^{*^{-1}}g)-a(s; (P(t)-P(s))A(t)^{-1}f, A(s)^{*^{-1}}g).\end{aligned}$$

This can be verified from the following relations;

$$\begin{aligned}&a(t; A(t)^{-1}f, A(s)^{*^{-1}}g) = a(t; A(t)^{-1}f, Q(t)A(s)^{*^{-1}}g)\\ &+a(t; A(t)^{-1}f, (I-Q(t))A(s)^{*^{-1}}g)\\ &=(f, Q(t)A(s)^{*^{-1}}g)+a(t; A(t)^{-1}f, (Q(s)-Q(t))A(s)^{*^{-1}}g),\\ &a(s; A(t)^{-1}f, A(s)^{*^{-1}}g)\\ &=a(s; P(s)A(t)^{-1}f, A(s)^{*^{-1}}g)+a(s; (I-P(s))A(t)^{-1}f, A(s)^{*^{-1}}g)\\ &=(P(s)A(t)^{-1}f, g)+a(s; (P(t)-P(s))A(t)^{-1}f, A(s)^{*^{-1}}g), \quad \text{etc.}\end{aligned}$$

Next, we devide both sides of (7.12) by $t-s$ and then let $s\to t$. We can easily calculate the limit of each term on the right member. For example

$$\left|\frac{a(t\,;A(t)^{-1}f,\,A(s)^{*^{-1}}g)-a(s\,;A(t)^{-1}f,\,A(s)^{*^{-1}}g)}{t-s}-\dot{a}(t\,;A(t)^{-1}f,\,A(t)^{*^{-1}}g)\right|$$
$$\leqq\left|\frac{a(t\,;A(t)^{-1}f,\,(A(s)^{*^{-1}}-A(t)^{*^{-1}})g)-a(s\,;A(t)^{-1}f,\,(A(s)^{*^{-1}}-A(t)^{*^{-1}})g)}{t-s}\right|$$
$$+\left|\frac{a(t\,;A(t)^{-1}f,\,A(t)^{*^{-1}}g)-a(s\,;A(t)^{-1}f,\,A(t)^{*^{-1}}g)}{t-s}-\dot{a}(t\,;A(t)^{-1}f,\,A(t)^{*^{-1}}g)\right|.$$

By (K. 3) and Lemma 7.3, the first term is domainated by

$$C|t-s|\,\|A(t)^{-1}f\|\,|g|\leqq C|t-s|\,|f|\,|g|,$$

and tends to 0 for $s\to t$, and so does the second term. Similarly,

$$\left\|\frac{Q(t)-Q(s)}{t-s}A(s)^{*^{-1}}g-\dot{Q}(t)A(t)^{*^{-1}}g\right\|$$
$$\leqq\left\|\frac{Q(t)-Q(s)}{t-s}(A(s)^{*^{-1}}g-A(t)^{*^{-1}}g)\right\|+\left\|\left(\frac{Q(t)-Q(s)}{t-s}-\dot{Q}(t)\right)A(t)^{*^{-1}}g\right\|$$
$$\leqq C|t-s|\,|g|+\left\|\left(\frac{Q(t)-Q(s)}{t-s}-\dot{Q}(t)\right)A(t)^{*^{-1}}g\right\|\to 0.$$

Dealing analogously with the remaining terms, we obtain

$$\left(\frac{d}{dt}A(t)^{-1}f,g\right)=-\dot{a}(t\,;A(t)^{-1}f,\,A(t)^{*^{-1}}g)+(\dot{P}(t)A(t)^{-1}f,\,g)+(f,\,\dot{Q}(t)A(t)^{*^{-1}}g)$$
$$-a(t\,;A(t)^{-1}f,\,\dot{Q}(t)A(t)^{*^{-1}}g)-a(t,\,\dot{P}(t)A(t)^{-1}f,\,A(t)^{*^{-1}}g)\,. \tag{7.13}$$

Therefore, for any $\lambda\in\Sigma$ we get

$$\left(\frac{\partial}{\partial t}(\lambda I+A(t))^{-1}f,g\right)=\left(A(t)(\lambda I+A(t))^{-1}\frac{dA(t)^{-1}}{dt}A(t)(\lambda I+A(t))^{-1}f,\,g\right)$$
$$=(dA(t)^{-1}/dt\cdot A(t)(\lambda I+A(t))^{-1}f,\,A(t)^*(\bar{\lambda}I+A(t)^*)^{-1}g)$$
$$=-\dot{a}(t\,;(\lambda I+A(t))^{-1}f,\,(\bar{\lambda}I+A(t)^*)^{-1}g)$$
$$+(\dot{P}(t)(\lambda I+A(t))^{-1}f,\,A(t)^*(\bar{\lambda}I+A(t)^*)^{-1}g)$$
$$+(A(t)(\lambda I+A(t))^{-1}f,\,\dot{Q}(t)(\bar{\lambda}I+A(t)^*)^{-1}g)$$
$$-a(t\,;(\lambda I+A(t))^{-1}f,\,\dot{Q}(t)(\bar{\lambda}I+A(t)^*)^{-1}g)$$
$$-a(t\,;\dot{P}(t)(\lambda I+A(t))^{-1}f,\,(\bar{\lambda}I+A(t)^*)^{-1}g)\,,$$

and so

$$\begin{aligned}&|((\partial/\partial t)(\lambda I+A(t))^{-1}f, g)| \leqq C\{\|(\lambda I+A(t))^{-1}f\| \, \|(\bar{\lambda} I+A(t)^*)^{-1}g\| \\ &+\|(\lambda I+A(t))^{-1}f\| \, |A(t)^*(\bar{\lambda} I+A(t)^*)^{-1}g| \\ &+|A(t)(\lambda I+A(t))^{-1}f| \, \|(\bar{\lambda} I+A(t)^*)^{-1}g\|\} . \end{aligned} \tag{7.14}$$

Next we notice that

$$\|(\lambda I+A(t))^{-1}f\| \leqq C|\lambda|^{-1/2}|f| , \tag{7.15}$$

$$\|(\bar{\lambda} I+A(t)^*)^{-1}g\| \leqq C|\lambda|^{-1/2}|g| . \tag{7.16}$$

The inequality (7.15) follows from

$$\begin{aligned}&\delta\|(\lambda I+A(t))^{-1}f\|^2 \leqq \operatorname{Re} a(t ; (\lambda I+A(t))^{-1}f, (\lambda I+A(t))^{-1}f) \\ &= \operatorname{Re} (A(t)(\lambda I+A(t))^{-1}f, (\lambda I+A(t))^{-1}f) \leqq C|\lambda|^{-1}|f|^2 ,\end{aligned}$$

and similarly for (7.16).

By (7.14), (7.15) and (7.16), we obtain

$$|((\partial/\partial t)(\lambda I+A(t))^{-1}f, g)| \leqq C|\lambda|^{-1/2}|f| \, |g| ,$$

which completes the proof of the theorem.

In order that $dA(t)^{-1}/dt$ be Hölder continuous, we must make some additional assumptions, which we state below.

(K. 5) $\dot{P}(t)$ and $\dot{Q}(t)$ are Hölder continuous in $t\in[0, T]$:

$$\|\dot{P}(t)-\dot{P}(s)\| \leqq M_3|t-s|^{\alpha} , \quad \|\dot{Q}(t)-\dot{Q}(s)\| \leqq M_3|t-s|^{\alpha} . \tag{7.17}$$

(K. 6) $\dot{a}(t ; u, v)$ is Hölder continuous in $t\in[0, T]$ for any $u, v\in K$:

$$|\dot{a}(t ; u, v)-\dot{a}(s ; u, v)| \leqq M_4|t-s|^{\alpha} \|u\| \, \|v\| .$$

In the remaining part of this section, we denote by C constants which depend only on M_0, M_1, M_2, M_3, M_4, δ and α.

Theorem 7.2. *Under the assumptions* (K. 1) $\sim$ (K. 6) $dA(t)^{-1}/dt$ *is Hölder continuous in* $t\in[0, T]$:

$$|dA(t)^{-1}/dt-dA(s)^{-1}/ds| \leqq C|t-s|^{\alpha} \tag{7.18}$$

Proof. By (7.13),

$$\begin{aligned}&((dA(t)^{-1}/dt-dA(s)^{-1}/ds)f, g) \\ &= -\{\dot{a}(t ; A(t)^{-1}f, A(t)^{*^{-1}}g)-\dot{a}(s ; A(s)^{-1}f, A(s)^{*^{-1}}g)\} \\ &+(\dot{P}(t)A(t)^{-1}f-\dot{P}(s)A(s)^{-1}f, g)+(f, \dot{Q}(t)A(t)^{*^{-1}}g-\dot{Q}(s)A(s)^{*^{-1}}g) \\ &-\{a(t ; A(t)^{-1}f, \dot{Q}(t)A(t)^{*^{-1}}g)-a(s ; A(s)^{-1}f, \dot{Q}(s)A(s)^{*^{-1}}g)\} \\ &-\{a(t ; \dot{P}(t)A(t)^{-1}f, A(t)^{*^{-1}}g)-a(s ; \dot{P}(s)A(s)^{-1}f, A(s)^{*^{-1}}g)\} .\end{aligned}$$

The absolute value of the first term is estimated by

$$\begin{aligned}&|\dot{a}(t\,;A(t)^{-1}f, A(t)^{*^{-1}}g)-\dot{a}(s\,;A(s)^{-1}f, A(s)^{*^{-1}}g)|\\ &\leqq|\dot{a}(t\,;A(t)^{-1}f, A(t)^{*^{-1}}g)-\dot{a}(s\,;A(t)^{-1}f, A(t)^{*^{-1}}g)|\\ &+|\dot{a}(s\,;A(t)^{-1}f-A(s)^{-1}f, A(t)^{*^{-1}}g)|\\ &+|\dot{a}(s\,;A(s)^{-1}f, A(t)^{*^{-1}}g-A(s)^{*^{-1}}g)|\leqq C|t-s|^{\alpha}|f|\,|g|.\end{aligned}$$

Using

$$\begin{aligned}&\|\dot{P}(t)A(t)^{-1}f-\dot{P}(s)A(s)^{-1}f\|\\ &\leqq\|(\dot{P}(t)-\dot{P}(s))A(t)^{-1}f\|+\|\dot{P}(s)(A(t)^{-1}f-A(s)^{-1}f)\|\leqq C|t-s|^{\alpha}|f|,\end{aligned}$$

we can obtain similar estimates for the remaining terms. Thus, we get

$$|((dA(t)^{-1}/dt-dA(s)^{-1}/ds)f, g)|\leqq C|t-s|^{\alpha}|f|\,|g|,$$

which completes the proof of the theorem.

Summing up, we have proved

Theorem 7.3. *Suppose that* (K.1)$\sim$(K.4) *hold. Then the assumptions* (E.1)$\sim$(E.3) *are all satisfied for* $A(t)$. *If we make the additional assumptions of* (K.5) *and* (K.6), *then* (E.4) *is also satisfied. Thus, we can apply all the results in* § 3 *or* § 4 *to the equation*

$$du(t)/dt+A(t)u(t)=f(t)$$

under the assumptions (K.1)$\sim$(K.4) *or* (K.1)$\sim$(K.6).

Remark. This theorem strengthens, in some respects, the results of Lions [6], Chapter VII.

§ 8. Example of the spaces $V(t)$

In this section we continue to use the notations in Lions [6] as in the last section. Let Ω be an open set in R^n whose boundary is a sufficiently smooth $(n-1)$-dimensional manifold. Let $L^2(\Omega)$ be the space of all square integrable complex-valued functions in Ω provided with the usual inner product. Let $H^m(\Omega)$ be the space of all complex-valued functions which belong to $L^2(\Omega)$ together with all of their distribution derivatives of order up to m. We provide $H^m(\Omega)$ with the usual inner product. Then $L^2(\Omega)$ and $H^m(\Omega)$ are both Hilbert spaces.

For any $v\in H^m(\Omega)$, we can determine the boundary values of its normal derivatives of order up to $m-1$ in the usual manner. We denote by $\gamma_j u$ the value on Γ of the j-th normal derivative of u

$$\gamma_j u=(\partial/\partial n)^j u|_{\Gamma}.$$

$\gamma_j u$ belongs to $H^{m-j-1/2}(\Gamma)$ in the notations of Lions [6]. For any $0 \leqq j \leqq m-1$, γ_j defines a linear bounded mapping on $H^m(\Omega)$ into $H^{m-j-1/2}(\Gamma)$.

As H and K in the last section we choose $L^2(\Omega)$ and $H^m(\Omega)$ respectively, and as $V(t)$ the space of all u in $H^m(\Omega)$ satisfiying

$$\gamma_j u = \sum_{k=0}^{k_0} a_{jk}(t)\gamma_k u\,, \quad 0 \leqq k_0 \leqq m-1,\ j \in J \subset [k_0+1, \cdots, m-1]\,, \tag{8.2}$$

where each $a_{jk}(t)$ is assumed to satisfy

$$a_{jk}(t) \in L(H^{m-k-1/2}(\Gamma)\,;\ H^{m-j-1/2}(\Gamma)) \tag{8.3}$$

and

$$\left.\begin{array}{l} (a_{jk}(t)\varphi, \psi)_{H^{m-j-1/2}(\Gamma)} \quad \text{is continuonusly differentiable} \\ \text{in} \quad t \in [0, T] \quad \text{for each fixed} \quad \varphi \in H^{m-k-1/2}(\Gamma) \quad \text{and} \\ \psi \in H^{m-j-1/2}(\Gamma)\,. \end{array}\right\} \tag{8.4}$$

Theorem 8.1 (Lions). *Under the assumptions made above the orthogonal projection $P(t)$ on K onto $V(t)$ satisfies*:

(i) *$P(t)u$ is continuous in the strong topology of K in $[0, T]$ for each fixed $u \in H^m(\Omega)$,*

(ii) *$h^{-1}(P(t+h)u - P(t)u) \rightarrow \dot{P}(t)$ weakly in K as $h \rightarrow 0$,*

(iii) *$\dot{P}(t)u$ is continuous in the weak topology of K in $[0, T]$.*

By examining the proof of the above theorem, we can easily prove

Theorem 8.2. *If in addition to the assumption in Theorem 8.1 we assume that each $\dot{a}_{jk}(t)$ is Hölder continuous in t:*

$$\|\dot{a}_{jk}(t) - \dot{a}_{jk}(s)\| \leqq M_5 |t-s|^{\alpha}\,,$$

then $\dot{P}(t)$ is Hölder continuous in t:

$$\|\dot{P}(t) - \dot{P}(s)\| \leqq C|t-s|^{\alpha}\,.$$

Using Theorem 8.2, we can give a partial improvement to a result in Lions [6], Chapter VIII.

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