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On Alternating Knots

By Kunio MURASUGI

Introduction

Let k be a knot[1] in 3-sphere S^3. Let k be an image of the regular projection of k onto $S^2 \subset S^3$. A knot projection K is said to be *alternating* if and only if it is connected and, as one follows along this knot, undercrossings and overcrossings alternate[2]. A knot is said to be *alternating* if it prossesses an alternating projection. There will be, of course, non-alternating knots. In fact, the first proof of their existence was given by Bankwitz in 1930 [3]. We do not know the general method by which we can decide whether or not a given knot projection represents an alternating knot. But some good methods have been found up to present. Recently, R. H. Crowell proved the theorem (cf. Theorem (6. 5) [5]) which much improved the Bankwitz's theorem (cf. Satz, p. 145 [3]). He showed by means of this theorem that seven of eleven non-alternating projections, in the Knot Table at the end of Reidemeister's Knotentheorie [11], represent non-alternating knots.

In a previous paper [9] we gave a necessary condition for a given knot to be alternating by means of its Alexander polynomial (cf. [4], [8]). In the present paper, in order to characterize the Alexander polynomial of the alternating knot, we shall assign a matrix, called the *knot matrix*, to an alternating knot. The relation between the knot matrix and its Alexander polynomial is expressed in Theorem 1. 17 which is the fundamental theorem of the present paper. From this theorem simply follow the main theorems in [4], [7], [8], [9] (Theorem 3. 8, Theorem 3. 13). Theorem 3. 12 is a simple application of Theorem 1. 17 and it plays a particular rôle in § 3.

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1) A *knot* is an oriented polygonal simple closed curve. A *link* of multiplicity μ is the union of μ ordered, pairwise disjoint knots. In the present paper we do not distinct exactly between links and knots, except the cases 3.7 and 3.8. Thus, by a *knot* (of multiplicity μ) is meant an ordinary knot or link according to $\mu=1$ or $\mu>1$.

2) For any knot we may select a "point at infinity" $\infty \in S^3-k$ and consider a Cartesian coordinate system $R\times R\times R=S^3-\infty$. The projection $p: S^3\rightarrow S^2$ is defined by $p(\infty)=\infty$ and $p(x, y, z)=(x, y)$. For each double point $p(a)=p(b)$, one of a and b with the larger z-coordinate is called the *overcrossing* and the other the *undercrossing*.

As an application of these theorems, it will be shown, moreover, that eleven non-alternating projections in the Knot Table [11] really represent non-alternating knots.

§ 1. Knot Matrix

In this section we shall assign a matrix to an alternating knot.

1.1. Let k be a knot and K be a regular projection of k. Let K be oriented by the orientation induced by that of k. Let K have n double points $D_1, D_2, \cdots, D_n$. Throughout the present paper, we may assume that

(1.1) *K has no trivial double point.*

By a *trivial double point* is meant a double point D as is shown in Fig. 1.

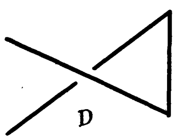

Fig. 1.

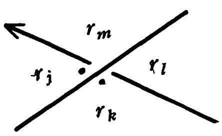

Fig. 2.

K divides S^2 into $n+2$ regions $r_0, r_1, \cdots, r_{n+1}$. At each double point D_i, just four corners of four regions, say, r_j, r_k, r_l and r_m, meet. Two corners among these four corners are marked with *dots* as is shown in Fig. 2.

The segments of K connecting two consecutive double points are called *sides* of K.

Now, let us divide K into some oriented loops[3], called the *standard loops,* as follows. Imagine an insect crawling along K in the positive direction of K. This insect must always turn to the left or to the right at a double point in the positive direction of K. Then, it will traverse a loop L, called a standard loop. It is clear that

(1.2) *Each side of K is contained in one and only one standard loop.*

If L bounds a region r_i, we say L is of the *first kind* and r_i is bounded by L. Otherwise, L is of *the second kind.* If K has no standard loop of the second kind, the alternating knot k is called a *special alternating* one. From the rule of the marking with dots, we have immediately,

(1.3) *The corners of the regions bounded by the standard loops of*

3) By a loop is meant a simple closed curve.

the first kind are either all dotted or all undotted. The converse is also true (cf. Lemma 6. 3 [8]).

Let m be the number of the standard loops of the second kind of K. We may assume that these m loops are disjoint. (We have only to deform slightly some of these loops, if necessary.) Then, it follows

(1. 4) *m standard loops of the second kind divide S^2 into $m+1$ domains*[4] (cf. Lemma 2. 4 [8]).

By the *genus* of a domain is meant the number of the standard loops of the second kind bounding the domain minus 1. Then it is easy to show that

(1. 5) *The sum of the genera of $m+1$ domains is equal to $m-1$.*

Now we can introduce the rules on the numbering of m standard loops of the second kind $C_1, \cdots, C_m$ and $m+1$ domains, $E_1, E_2, \cdots, E_{m+1}$ in such a way that the following conditions hold (cf. § 3 [8]):

(1. 6) *If $\dot{E}_i$[5]$=C_{i_0} \cup C_{i_1} \cup \cdots \cup C_{i_j}$, then $i_0=i$ and $i_1, \cdots, i_j < i$, for $i=1, \cdots, m$.*

C_i is called the *outer boundary* of E_i. We denote the remaining domain by E_{m+1}, and the outer boundary of it is defined as follows: Let $\dot{E}_{m+1}=C_{\lambda_1} \cup \cdots \cup C_{\lambda_p}$. Denoting $\lambda=\max(\lambda_1, \cdots, \lambda_p)$, C_λ is the outer boundary of E_{m+1}.

Let us denote $(E_i \cup \dot{E}_i) \cap K=K_i$.

A double point such that at least two of the four regions meeting at it are contained in E_i is called a double point *contained in K_i* (or simply *in K_i*).

(1. 7) *Each double point is necessarily contained in only one of these K_i.*

By the *sides* of K_i are meant segments of K_i connecting two consecutive double points in K_i. Then it is clear that

(1. 8) *Each K_i is a knot projection of a knot, say k_i, and, in particular, each k_i is special alternating.*

And from (1. 1), we have immediately

(1. 9) *K_i have no trivial double points.*

Moreover, we can prove easily

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4) A *domain* is a connected open subset of S^2.

5) A dot over the symbol denotes the set of boundary points.

Lemma 1.10. *The regions contained in E_i can be classified into two classes, called "black" or "white", in such a way that every side of K_i is the common boundary of a black and a white region and that the region having some sides in common with $\dot{E}_i$ is a white region and that each black region is bounded by a standard loop of the first kind* (cf. Lemma 2.6 [8]).

1.2. The Alexander matrix A of a knot is defined as follows. Let four regions meet at a double point D_i of K as is shown in Fig. 2. Then we assign to D_i a linear equation:

$$D_i(r) = tr_j - tr_k + r_l - r_m = 0 .$$

From (1.1) we see j, k, l and m are different from one another. Then A is defined to be the matrix constructed by all coefficients of these equations. A has n rows and $n+2$ columns. Each row and each column correspond to a double point and a region respectively. Then the determinant of the square matrix obtained from A by striking out two columns corresponding to a pair of regions, which have a side in common, is uniquely determined, freed from the factor $\pm t^{\lambda}$. This is a knot invariant. We call it the *A-polynomial* and denote it by $\Delta(t)$. Hereafter we may assume without loss of generality that the constant term $\Delta(0)>0$. Here we should notice that

(1.11) *If the multiplicity μ of a knot k is one, then the A-polynomial and the ordinary Alexander polynomial are the same. If $\mu>1$, then $\Delta(t) =(1-t)\Delta(t, \cdots, t)$, where $\Delta(t_1, \cdots, t_\mu)$ is the ordinary Alexander polynomial of k.*

Moreover, if the knot projection is separated into two disjoint parts, the A-polynomial is zero (cf. [1]). We have to exclude such a knot, because we treat only the knots whose projection are *connected* as stated in the introduction.

Let k be a special alternating knot. We shall assign a matrix M to k by means of its projection K.

Definition 1.12. *Let $W_1, W_2, \cdots, W_p$ be the white regions in K. Then a matrix $M=(a_{ij})_{i,j=1,2,\cdots,p}$, called the knot matrix, is defined as follows:*

a_{ii} is one half of the number of the double points lying on $\dot{W}$[6].

$-a_{ij}(i \neq j)$ is the number of the dotted corners of W_i at the double points lying on $\dot{W}_i \cap \dot{W}_j$.

Then it is easy to prove the following

6) It is clear that a_{ii} is an integer. (cf. Lemma 3.6 [7])

Theorem 1.13. *Two special alternating knots of s-equivalent knot matrices are equivalent.*

Two matrices M, M' are said to be *s-equivalent* if one can be transformed into the other by applying finitely many times the following operations: To exchange i-th row and j-th row and, at the same time, to exchange i-th column and j-th column $(i, j=1, \cdots, p)$.

EXAMPLE 1. The knot matrix M of the knot as is shown in Fig. 3 is the following:

$$M=\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

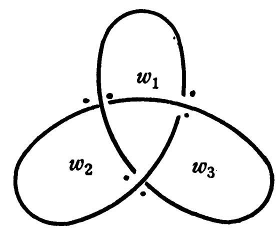

Fig. 3.

(1.14) *M is of the following properties*:

(i) $a_{ii}>0$, $a_{ij}\leqq 0$, $(i, j=1, \cdots, p,\ i\neq j)$,

(ii) $\sum_{i=1}^{p} a_{ij}=0$, $\sum_{j=1}^{p} a_{ij}=0$,

(iii) $|a_{ij}-a_{ji}|\leqq 1$.

Next, we shall assign a matrix to an alternating knot.

Let K have m standard loops of the second kind $C_1, C_2, \cdots, C_m$. Let E_i, K_i be the same as in 1.1. Then

Definition 1.15. *The knot matrix M of k is defined as follows*:

$$M=\begin{pmatrix} M_{11}\ M_{12} & \cdots\ M_{1,m+1} \\ M_{21}\ M_{22} & \cdots\ M_{2,m+1} \\ \cdots & \\ M_{m+1,1}\ M_{m+1,2} & \cdots\ M_{m+1,m+1} \end{pmatrix}$$

(i) *M_{ii} are the special knot matrices assigned to the special alternating knots k_i.*

(ii) *$M_{ij}=(b_{\alpha\beta})$ are defined as follows*:

(a) *If $\dot{E}_i\frown\dot{E}_j=\phi$, $M_{ij}=M_{ji}=0$.*

(b) *Let $\dot{E}_i\frown\dot{E}_j=C_l$, $l=min(i\ j)$.*

(i) *If E_i stands on the right hand side in the positive direction of C_l, then $M_{ij}=0$.*

(ii) *Otherwise, $W_\alpha\subset E_i$, $W_\beta\subset E_j$, $b_{\alpha\beta}=\lambda-\mu$, where λ, μ denote the numbers of the dotted and undotted corners of W_α at all double points in K_i lying on $W_\alpha\frown W_\beta$ respectively.*

EXAMPLE 2. The knot matrix M of the knot as is shown in Fig. 4 is the following:

$$M=\left(\begin{array}{cc|cccc} 1 & -1 & & & & \\ -1 & 1 & & 0 & & \\ \hline 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ -1 & 1 & 0 & 0 & -1 & 1 \end{array}\right)$$

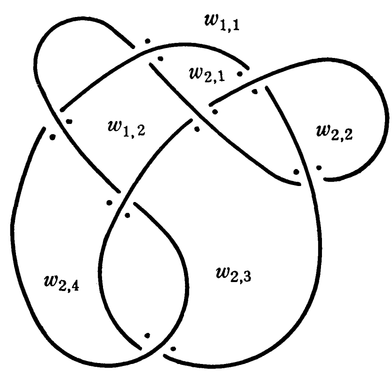

Fig. 4.

(1.16) *M_{ij} $(i \neq j)$ have the following properties*:

(1) *At least one of M_{ij}, or M_{ji} is equal to* 0.

(2) *Each row and each column of M_{ij} contain only two elements different from* 0, *if these row and column are different from* 0. *And one is* 1, *the other is* -1.

(3) $\sum_{\alpha} b_{\alpha\beta}=0, \quad \sum_{\beta} b_{\alpha\beta}=0.$

1.3. We shall introduce the notations on matrices and determinants used in the present paper from now on.

Let $M=(a_{ij})_{i=1,\cdots,n,\ j=1,\cdots,m}$ be a matrix. By $M\begin{pmatrix} i_1 i_2 \cdots i_p \\ j_1 j_2 \cdots j_q \end{pmatrix}$ is denoted the matrix *consisting of* i_1^{th} row, i_2^{th} row, $\cdots$, i_p^{th} row, and j_1^{th} column, j_2^{th} column, $\cdots$, j_q^{th} column, of M:

$$M\begin{pmatrix} i_1 & i_2 \cdots i_p \\ j_1 & j_2 \cdots j_q \end{pmatrix} = \begin{pmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_q} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_q} \\ & & \cdots & \\ a_{i_pj_1} & a_{i_pj_2} & \cdots & a_{i_pj_q} \end{pmatrix}.$$

In particular, we may denote $M\begin{pmatrix} i \\ j \end{pmatrix}$ by $M(i, j)$. By $\tilde{M}\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_q \end{pmatrix}$ is denoted the matrix obtained from M by *striking out* i_1^{th} row, $\cdots$, i_p^{th} row, and j_1^{th} column, $\cdots$, j_q^{th} column :

$$\tilde{M}\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_q \end{pmatrix} = M\begin{pmatrix} r_1 \cdots r_{n-p} \\ s_1 \cdots s_{m-q} \end{pmatrix},$$

where $r_1 < \cdots < r_{n-p}$, $s_1 < \cdots < s_{m-q}$, and $(r_1, \cdots, r_{n-p})$, $(s_1, \cdots, s_{m-q})$ denote the sets of $(n-p)$, $(m-q)$ integers obtained from the sets of n, m integers $(1, 2, \cdots, n)$, $(1, 2, \cdots, m)$ by striking out the sets of p, q integers $(i_1, i_2, \cdots, i_p)$, $(j_1, j_2, \cdots, j_q)$ respectively. By M^t is denoted the *transposed matrix* of M:

$$M^t = (b_{ij})_{i=1,\cdots,m,\ j=1,\cdots,n}, \text{ where } b_{ij} = a_{ji}.$$

Using these notations, the fundamental theorem of the present paper can be stated as follows:

Theorem 1.17. *Let $\Delta(t)$ denote the A-polynomial of an alternating knot. Then,*

$$\pm t^{\lambda}\Delta(t) = det\left\{\tilde{M}\begin{pmatrix} i_1 \; i_2 \cdots i_{m+1} \\ i_1 \; i_2 \cdots i_{m+1}\end{pmatrix} - t\tilde{M}^T\begin{pmatrix} i_1 \; i_2 \cdots i_{m+1} \\ i_1 \; i_2 \cdots i_{m+1}\end{pmatrix}\right\},$$

where (i) λ *is a suitable integer,* (ii) *each* i_p *is chosen arbitrarily such that* $M\begin{pmatrix} i_p \\ i_p \end{pmatrix}$ *is contained in* M_{pp} *for* $p=1, \cdots, m+1$, *and* (iii) M^T *denotes the following matrix:*

$$M^T = \begin{pmatrix} M^t_{11} & -M^t_{21} & -M^t_{31} & \cdots -M^t_{m+1,1} \\ -M^t_{12} & M^t_{22} & -M^t_{32} & \cdots -M^t_{m+1,2} \\ -M^t_{13} & -M^t_{23} & M^t_{33} & \cdots -M^t_{m+1,3} \\ \cdots & & & \\ -M^t_{1,m+1} & -M^t_{2,m+1} & -M^t_{3,m+1} & \cdots \quad M^t_{m+1,m+1} \end{pmatrix}.$$

Proof will be given in 1.4, 1.5.

1.4. Let $W_{i,1}, \cdots, W_{i,p_i}$ and $B_{i,1}, \cdots, B_{i,q_i}$ denote the white and black regions in K_i respectively. Let $D_{i,1}, \cdots, D_{i,n_i}$ denote the double points in K_i and let s_i denote the genera of $E_i (i=1, \cdots, m+1)$. Then we see immediately,

$$p_i + q_i + s_i - 1 = n_i. \tag{1.18}$$

We may assume without loss of generality that

(1.19) (i) $(p_1+q_1)+ \cdots +(p_{i-1}+q_{i-1})+j^{\text{th}}$ *column corresponds to the white region* $W_{i,j}$,

(ii) $(p_1+q_1)+ \cdots +(p_{i-1}+q_{i-1})+p_i+l^{\text{th}}$ *column corresponds to the black region* $B_{i,l}$,

(iii) $n_1+ \cdots +n_{i-1}+r^{\text{th}}$ *row corresponds to* $D_{i,r}$, *for* $i=1, \cdots, m+1$.

(We have only to change the permutation of columns and rows, if necessary.) For the sake of brevity, we say, for example, the column corresponding to $W_{i,j}$ $W_{i,j}$*-column* and say the row corresponding to $D_{i,j}$ $D_{i,j}$*-row*. And we denote, for example, $M\begin{pmatrix}1\\1\end{pmatrix}$ by $M\begin{pmatrix}D_{1,1}\\W_{1,1}\end{pmatrix}$.

Now, let us take a point, called the *center*, from each white region and a point from each black region, and fix them. Moreover, we take a point from each component of the complementary domain of $E_i (i=1, \cdots, m+1)$, and fix them. We call it the *center* of the complementary domain. The subset G_i(or G_i^*) of S^2 obtained by connecting the centers of all the white regions (or all the black regions) with the double points lying on their boundaries will be called the *graph* (or the *dual* graph) of K_i. The center of each white region (or each black region) is called the *vertex* of G_i(or G_i^*) and the segments of G_i(or G_i^*) connecting two consecutive vertices are called the *sides* of G_i(or G_i^*).

Let T be a tree[7] in G_i. The subset of G_i^* consisting of sides disjoint to T will be called the *dual* of T, denoted by T^*. Then, we see

(1.20) *If T is a maximal tree in G_i, then T^* is also a maximal tree in G_i^*.*

Let T_i be a maximal tree in G_i and let us fix it. Since T_i contains p_i vertices, it contains p_i-1 sides. Then it follows

Lemma 1.21. *There exists 1-1-correspondence φ_i between p_i vertices except one vertex, the center of W_{i,p_i} say, and p_i-1 sides of T_i in such a way that each side of T_i corresponds to one of vertices lying on its side by φ_i. Moreover a correspondence φ_i is unique. In the same way, there exists one and only one $1-1$ correspondence φ_i^* between q_i+s_i+1 vertices except one vertex, one of the centers of components of the complementary domain S^2-E_i, and q_i+s_i sides of T_i^*.*

Since this lemma will be proved by induction on the number of the sides in T_i, we omit the details of proof.

φ_i naturally gives rise to $1-1$ correspondence between $W_{i,1},\cdots,W_{i,p_i-1}$, and $D_{i,1},\cdots,D_{i,p_i-1}$, and φ_i^* gives rise to $1-1$ correspondence between $B_{i,1},\cdots,B_{i,q_i}$ and s_i components of complementary domain $F_{i,1},\cdots,F_{i,s_i}$ and q_i+s_i double points $D_{i,p_i},\cdots,D_{i,n_i}$ $(n_i=p_i+q_i+s_i-1)$. We may assume that $\varphi_i(W_{i,\lambda})=D_{i,\lambda}$, $\varphi_i^*(B_{i,\mu})=D_{i,p_i+\mu-1}$, $\varphi_i^*(F_{i,\nu})=D_{i,p_i+q_i+\nu-1}$. (We may have only to change the numbering of the double points, if necessary.) We should note that there is no double point corresponding to $m+1$ white regions $W_{1,p_1},\cdots,W_{m+1,p_{m+1}}$. We may assume, moreover, that W_{r,p_r} and $W_{m+1,p_{m+1}}$ have a side in common, because the outer boundary C_r of E_r for some r is also the outer boundary of E_{m+1}. See 1.1.

1.5. We say the transformation from a matrix $M=(a_{ij})_{i=1,\cdots,n,\ j=1,\cdots,m}$ to a matrix M':

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7) By a *tree* is meant the connected subset of G which contains no loop. A tree is called *maximal* if it contains all vertices of G.

$$M' = \begin{array}{cl}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \\ \sum_{i=1}^{n} \alpha_{1i} a_{i1} & \sum_{i=1}^{n} \alpha_{1i} a_{i2} & \cdots & \sum_{i=1}^{n} \alpha_{1i} a_{im} \\ \vdots & & & \\ \sum_{i=1}^{n} \alpha_{2i} a_{i1} & \sum_{i=1}^{n} \alpha_{2i} a_{i2} & \cdots & \sum_{i=1}^{n} \alpha_{2i} a_{im} \\ \vdots & & & \\ \sum_{i=1}^{n} \alpha_{li} a_{i1} & \sum_{i=1}^{n} \alpha_{li} a_{i2} & \cdots & \sum_{i=1}^{n} \alpha_{li} a_{im} \\ \cdots & & & \\ a_{n1} & a_{n2} & & a_{nm} \end{pmatrix} & \begin{array}{l} \\ \\ j_1^{\text{th}} \\ \\ j_2^{\text{th}} \\ \\ j_l^{\text{th}} \\ \\ \\ \end{array}\end{array}$$

is obtained by adding essentially α_{11} *times of the first row,* $\cdots$ *,* α_{1n} *times of the* n^{th} *row to* j_1^{th} *row and by adding essentially* α_{21} *times of the first row,* $\cdots$ *,* α_{2n} *times of the* n^{th} *row to* j_2^{th} *row,* $\cdots$ Then it will be easily proved

(1.22) *The essential addition is the elementary transformation, if* $\alpha_{1,j_1}, \cdots, \alpha_{l,j_l}$ *are all* ± 1.

By *the elementary transformation* is meant the transformation obtained by applying a finite number of times the following three operations E_i(or E_i^*):

$E_1(E_1^*)$: To exchange two rows (or two columns).

$E_2(E_2^*)$: To add λ times of a row (or a column) to the other row (or column), λ being integer.

$E_3(E_3^*)$: To multiply a row (or a column) by ± 1.

Two matrices are called *equivalent* if one can be transformed into the other by elementary transformations.

We shall transform the Alexander matrix A.

Let $D_{i,j_1}, \cdots, D_{i,j_\lambda}$, in which $D_{i,j}$ is contained, be double points in K_i lying on $\dot{W}_{i,j}$ for $j=1, \cdots, p_i-1$. Then we see

(1.23) λ *is even, and just one half of all corners of* W_{ij} *at these double points are dotted.* (cf. Lemma 6.3 [8])

Moreover, we have

(1.24) *The signs of all elements in* $A\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{i,1} \cdots W_{i,p_i} \end{pmatrix}$ *are constant.* (cf. Lemma 3.1 [7])

Hence we can denote these elements different from 0 by $\varepsilon_i t$ or $\varepsilon_i \cdot 1$, $\varepsilon_i = \pm 1$. Let us assume, then, $A(D_{i,j_1}, W_{i,j}) = A(D_{i,j_2}, W_{i,j}) = \cdots = A(D_{i,j_\mu}, W_{i,j}) = \varepsilon_i \cdot 1$ and $A(D_{i,j_{\mu+1}}, W_{i,j}) = A(D_{i,j_{\mu+2}}, W_{i,j}) = \cdots = A(D_{i,j_\lambda}, W_{i,j}) = \varepsilon_i t$, where $\mu = \lambda/2$. Now we *essentially* add ε_i times of D_{i,j_1}-row, $\cdots$, D_{i,j_μ}-row to $D_{i,j}$-row, and *essentially* add $-\varepsilon_i$ times of $D_{i,j_{\mu+1}}$-row, $\cdots$, D_{i,j_λ}-row to $D_{i,j}$-row. Thus A is transformed into a matrix A'. Then it follows

(1.25) *A' is equivalent to A.*

Moreover, we have

(1.26) $A'(D_{i,j}, W_{i,j}) = a_{i_j i_j}(1-t)$, $A'(D_{i,j}, W_{i,l}) = a_{i_j i_l} - t a_{i_l i_j}$,

where $a_{i_j i_j}$ denotes the number of the undotted corners of $W_{i,j}$ at all double points in K_i lying on $\dot{W}_{i,j}$, and where $a_{i_j i_l}$ denotes the number of the dotted corners of $W_{i,j}$ at all double points in K_i lying on $\dot{W}_{i,l} \cap \dot{W}_{i,j}$.

Thus it follows

(1.27) $$A'\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{i,1} \cdots W_{i,p_i-1} \end{pmatrix} = \tilde{M}_{ii}\begin{pmatrix} p_i \\ p_i \end{pmatrix} - t\tilde{M}^t_{ii}\begin{pmatrix} p_i \\ p_i \end{pmatrix}.$$

Moreover, we can prove

(1.28) $$A'\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ B_{i,1} \cdots B_{i,q_i} \end{pmatrix} = 0, \quad \textit{for } i = 1, \cdots, m+1.$$

Proof. Because the common part of a black region and a white region has only to contain the consecutive two double points, q.e.d.

Since double points lying on the boundary of a black region of E_i are contained in K_i, we have obviously

(1.29)
$$A'\begin{pmatrix} D_{1,1} \cdots D_{1,n_1} \cdots D_{i-1,1} \cdots D_{i-1,n_{i-1}}, D_{i+1,1} \cdots D_{i+1,n_{i+1}} \cdots D_{m+1,1} \cdots D_{m+1,n_{m+1}} \\ B_{i,1} \cdots B_{i,q_i} \end{pmatrix} = 0,$$
$$\textit{for } i = 1, \cdots, m+1.$$

Next, we shall decide the form of the matrix

$$P'_{ij} = A'\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{j,1} \cdots W_{j,p_j} \end{pmatrix} \quad (i \neq j).$$

Since P'_{ij} is transformed from P_{ij}:

$$P_{ij} = A\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{j,1} \cdots W_{j,p_j} \end{pmatrix},$$

we shall first decide the form of P_{ij}. From the definition, we have

(1.30) $$\textit{If } \dot{E}_i \cap \dot{E}_j = \phi, \qquad A\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{j,1} \cdots W_{j,p_i} \end{pmatrix} = 0.$$

Let $\dot{E}_i \cap \dot{E}_j = C_l$, $l = \min(i, j)$. Then elements of P_{ij} are given as follows:

(1.31) (i) *If $D_{i,\lambda}$ does not lie on $\dot{W}_{j,\mu}$, then*
$$A(D_{i,\lambda}, W_{j,\mu}) = 0.$$

(ii) *The case where $D_{i,\lambda}$ lies on $\dot{W}_{j,\mu}$.*

(a) *If E_i stands on the left hand side in the positive direction on C_l,*

$$A(D_{i,\lambda}, W_{j,\mu}) = \varepsilon_j \cdot 1 .$$

(b) *Otherwise,* $A(D_{i,\lambda}, W_{j,\mu}) = \varepsilon_j \cdot t.$

Hence, we can decide the elements of P'_{ij} as follows:

(1.32) *Let $D_{i,\xi}, \cdots$ be the double points except $D_{i,\lambda}$ in K_i lying on $\dot{W}_{i,\mu} \frown \dot{W}_{j,\mu}$.*

(i) *If E_i stands on the left hand side in the positive direction on C_l,* $A'(D_{i,\lambda}, W_{j,\mu}) = A(D_{i,\lambda}, W_{j,\mu}) + \varepsilon_i A(D_{i,\xi}, W_{j,\mu}) + \cdots$.

(ii) *Otherwise,* $A'(D_{i,\lambda}, W_{j,\mu}) = A(D_{i,\lambda}, W_{j,\mu}) - \varepsilon_i A(D_{i,\xi}, W_{j,\mu}) - \cdots$.

Thus, we have from (1.32),

(1.33)

(i) *If E_i stands on the left hand side in the positive direction on C_l, then* $A'(D_{i,\lambda}, W_{j,\mu}) = \alpha - \beta$,

(ii) *Otherwise,* $A'(D_{i,\lambda}, W_{j,\mu}) = (\alpha - \beta)t$,

where α, β denote the numbers of the dotted and undotted corners of $\dot{W}_{i,\lambda}$ at all double points in K_i lying on $\dot{W}_{i,\lambda} \frown \dot{W}_{j,\mu}$ respectively and $\mu \neq p_i$.

Thus it follows

$$(1.34) \qquad A'\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{j,1} \cdots W_{j,p_j-1} \end{pmatrix} = \tilde{M}_{ij}\begin{pmatrix} p_i \\ p_j \end{pmatrix} + t\tilde{M}^t_{ji}\begin{pmatrix} p_j \\ p_i \end{pmatrix}, \qquad (i \neq j).$$

Let A'' be a matrix obtained from A' by adding all $W_{i,1}$-column, $\cdots$ W_{i,p_i-1}-column to W_{i,p_i}-column for $i=1, \cdots, m+1$. Then it is clear that

$$(1.35) \qquad A''\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{i,p_i} \end{pmatrix} = 0 .$$

Moreover, we have

(1.36)

$$A''\begin{pmatrix} D_{1,1} \cdots D_{1,p_i-1} \cdots D_{j-1,1} \cdots D_{j-1,p_{j-1}-1}, D_{j+1,1} p_{j+1}-1 \cdots D_{m+1,1} \cdots D_{m+1,p_{m+1}-1} \\ W_{j,p_j} \end{pmatrix} = 0 .$$

Because $A''(D_{i,\lambda}, W_{j,p_j}) = \sum_{\mu=1}^{p_j} A'(D_{i,\lambda}, W_{j,\mu}) = 0$.

Let $\dot{E}_i$ consist of the outer boundary and s_i standard loops of the second kind $C_{i_1}, \cdots, C_{i_{s_i}}$. C_{i_l} are the outer boundaries of E_{i_l}. Then let A''' be a matrix obtained from A'' by arranging s_i columns $W_{i_1,p_{i_1}}$-column, $\cdots$, $W_{i_{s_i},p_{s_i}}$-column, after B_{i,q_i}-column. We should note that W_{r,p_r}-column and $W_{m+1,p_{m+1}}$-column do not move. Let us denote

$$\tilde{A}'''(W_{r,p_r}, W_{m+1,p_{m+1}}) = A_0 ,$$

Then it follows by noting that φ_i^* in Lemma 1.21 is unique,

$$(1.37) \qquad det\ A_0\begin{pmatrix} D_{i,p_i} \cdots D_{i,n_i} \\ B_{i,1} \cdots B_{i,q_i} W_{i_1,p_{i_1}} \cdots W_{i_{s_i},p_{i_{s_i}}} \end{pmatrix} = \pm t^{\nu},$$

where ν is a suitable integer.

Moreover, if $g(E_i)=0$, E_i is bounded only by its outer boundary. Hence we have

(1.38) *If the genus $g(E_i)$ of E_i is zero, then*

$$A_0\begin{pmatrix} D_{1,1}\cdots D_{1,p_1-1}\cdots D_{i-1,1}\cdots D_{i-1,p_{i-1}-1}, D_{i+1,1}\cdots D_{i+1,p_{i+1}-1} D_{m+1,1}\cdots D_{m+1,p_{m+1}-1} \\ B_{i,1} \cdots B_{i,q_i} \end{pmatrix} = 0.$$

If $\dot{E}_i(i \neq m+1)$ are bounded by $C_{i_1}, \cdots, C_{i_{s_i}}, C_i$, then $i_1, i_2, \cdots, i_{s_i} < i$, and hence $A_0\begin{pmatrix} D_{j,p_j} \cdots D_{j,n_j} \\ B_{i,1} \cdots B_{i,q_i} W_{i_1,p_{i_1}} \cdots W_{i_{s_i},p_{s_i}} \end{pmatrix} = 0$. Thus we have at last the following:

$$(1.39) \qquad det\ A = det \begin{pmatrix} A_1(t) & B_{12}(t) & \cdots B_{1,m+1}(t) \\ B_{21}(t) & A_2(t) & \cdots B_{2,m+1}(t) \\ & \cdots & \\ B_{m+1,1}(t) & B_{m+1,2}(t) & \cdots A_{m+1}(t) \end{pmatrix}$$

where

$$A_i(t) = A_0\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{i,1} \cdots W_{i,p_i-1} \end{pmatrix} = A'\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{i,1} \cdots W_{i,p_i-1} \end{pmatrix},$$

$$B_{ij}(t) = A_0\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{j,1} \cdots W_{j,p_j-1} \end{pmatrix} = A'\begin{pmatrix} D_{i,1} \cdots D_{i,p_i-1} \\ W_{j,1} \cdots W_{j,p_j-1} \end{pmatrix}.$$

Thus the proof of Theorem 1.17 is complete.

1.6.

Corollary 1.40. *If we set*

$$D(t) = det\left\{\tilde{M}\begin{pmatrix} i_1 \cdots i_{m+1} \\ i_1 \cdots i_{m+1} \end{pmatrix} - t\tilde{M}^T\begin{pmatrix} i_1 \cdots i_{m+1} \\ i_1 \cdots i_{m+1} \end{pmatrix}\right\},$$

then

$$D(0) = \prod_{j=1}^{m+1} det\ \tilde{M}_{jj}\begin{pmatrix} i_j \\ i_j \end{pmatrix}.$$

Proof. It is sufficient to prove that $D(0)=\prod_{j=1}^{m+1} A_j(0)$. Since one of $B_{ij}(0)$ or $B_{ji}(0)$ is always 0, noting the numbering of E_j, we obtain the required result.

This corollary expresses that the special alternating knots play an important rôle in the studying of $D(0)$.

§ 2. Determinant

2. 1. The square matrix $M=(a_{ij})_{i,j=1,\cdots,n}$ is called the matrix of *special type on the rows* (or *the columns*) if it satisfies the following conditions (i), (ii) (or (ii)*):

(i) $a_{ii}>0,\ a_{ij}\leqq 0,\quad \text{for } i, j=1,\cdots,n,\quad i\neq j.$

(ii) $\sum_{j=1}^{n} a_{ij}\geqq 0,\quad \text{for } i=1,\cdots,n,$

((ii)* $\sum_{i=1}^{n} a_{ij}\geqq 0,\quad \text{for } j=1,\cdots,n).$

Moreover, M is said to be of *(P)-property on the row* (or *the column*) if it satisfies the following condition:

(iii) There exists an i such that $\sum_{j=1}^{n} a_{ij}\neq 0$.

((iii)* There exists a j such that $\sum_{i=1}^{n} a_{ij}\neq 0$.)

M is called the matrix of *strongly special type on the row* (*or the column*), if it is the matrix of special type on the row (or the column) and $M\begin{pmatrix} i_1 \cdots i_p \\ i_1 \cdots i_p\end{pmatrix}$ $(1\leqq i_1<\cdots<i_p\leqq n)$ are of (*P*)-property on the row (or the column). We first state a well-known result about matrices of this type.

Lemma 2. 1. *If M is a matrix of special type on the row* (*or the column*) *and if M is of* (*P*)*-property on all rows* (*or all columns*):

$$\sum_{j=1}^{n} a_{ij}>0 \quad \textit{for all } i\,, \tag{2. 2}$$

then $\qquad\qquad det\, M>0\,.$

From this Lemma, it follows

Lemma 2. 3. *If $M=(a_{ij})_{i,j=1,\cdots,n}$, is a matrix of special type on the row* (*or the column*), *then det $M\geqq 0$.*

Proof. Let $N(t)=(b_{ij})_{i,j=1,\cdots,n}$, where $b_{ii}=a_{ii}$, $b_{ij}=ta_{ij}$. Since $N(t)$ satisfies the condition (2. 2) for $0\leqq t<1$, it follows det $N(t)>0$. Moreover, since det $N(t)$ is a continuous function of t, it follows $\det M=\det N(1)=\lim_{t\to 1-} N(t)\geqq 0$, q.e.d.

We shall prove the following

Lemma 2. 4. *If M is of strongly special type on the row* (*or the column*), *then*

(i) $det\, M>0,$

(ii) $(-1)^{i+j}\, det\, \tilde{M}\binom{i}{j}\geqq 0,\quad (i, j=1,\cdots,n)$

(iii) $det\, \tilde{M}\binom{i}{i}\geqq \left| det\, \tilde{M}\binom{i}{j}\right|,\quad (i, j=1,\cdots,n).$

Proof. Since (ii), (iii) are proved in the same way as used in the proofs of Satz 2, Satz 3 in [3], we omit the details. We shall prove (i) by induction on n. The case $n=1$ is clear. Suppose that Lemma is true for the case $n-1$. We may assume without loss of generality

$$a_{11}+a_{12}+\cdots+a_{1n}>0\,. \tag{2.5}$$

We may assume that at least one of $a_{21}, a_{31}, \cdots, a_{n1}$ is different from 0. For otherwise, it follows $\det M=a_{11}\det \tilde{M}\binom{1}{1}>0$. Let $a_{21}\neq 0$.

Now, let us denote

$$\det M = \det\begin{pmatrix} a_{11} & 0 & 0 & \cdots 0 \\ a_{21} & b_{22} & b_{23} & \cdots b_{2n} \\ a_{31} & b_{32} & b_{33} & \cdots b_{3n} \\ & \cdots & & \\ a_{n1} & b_{n2} & b_{n3} & \cdots b_{nn} \end{pmatrix} = a_{11}\det M' = a_{11}\, det\,(b_{ij})\,,$$

where

$$b_{ii} = a_{ii}-\frac{a_{i1}a_{1i}}{a_{11}} \qquad (i \neq 1)\,, \tag{2.6}$$

$$b_{ij} = a_{ij}-\frac{a_{i1}a_{1j}}{a_{11}} \qquad (i\neq j)\,.$$

To prove Lemma, it is sufficient to show that M' is a matrix of *strongly* special type on the row. From (2.6), it follows immediately that $b_{ii}>0$, $b_{ij}\leqq 0$. Moreover, it follows $\sum_{j=2}^{n} b_{ij}=\sum_{j=2}^{n}\left(a_{ij}-\frac{a_{i1}a_{1j}}{a_{11}}\right)=\sum_{j=2}^{n} a_{ij} -\frac{a_{i1}}{a_{11}}\sum_{j=2}^{n} a_{1j}=\sum_{j=1}^{n} a_{ij}-\frac{a_{i1}}{a_{11}}\sum_{j=1}^{n} a_{1j}\geqq 0$. In particular, $b_{22}+\cdots+b_{2n}>0$. Thus we see M' is of (P)-property on the row. Next, we shall prove that $M'\binom{i_1\cdots i_p}{i_1\cdots i_p}$ are of (P)-property on the row. Suppose $M'\binom{i_1\cdots i_p}{i_1\cdots i_p}$ be not of (P)-property on the row. For the sake of brevity, we shall prove the case on $M'\binom{23\cdots p}{23\cdots p}$. Then we see

$$\left\{\begin{array}{llll} b_{33} & +b_{34} & +\cdots+b_{3,p+1} & =0\,, \\ b_{43} & +b_{44} & +\cdots+b_{4,p+1} & =0\,, \\ & \cdots & & \\ b_{p+1,3} & +b_{p+1,4} & +\cdots+b_{p+1,p+1} & =0\,. \end{array}\right. \tag{2.7}$$

Since $b_{j3}+\cdots+b_{j,p+1}=a_{j3}+\cdots+a_{j,p+1}-\frac{a_{j1}}{a_{11}}(a_{13}+\cdots+a_{1,p+1})=0$, for $j=3, \cdots, p+1$, and since $a_{11}>-(a_{12}+\cdots+a_{1n})$, we have

$$a_{j3}+a_{j4}+\cdots+a_{j,p+1} = 0\,, \qquad \text{for } j = 3, \cdots, p+1\,. \tag{2.8}$$

Thus, it follows $M\begin{pmatrix}34\cdots p+1\\34\cdots p+1\end{pmatrix}$ is not of (P)-property, which contradicts the assumption. Thus, M' is a matrix of strongly special type, q.e.d.

2.2. A matrix $M=(a_{ij})$ of strongly special type is said to be *k-strongly special type* if it satisfies the conditions

$$|a_{ij}-a_{ji}|\leqq 1\,,\quad \text{for all } i, j\,. \tag{2.9}$$

We can first prove

Lemma 2.10. *Let* $M=\begin{pmatrix}ab\\cd\end{pmatrix}$ *be a matrix of k-strongly special type on the row and the column, a, b, c, d being integers. If* $\det M=p$, *then a or* $d\leqq p$. *If, in particular,* $p=3$, *then matrices without s-equivalent admit only the following 7 matrices*:

$$\begin{pmatrix}1&-1\\0&3\end{pmatrix},\ \begin{pmatrix}1&-1\\-1&4\end{pmatrix},\ \begin{pmatrix}3&-1\\0&1\end{pmatrix},\ \begin{pmatrix}3&0\\0&1\end{pmatrix},\ \begin{pmatrix}2&-1\\-1&2\end{pmatrix},$$
$$\begin{pmatrix}3&-3\\-2&3\end{pmatrix},\ \begin{pmatrix}3&-3\\-3&4\end{pmatrix}.$$

Proof. If $a=|b|$ or $d=|c|$, then it is clear that $a=|b|\leqq p$ or $|c|=d\leqq p$. Let $a>|b|$, $d>|c|$. Set $a=-b+\varepsilon$, $d=-c+\eta$, where $\varepsilon, \eta\geqq 1$. Since $p=ad-bc=a\eta-c\varepsilon\geqq a\eta$, it follows $a\leqq p$. The latter half of this Lemma will be easily shown.

Our main theorem in this section, which is an extension of Lemma 2.4, is the following

Theorem 2.11. *If* $M=(a_{ij})_{i,j=1,\cdots,n}$, a_{ij} *being integers, is a matrix of k-strongly special type on the row and the column, then*

$$\det M\geqq \min\,\{a_{11}, a_{22}, \cdots, a_{nn}\}\,.$$

Proof. Let $\det M=p$. Theorem will be proved by induction on p and n. The case $n=2$, the theorem is the same as Lemma 2.10.

Suppose that the theorem holds for the matrix N such that *degree of* $N=n-1$ and that $\det N<p$. We may assume without loss of generality that $a_{11}+a_{12}+\cdots+a_{1n}>0$. Let us set $p_i=(-1)^{i+1}\det \tilde{M}\begin{pmatrix}i\\1\end{pmatrix}$, for $i=1,\cdots,n$. Then, it follows from Lemma 2.4 that $p_1\geqq p_i$ and $p_i\geqq 0$. Now, if $p_1\leqq p$, then the theorem is proved by applying the assumption to $\tilde{M}\begin{pmatrix}1\\1\end{pmatrix}$. Otherwise, it follows $a_{11}\leqq p$. For, let $a_{11}>p$. Then since $p_1\geqq p_i$ and $a_{1i}\leqq 0$ $(i\neq 1)$, we have $a_{1i}p_1\leqq a_{1i}p_i$. Thus it follows $p=a_{11}p_1+a_{12}p_2+\cdots+a_{1n}p_n\geqq(a_{11}+a_{12}+\cdots+a_{1n})p_1\geqq p_1>p$. This is a contradiction. Thus we have $a_{11}\leqq p$, q.e.d.

Moreover, it follows

Lemma 2.12. *If $M=(a_{ij})_{i,j=1,\cdots,n}$ is a matrix of strongly special type on the row and the column, then*

$$(2.13) \qquad \sum_{1\leqq i_1<\cdots<i_p\leqq n} \det N_{i_1,\cdots,i_p}>0\,, \qquad \text{for } p=1,\cdots,n\,,$$

where

$$N_{i_1,\cdots,i_p}=\begin{pmatrix} a_{11} & \cdots & a_{i_1,1} & \cdots & a_{i_p,1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{i_1,2} & \cdots & a_{i_p,2} & \cdots & a_{2n} \\ & & & \cdots & & & \\ a_{n1} & & a_{i_1,n} & & a_{i_p,n} & & a_{nn} \end{pmatrix}$$

Proof. We may assume without loss of generality that

$$(2.14)_1 \qquad a_{11}+a_{12}+\cdots+a_{1n}>0\,,$$
$$(2.14)_2 \qquad a_{22}+\cdots+a_{2n}>0\,,$$
$$\cdots\cdots$$
$$(2.14)_{n-1} \qquad a_{n-1,n-1}+a_{n-1,n}>0\,,$$
$$(2.14)_n \qquad a_{n,n}>0\,.$$

First, $N_{i_1,\cdots,i_p}\begin{pmatrix}\cdots\\\cdots\end{pmatrix}$ is a matrix of special type on the column, because M is a matrix of special type on the row and the column. Thus it follows from Lemma 2.3 that

$$\det N_{i_1,\cdots,i_p}\begin{pmatrix}\cdots\\\cdots\end{pmatrix}\geqq 0\,.$$

(We should note that $N_{i_1,\cdots,i_p}\begin{pmatrix}\cdots\\\cdots\end{pmatrix}$ is not necessarily a matrix of *strongly* special type.) We have to prove that one of the terms in summation has a positive value, that is, it is of *strongly* special type. To do this, we shall prove that $N_{12\cdots p}$ (for $p=1,\cdots,n$) is a matrix of strongly special type. Set $N=N_{12\cdots p}$. Then N is of (P)-property on the column from the assumption $(2.14)_1$. Next, $\tilde{N}\begin{pmatrix}1\\1\end{pmatrix}$ is of (P)-property on the column from $(2.14)_2$. In the same way, we can generally prove that $\tilde{N}\begin{pmatrix}12\cdots q\\12\cdots q\end{pmatrix}$ $(q<p)$ are of (P)-property. Moreover, $\tilde{N}\begin{pmatrix}12\cdots p\\12\cdots p\end{pmatrix}$ is of (P)-property, because it is the same as $\tilde{M}\begin{pmatrix}12\cdots p\\12\cdots p\end{pmatrix}$. In the same way, $N\begin{pmatrix}r\cdots s\\r\cdots s\end{pmatrix}$, $(r>1)$ is of (P)-property from $(2.14)_1$. Thus it follows that N is a matrix of strongly special type, q.e.d.

§ 3. Applications to knot theory

In this section, we shall apply the results obtained in §§ 1, 2 to the knot theory.

3. 1. When an alternating knot k is divided into $m+1$ special alternating knots $k_1, \cdots, k_{m+1}$ by m standard loops of the second kind, we say k is the *$*$-product* of $k_1, \cdots, k_{m+1}$, and denote by

$$k = k_1 * k_2 * \cdots * k_{m+1}. \tag{3.1}$$

We have first to prove the following

Theorem 3. 2. *Let $M=(a_{ij})_{i,j=1,\cdots,n}$, be the knot matrix of a special alternating knot k. Then $\tilde{M}\binom{i}{i}$ $(i=1, \cdots, n)$ are matrices of k-strongly special type on the row and the column.*

Proof. It will be shown from (1. 14) that $\tilde{M}\binom{i}{i}$ are matrices of special type and that these are of k-property. Hence we have only to show that $\tilde{M}\binom{i}{i}$ are matrices of *strongly* special type. We shall show that $\tilde{M}\begin{pmatrix} ij_1 \cdots j_{\lambda-1} \\ ij_1 \cdots j_{\lambda-1} \end{pmatrix}$ is of (P)-property on the row and the column. Now suppose the contrary. For the sake of brevity, we assume that $i=1$, $j_1=2, \cdots, j_{\lambda-1}=\lambda$. Then it follows

$$\left\{\begin{array}{l} a_{\lambda+1,\lambda+1}+a_{\lambda+1,\lambda+2}+\cdots+a_{\lambda+1,n}=0, \\ a_{\lambda+2,\lambda+1}+a_{\lambda+2,\lambda+2}+\cdots+a_{\lambda+2,n}=0, \\ \cdots \\ a_{n,\lambda+1}\quad+a_{n,\lambda+2}\quad+\cdots+a_{n,n}\quad=0, \end{array}\right. \tag{3.2}$$

or

$$\left\{\begin{array}{l} a_{\lambda+1,\lambda+1}+a_{\lambda+2,\lambda+1}+\cdots+a_{n,\lambda+1}=0, \\ a_{\lambda+1,\lambda+2}+a_{\lambda+2,\lambda+2}+\cdots+a_{n,\lambda+2}=0, \\ \cdots \\ a_{\lambda+1,n}\quad+a_{\lambda+2,n}\quad+\cdots+a_{n,n}\quad=0. \end{array}\right. \tag{3.3}$$

And it follows, moreover,

(3. 4) *If one of* (3. 2) *and* (3. 3) *holds, then it follows the other.*

Because, let (3. 2) hold. Then, it follows $(a_{\lambda+1,\lambda+1}+\cdots+a_{n,\lambda+1})+\cdots+(a_{\lambda+1,n}+\cdots+a_{n,n})=0$. Since $\tilde{M}\binom{1}{1}$ is a matrix of special type, the value of the sum in each bracket is $\geqq 0$. Hence we have (3. 3).

Thus, we have from (3. 2), (3. 3)

$$\left\{\begin{array}{l} a_{\lambda+1,1}=a_{\lambda+1,2}=\cdots=a_{\lambda+1,\lambda}=0, \\ \cdots \\ a_{n,1}\quad=a_{n,2}\quad=\cdots=a_{n,\lambda}\quad=0, \end{array}\right.$$

and

$$\begin{cases} a_{1,\lambda+1} = a_{2,\lambda+1} = \cdots = a_{\lambda,\lambda+1} = 0\,, \\ \quad\cdots \\ a_{1,n} \quad = a_{2,n} \quad = \cdots = a_{\lambda,n} \quad = 0\,. \end{cases}$$

Hence M is of the form $M=\left(\frac{*\,|\,0}{0\,|\,*}\right)$. This shows that the knot projection is separated into two parts, which contradicts the assumption in 1.2. The proof of Theorem 3.2 is thus complete.

From (1.14), it follows, moreover,

$$(3.5) \qquad det\,\tilde{M}\binom{1}{1} = det\,\tilde{M}\binom{2}{2} = \cdots = det\,\tilde{M}\binom{n}{n}\,.$$

3.2. Let $k=k_1 * \cdots * k_{m+1}$, and denote the A-polynomial of k by $\Delta(t)$. Then, it follows in the same notations as used in corollary 1.40,

Lemma 3.6. $\qquad D(0)>0\,.$

Proof. Since $D(0)=\prod_{i=1}^{m+1} A_i(0)$, it is sufficient to prove $A_i(0)=\det\tilde{M}_{ii}\binom{p_i}{p_i}>0$, where M_{ii} denote the special knot matrices of k_i. From Theorem 3.2, it follows $\tilde{M}_{ii}\binom{p_i}{p_i}$ are matrices of k-strongly special type on the row and the column. Thus, we have $\det\tilde{M}_{ii}\binom{p_i}{p_i}>0$, q.e.d.

On account of this Lemma, the fundamental theorem 1.17 can be stated as follows:

$$(3.7) \quad \Delta(t) = D(t) = det\left\{\tilde{M}\binom{i_1 \cdots i_{m+1}}{i_1 \cdots i_{m+1}} - t\tilde{M}^T\binom{i_1 \cdots i_{m+1}}{i_1 \cdots i_{m+1}}\right\}.$$

Now, Lemma 3.6 is equivalent to the following theorem which is obtained in [4, p. 265] [7], [8].

Theorem 3.8. *The degree of the A-polynomial of an alternating knot k plus one equals twice its genus plus its multiplicity* μ.

Proof. Applying the same notations as used in §1, Lemma 3.6 is equivalent to the following.

(3.9) *The degree of* $\Delta(t)$ *is equal to* $(p_1+1)+\cdots+(p_{m+1}-1)=(p_1+\cdots+p_{m+1})-(m+1)$.

On the other hand, Seifert showed that k is spanned by an orientable surface without singularity with genus h, where $2h=n_1+\cdots+n_{m+1}-(q_1+\cdots+q_{m+1}+m)-\mu+2$. (See [12].) Hence it follows,

$$\begin{aligned} 2h &= (n_1-q_1)+\cdots+(n_{m+1}-q_{m+1})-m-\mu+2 \\ &= (p_1+s_1-1)+\cdots+(p_{m+1}+s_{m+1}-1)-m-\mu+2 \\ &= (p_1+\cdots+p_{m+1})+(s_1+\cdots+s_{m+1})-(m+1)-m-\mu+2 \\ &= p_1+\cdots+p_{m+1}+m-1-m-1-m-\mu+2 \\ &= p_1+\cdots+p_{m+1}-m-\mu\,. \end{aligned}$$

Thus we have

(3.10) $$2h+\mu-1 = p_1+\cdots+p_{m+1}-m-1\,.$$

Denoting the genus of k by g, it follows (cf. [12])

(3.11) *The degree of* $\Delta(t)\leqq 2g+\mu-1$.

Hence, it follows from (3.9) (3.10) (3.11), $2g+\mu-1\geqq$ *the degree of* $\Delta(t)=(p_1+\cdots+p_{m+1})-m-1=2h+\mu-1\geqq 2g+\mu-1$, which is the required result, q.e.d.

Theorem 3.12. *If k is a special alternating knot of multiplicity* 1, *then* $\Delta(1)=1$.

Proof. As is well-known, $\Delta(1)=\pm 1$. Thus it is sufficient to prove $\Delta(1)\geqq 0$. It follows from (3.7),

$$\Delta(1) = \det\left\{\tilde{M}\binom{p}{p}-\tilde{M}^t\binom{p}{p}\right\} = \det\begin{pmatrix} 0 & \varepsilon_{12}\cdots\cdots\cdots\varepsilon_{1,p-1} \\ \varepsilon_{21} & 0\cdots\cdots\cdots\varepsilon_{2,p-1} \\ & \cdots \\ \varepsilon_{p-1,1}\varepsilon_{p-1,2}\cdots & 0 \end{pmatrix} \geqq 0\,,$$

where $\varepsilon_{ij}=a_{ij}-a_{ji}$, $\varepsilon_{ij}=-\varepsilon_{ji}$, q.e.d.

We see from the above Lemma that there are knots which cannot be transformed into the special alternating. The knot shown in Fig. 4 is one of these knots, for $\Delta(t)=1-5t+7t^2-5t^3+t^4$ and $\Delta(1)=-1$.

Theorem 3.13. *Let $\Delta(t)=c_0+c_1t+\cdots+c_{p-1}t^{p-1}$ be the A-polynomial of a special alternating knot, then it follows $(-1)^jc_j>0$, for $j=0,1,\cdots,p-1$.*

Proof. From (3.7), we have

$$\Delta(t) = \det\left\{\tilde{M}\binom{p}{p}-t\tilde{M}^t\binom{p}{p}\right\}.$$

Then since $(-1)^jc_j=\dfrac{1}{j!}\left[\dfrac{d^j}{dt^j}\Delta(t)\right]_{t=0}$, it follows from Lemma 2.12 that

(3.14) $$(-1)^jc_j = \frac{1}{j!}\sum_{1\leqq i_1<\cdots<i_j\leqq p-1} N_{i_1,\ldots,i_j}>0\,,$$ q.e.d.

3. 3. Let $\mathfrak{R}$ be a ring of all polynomials $f(t)=a_0+a_1t+\cdots+a_nt^n$, a_i being integers, satisfying the following conditions:

$$(3.15)\qquad a_0>0\,,\quad a_i=(-1)^n a_{n-i}\,,\quad \text{for}\quad i=0,\cdots,n\,.$$

We introduce a semi-order, denoted by $\geqq$, in $\mathfrak{R}$. Let $f(t)=a_0+a_1t+\cdots+a_nt^n$, $g(t)=b_0+b_1t+\cdots+b_mt^m$. By $f(t)\geqq g(t)$ is meant that

$$(3.16)\qquad n\geqq m \quad\text{and}\quad |a_i|\geqq|b_i| \quad \text{for}\quad i=0,\cdots,m\,.$$

Then, as is shown in [6], [12], it follows

Lemma 3.17. *$\mathfrak{R}$ coincides with a ring of all A-polynomials of knots.*

Proof has been given in [12] in the case that the multiplicity μ of knots is equal to 1, and given in [6] in the case $\mu>1$.

Let us denote, moreover, the subring of all polynomials satisfying the conditions

$$(3.18)\qquad (-1)^i a_i>0\,,\quad \text{for all } i\,,$$

by $\mathfrak{A}$.

Now, let an alternating knot $k=k_1*\cdots*k_{m+1}$. Let the A-polynomials of $k, k_1,\cdots,k_{m+1}$ be denoted by $\Delta(t), \Delta_1(t),\cdots,\Delta_{m+1}(t)$. Then it follows,

Lemma 3.19. $\qquad \Delta(t)\geqq\Delta_1(t)\cdots\Delta_{m+1}(t)\,.$

Since the proof of this Lemma has been given in [8, pp. 247-248], [9, pp. 181-185], we omit the details.

Hereafter we shall symbolize these as follows:

$$(3.20)\qquad \Delta(t)=\Delta_1(t)*\cdots*\Delta_{m+1}(t)\,.$$

From Theorem 3.13, Lemma 3.19, we have immediately,

$$(3.21)\qquad \Delta(-1)\geqq\Delta_1(-1)\cdots\Delta_{m+1}(-1)\,.$$

Moreover, we have,

Lemma 3.22. *The totality $\mathfrak{B}$ of all A-polynomials of alternating knots is contained in $\mathfrak{A}$.* [9]

In the following, it will be shown

$$(3.23)\qquad \mathfrak{A}\neq\mathfrak{B}\,.$$

3.4. The main theorem of this section is the following

Theorem 3.24. *The special alternating knots with $\Delta(0)=1$ admit only elementary torus knots or their products.*

By an *elementary torus knot* is meant a special alternating knot whose graph or dual graph of its projection is a polygon. (See Fig. 5)

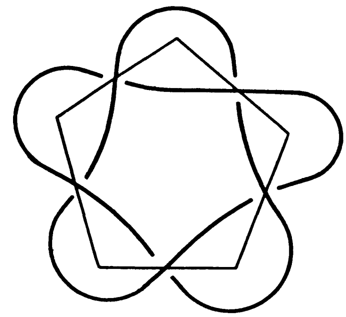

Fig. 5.

Proof. Let $M=(a_{ij})_{i,j=1,\cdots,n}$ be the knot matrix of a knot satisfying the condition in this theorem. Then, we see from Theorem 2.11 that at least two of $a_{11},\cdots,a_{nn}$ are equal to 1. We shall prove the theorem by induction on n. If $n=2$, it follows $M=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$, from which the theorem follows. Now suppose that the theorem is proved in the case $n-1$. We may assume without loss of generality that

$$(3.25)\qquad a_{11}=1\,,\quad a_{12}=-1\,,\quad a_{13}=\cdots=a_{1n}=0\,.$$

Then it follows that $\Delta(0)=det\,\tilde{M}\binom{2}{2}=det\,\tilde{M}\binom{12}{12}=1$.

Let
$$N=\begin{pmatrix} a_{21}+a_{22} & a_{23}\cdots a_{2n}\\ a_{31}+a_{32} & a_{33}\cdots a_{3n}\\ \cdots & \\ a_{n1}+a_{n2} & a_{n3}\cdots a_{nn}\end{pmatrix}.$$

N is the knot matrix of a special alternating knot k_1, where k_1 is transformed from k by applying in its projection the following operation as is shown in Fig. 6:

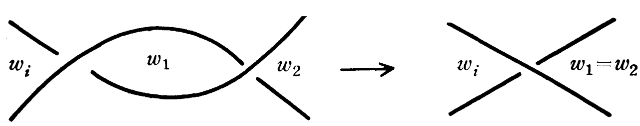

Fig. 6.

Denoting the A-polynomial of k_1 by $\Delta_1(t)$, it follows

$$\Delta_1(0)=det\,\tilde{N}\binom{1}{1}=det\,\tilde{M}\binom{12}{12}=1\,.$$

Thus we see from the assumption of the induction that k_1 is an elementary torus knot or their product. Hence k must be also an elementary torus knot or their product, q.e.d.

(3.26) *The A-polynomial of an elementary torus knot is of the form*: $1-t+\cdots+(-1)^i t^i+\cdots+(-1)^n t^n$.

The converse of (3.26) is also true. That is,

Lemma 3.27. *An alternating knot whose A-polynomial is of the form, $1-t+\cdots+(-1)^n t^n$, is an elementary torus knot.*

Proof. Let $k=k_1*\cdots*k_{m+1}$, and denote the A-polynomials of k, $k_1, \cdots, k_{m+1}$ by $\Delta(t), \Delta_1(t), \cdots, \Delta_{m+1}(t)$. Then, we see from Lemma 3.19 and (1.40) that $\Delta_1(0)=\cdots=\Delta_{m+1}(0)=1$. Hence it follows from Theorem 3.24 that k_i are elementary torus knots or their products. Then,

$$\Delta(t) \leqq \Delta_1(t) \cdots \Delta_{m+1}(t)\,,$$

which implies $m=0$. Hence k is an elementary torus knot or their product. It is clear, however, that k is not a product knot, q.e.d.

3.5. From Theorem 3.24, (3.26), Lemma 3.27, we have the following

Theorem 3.28. *An alternating knot with $\Delta(0)=1$ is represented as the $*$-product of some elementary torus knots.*

This theorem will follow Lemma 3.30 about the number of the double points in a knot projection. Before Lemma 3.30 is stated, we state the following

Lemma 3.29. *Let $k=k_1*\cdots*k_{m+1}$, and denote the numbers of the double points in K_i by n_i. Then the number of the double points in K equals $n_1+\cdots+n_{m+1}$.*

Thus k possesses an alternating projection, in which there are at most $n_1+\cdots+n_{m+1}$ double points. From Lemma 3.29, it follows

Lemma 3.30. *An alternating knot k with $\Delta(0)=1$ possesses a projection, in which there are at most $2n$ double points, n denoting the degree of $\Delta(t)$.*

Proof. k can be represented as the $*$-product of at most n special alternating knots. Then, every $*$-component is an elementary torus knot whose A-polynomial is $1-t$. Since the number of the double points of it is 2, it follows the number of double points in K is $2n$. If k can be represented as the $*$-product of $m(<n)$ special elementary torus knots, then the number of double points in K is less than $2n$, q.e.d.

Corollary 3.31. *Alternating prime knots whose A-polynomials are of the form: $\Delta(t)=1-c_1t+c_2t^2-c_3t^3+t^4$, admit only the following knots:* 5_1, 6_2, 6_3, 7_6, 7_7, 8_{12}.

Hence it follows:

(3.32) *Projection* 8_{20}, 8_{21}, 9_{42}, 9_{44}, 9_{45}, 9_{48}, *really represent non-alternating knots.*

Last of all, we shall show that 3 projections[8] 9_{43}, 9_{47}, 9_{49} really represent non-alternating knots.

3.6. In the following, we denote knots 9_n and their A-polynomials by k_n and $\Delta_n(t)$ respectively. Let us denote the elementary torus knot whose A-polynomial is of the degree n, by $k_{(n)}$ and denote its A-polynomial by $\Delta_{(n)}(t)$.

Now, since $\Delta_{43}(t)=1-3t+2t^2-t^3+2t^4-3t^5+t^6$ and $\Delta_{47}(t)=1-4t+6t^2-5t^3+6t^4-4t^5+t^6$, it follows that $\Delta_{43}(1)=-1$, $\Delta_{43}(-1)=13$ and $\Delta_{47}(1)=1$, $\Delta_{47}(-1)=27$.

Suppose k_{43}, k_{47} be alternating. Then it follows from Lemma 3.29, that each of k_{43}, k_{47} has to be represented as one of the following $*$-products:

(1) $k_{(3)}*k_{(1)}*k_{(1)}*k_{(1)}$,
(2) $k_{(2)}*k_{(2)}*k_{(1)}*k_{(1)}$,
(3) $k_{(2)}*k_{(1)}*k_{(1)}*k_{(1)}*k_{(1)}$,
(4) $k_{(1)}*k_{(1)}*k_{(1)}*k_{(1)}*k_{(1)}*k_{(1)}$.

In each case we set $t=-1$. Then we have the following

(1) $\Delta_{(3)}(-1)\{\Delta_{(1)}(-1)\}^3 = 32 > 13, 27$,
(2) $\{\Delta_{(2)}(-1)\}^2\{\Delta_{(1)}(-1)\}^2 = 36 > 13, 27$,
(3) $\Delta_{(2)}(-1)\{\Delta_{(1)}(-1)\}^4 = 48 > 13, 27$,
(4) $\{\Delta_{(1)}(-1)\}^6 = 64 > 13, 27$.

All cases contradict (3.21). Thus we have

(3.33) k_{43}, k_{47} *represent non-alternating knots.*

To show that 9_{49} represents a non-alternating knot, we require some preparations.

3.7. In this and next paragraphs, we make an exact distinction between knots and links.

Let $M=(a_{ij})_{i,j=1,\dots,n}$, be the knot matrix of a special alternating *knot* k. Let $N=(b_{ij})_{i,j=1,\dots,n}$, be a matrix obtained from M as follows:

8) It will be easily shown from Theorem 3.13, Lemma 3.19 and by simple computations that two knot projections 8_{19}, 9_{46} represent non-alternating knots.

(3.34) $$b_{ij} = a_{ij} + \min(|a_{ij}|, |a_{ji}|), \qquad (i \neq j)$$
$$b_{ii} = a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \min(|a_{ij}|, |a_{ji}|).$$

Then, if follows

Lemma 3.35. *N is the knot matrix of a special alternating knot k_0.*

Proof. Let W_i, W_j be two regions in K such that $\dot{W}_i \frown \dot{W}_j \neq \phi$. Then, by applying, as much as posible, the operation as is shown in Fig. 7, we have a knot. It is clear that such a knot is k_0.

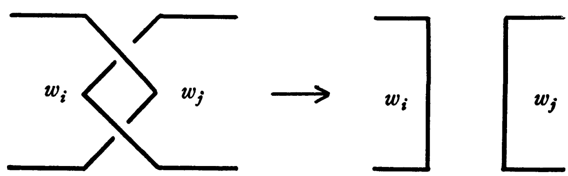

Fig. 7.

We shall call k_0 the *frame knot* of k. We can naturally extend this concept to links, but we should note that link projection may be separated into some parts. (See Fig. 8)

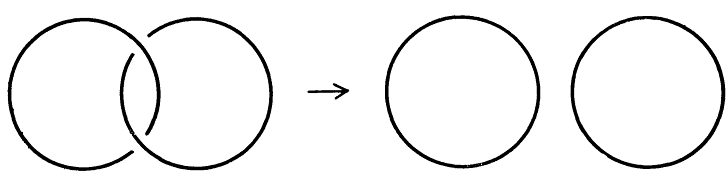
Fig. 8.

In all cases, it follows

(3.36) $$\mu = \mu_0,$$

where μ, μ_0 denote the multiplicities of k, k_0.

3.8. Since $\Delta_{49}(t) = 3 - 6t + 7t^2 - 6t^3 + 3t^4$, it follows $\Delta_{49}(1) = 1$, $\Delta_{49}(-1) = 25$. We shall first prove

(3.37) *If k_{49} is alternating, it must be a special alternating knot.*

Proof. By noting $\Delta(1) = 1$, it follows that if k_{49} is alternating and if it is not a special alternating knot, then $\Delta_{49}(t)$ must be represented as one of the following $*$-products:

(1) $\Delta_{(2)}(t)*(3-5t+3t^2)$,

(2) $\Delta_{(2)}(t)*3(1-t)^2$,

(3) $\Delta_{(2)}(t)*\Delta_{(1)}(t)*3(1-t)$,

(4) $3(1-t)^2*\Delta_{(1)}(t)*\Delta_{(1)}(t)$,

(5) $(3-5t+3t^2)*\Delta_{(1)}(t)*\Delta_{(1)}(t)$,

(6) $3(1-t)*\Delta_{(1)}(t)*\Delta_{(1)}(t)*\Delta_{(1)}(t)$.

In all cases we set $t=-1$. Then every value is larger than $\Delta_{49}(-1)=25$. This contradicts Lemma 3.21. Thus k_{49} must be a special alternating knot.

Now, for the frame knot of a special alternating knot, we can prove the following

Lemma 3.38. *The frame knot k_0 of a special alternating knot whose A-polynomial is of degree* 4, *admits only an elementary torus knot $k_{(4)}$ or a product of two knots $k_{(2)}$.*

This will be proved by simple computation, if we note that at least one of b_{ij}, b_{ji} in the knot matrix of k_0 $N=(b_{ij})$ is equal to 0.

From Lemma 3.38, we have, moreover, by computation,

(3.39) *In the knot matrix M of a special alternating knot whose A-polynomial $\Delta(t)$ is of the degree* 4 *and $\Delta(0)=3$, at least two of $a_{11}, \cdots, a_{55}$ are equal to* 1.

Thus, it follows from Lemma 2.10, (3.39) and from the fact that M is of k-strongly special type, without s-equivalent, that M admits only the following:

(3.40)

$$(1)\ \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix},$$

$$(2)\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -3 & 0 & 3 \end{pmatrix},$$

$$(3)\quad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 3 & -2 & 0 \\ 0 & 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$(4)\quad \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & -3 & -1 \\ -1 & 0 & -2 & 3 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

These matrices correspond to knot matrices of 8_{15}, 9_4, 9_7 and of a product of 3_1 and 7_2, respectively. Thus we have the following

(3.41) *Special alternating knots whose A-polynomials* $\Delta(t)$ *are of degree* 4 *and* $\Delta(0)=3$, *admit only the knots*: 8_{15}, 9_4, 9_7 *and a product of* 3_1 *and* 7_2.

This naturally follows

(3.42) 9_{49} *represents a non-alternating knot.*

Thus, it has been really shown that all non-alternating projections at the end of [11] represent non-alternating knots.

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