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Osaka Math. J.
12 (1960) 217-252.

Some Properties of Complex Analytic Vector Bundles over Compact Complex Homogeneous Spaces

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Introduction

Recently R. Bott [7] has introduced the concept of homogeneous vector bundles over C-manifolds. By a C-manifold we understand here a simply connected homogeneous compact complex manifold, which was the subject of Wang's exhaustive work [22]. Bott's theory is a natural extension of the preceding researches by A. Borel and A. Weil [6] and permits us to utilize the theory of Lie groups to study complex analytic vector bundles over C-manifolds. The purpose of this paper is to study several problems on complex analytic vector bundles over C-manifolds utilizing Bott's results. The present paper is divided into four chapters. In Chapter I are given preliminaries for the subsequent chapters. First we recall the method of Y. Matsushima and A. Morimoto (cf. [16], [17]) to define the homogeneous (not necessarily vector) bundles in a more natural and intrinsic way. Their definition of homogeneous bundles does not presuppose the simply-connectedness of the base spaces, and in the case of C-manifolds it agrees with Bott's definition. Next, after a résumé of the results of Wang and Bott, we shall prove that every complex line bundle over a C-manifold is homogeneous (Theorem 1). This result was proved very recently by S. Murakami [18], but our proof is more direct and endows us some other implications. In Chapter II we discuss an application of the so-called classification theorem of complex analytic vector bundles to homogeneous vector bundles using Bott's idea (Theorem 2 and Theorem 4). The classification theorem of general vector bundles is due to S. Nakano, K. Kodaira and J. P. Serre (cf. [3], [19]), but it has not been yet published in a complete from and so we shall state our classification theorem of homogeneous vector bundles considerably in detail. In this chapter we shall show that the main parts of the researches by Borel-Weil [6] and by M. Gotô [9] are two special cases of our Theorem 2, and, as another application of this theorem, we shall show that the classical theorem of F. Severi concerning the positive divisors of complex Grassman manifolds (cf. [13]) can be generalized to any kählerian C-manifold of which the 2nd Betti number is 1 (Theorem 3). Chapter

III is devoted to some studies of (not necessarily homogeneous) vector bundles over C-manifolds. Our principal aim in this chapter is to prove Theorem 5, which asserts that the complex projective line can be characterized among C-manifolds by the property that every complex analytic vector bundle is decomposed into the direct sum of complex line bundles. A. Grothendieck [10] has proved that the complex projective line satisfies the above property and he posed a conjecture to the effect that the complex projective line would be the unique projective algebraic variety satisfying the above property. Our Theorem 5 gives a partial answer to this conjecture. Finally in Chapter IV, we shall be concerned with the tangential vector bundles of C-manifolds. It will be an important problem to clarify the relation between the indecomposability of homogeneous vector bundles and the defining representations. We treat here this problem in the case of tangential bundles (Theorem 6). In the first part of this chapter some results on non-kählerian C-manifolds shall be discussed which are of interest for themselves. A few unsolved problems, in connection with Theorem 6, are presented at the end of Chapter IV.

Some of the results of Bott and ours are valid also for Hopf manifolds, which shall be discussed in the forthcoming paper.

The author wishes to express here his sincere thanks to Professors Y. Matsushima, S. Murakami and S. Nakano for their valuable instructions and criticisms during the preparation of this paper.

I. Preliminaries. Homogeneous Vector Bundles.

1. Homogeneous bundles.

Throughout the paper, we denote by X a compact complex manifold and by $A(X)$ the group of all complex analytic antomorphisms of X with the compact open topology. Its connected component $A^0(X)$ containing the identity element is a complex Lie group by a well-known theorem of Bochner-Montgomery, so we denote by $\mathfrak{a}(X)$ its Lie algebra. By $P(X, B, \varpi)$ (resp. $E(X, F, B, \varpi)$) is meant a complex analytic principal bundle over X with group B and projection ϖ (resp. a complex analytic fibre bundle over X with fibre F, group B and projection ϖ). We denote by $F(P)$ (resp. $F(E)$) the group of all bundle automorphisms of P (resp. the group of all fibre-preserving automorphisms of E). We note that, if E is an associated bundle of P and if B acts on F transitively, $F(P)$ and $F(E)$ are isomorphic in a natural manner. Now, let $F^0(P)$ denote the connected component of the identity element of $F(P)$ with the compact open topology. It is also a complex Lie group by a theorem of Morimoto [17]. The Lie algebra of $F^0(P)$ will be denoted by $\mathfrak{f}(P)$. The projection ϖ induces a complex Lie group homomorphism of $F^0(P)$ into $A^0(X)$ and a complex Lie algebra homomorphism of $\mathfrak{f}(P)$ into $\mathfrak{a}(X)$, which shall be denoted by the same symbol ϖ. If the group $\varpi(F^0(P))$ operates transitively on X, we say that P is *homogeneous.* Therefore if P is homogeneous, the base X must be a homogeneous complex manifold.

A simply connected homogeneous compact complex manifold is called a *C-manifold* [22]. Let X be a C-manifold. It is known that X is represented in the form $X=G/U$, where G is a connected complex semi-simple Lie group operating almost effectively on X [22]. Now let ρ be a holomorphic homomorphism of U into a complex Lie group B. Let $P(X, B, \varpi)$ be the complex analytic principal bundle associated to the coset bundle $G(X, U, \pi)$ (π being the canonical projection of G onto $X=G/U$) by the homomorphism ρ of U into B. We say that the bundle $P(X, B, \varpi)$ obtained in this manner is *homogeneous in the strong sense.*

Proposition 1 (MATSUSHIMA). *Let $P(X, B, \varpi)$ be a complex analytic principal bundle over a C-manifold X. Then $P(X, B, \varpi)$ is homogeneous, if and only if it is homogeneous in the strong sense.*

Let $P(X, B, \varpi)$ be homogeneous. Then $\varpi(F^0(P))$ is transitive on X. Since X is a C-manifold, any maximal complex semi-simple Lie subgroup of $\varpi(F^0(P))$ operates transitively on X [22]. We can choose a complex semi-simple Lie subgroup G of $F^0(P)$ in such a way that the restriction of the homomorphism $\varpi: F^0(P)\to A^0(X)$ on G is a local isomorphism and that $\varpi(G)$ operates transitively on X. For $f\in G$, $x\in X$, set $f\circ x=\varpi(f)\cdot x$. Then G operates on X transitively and almost effectively. Now let $u\in P$ and let U be the subgroup of G consisting of the elements $f\in G$ which leave the fibre $\varpi^{-1}(\varpi(u))$ invariant. U coincides with the subgroup of G of the elements $f\in G$ such that $f(\varpi(u))=\varpi(u)$. Therefore $X=G/U$. Let ρ_u be the complex analytic mapping of G into P defined by $\rho_u(f)=f(u)$. Let $f\in U$. Then $\rho_u(f)=u\cdot a$, for a definite element $a\in B$. We denote $\rho_u(f)=a$ for $f\in U$. Then $\rho_u: U\to B$ is a holomorphic homomorphism. It is easily seen that the mapping $\rho_u: G\to P$ and the homomorphism $\rho_u: U\to B$ define a bundle homomorphism of the coset bundle $G(X, U, \pi)$ into $P(X, B, \varpi)$. Therefore $P(X, B, \varpi)$ is homogeneous in the strong sense. Conversely, let $P(X, B, \varpi)$ be homogeneous in the strong sense. Suppose that $P(X, B, \varpi)$ is associated to the coset bundle $G(X, U, \pi)$ by a homomorphism ρ of U into B, where $X=G/U$ is a certain expression of X with semi-simple and almost effective G. Then $P=G\times_U B$ in the notation of Bott [7], that is, P is the quotient of $G\times B$ by the equivalence relation

$(g, b) \sim (g\cdot h, \rho(h^{-1})\cdot b)$ $(g\in G, b\in B, h\in U)$. Then the left translation by an element $g'\in G$ can be defined by setting $g'\cdot\alpha(g, b)=\alpha(g'\cdot g, b)$, where $\alpha(g, b)$ denotes the equivalence class containing (g, b). It is easily seen that the left translations by the elements of G are bundle automorphisms of $P(X, B, \varpi)$. It follows then that $P(X, B, \varpi)$ is homogeneous.

2. The structures of C-manifolds.

Let X be a C-manifold and let $X=G/U$, where G is a connected complex semi-simple Lie group G operating almost effectively on X. We assume hereafter these properties for the Klein form G/U of X without any specific comment, and in this case U is called a *C-subgroup* of G. If $X=G/U$ is not kählerian, there exists a C-subgroup $\hat{U}$ of G such that $\hat{X}=G/\hat{U}$ is a kählerian C-manifold, and that U is a closed normal subgroup of $\hat{U}$ and $\hat{U}/U$ is a complex toroidal group. The complex analytic principal fibering $X(\hat{X}, \hat{U}/U, \phi)$ thus obtained is called *the fundamental fibering* of a (non-kählerian) C-manifold $X=G/U$ and the base space $\hat{X}=G/\hat{U}$ is called *the associated kählerian C-manifold of X* (see [8], [18]). If $X=G/U$ is a kählerian C-manifold, there are a maximal C-subgroup U_m of G containing U and a maximal solvable subgroup (=a minimal C-subgroup) U_f of G contained in U. We have the corresponding coset fiberings $X_f(X, U/U_f, \psi_f)$ and $X(X_m, U_m/U, \psi_m)$, where $X_f=G/U_f$ is called *the flag manifold* associated to X and $X_m=G/U_m$ a *maximal C-manifold* associated to X respectively. We note here the fibres U/U_f and U_m/U are again kählerian C-manifolds (with non effective Klein forms). It is known that irreducible hermitian symmetric spaces are maximal C-manifolds.

Now we summarize the structure theory of C-manifolds due to Wang [22] and the main theorem of Bott concerning the homogeneous vector bundles ([17]. Let $X=G/U$ be a C-manifold and $\hat{X}=G/\hat{U}$ the associated kählerian C-manifold. The Lie algebras of G, U and $\hat{U}$ are denoted by $\mathfrak{g}$, $\mathfrak{u}$ and $\hat{\mathfrak{u}}$ respectively. Let K be a compact form of G and set $K\cap\hat{U}=\hat{V}$, and $K\cap U=V$. The Lie algebras of K, V and $\hat{V}$ are denoted by $\mathfrak{k}$, $\mathfrak{v}=\mathfrak{k}\cap\mathfrak{u}$ and $\hat{\mathfrak{v}}=\mathfrak{k}\cap\hat{\mathfrak{u}}$. We can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a fundamental root system $\{\mathring{\alpha}_1, \cdots, \mathring{\alpha}_l\}$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ satisfying the following conditions: There exists a subset S of $\{\mathring{\alpha}_1, \cdots, \mathring{\alpha}_l\}$ such that

$$
(1)\qquad
\begin{aligned}
&\mathfrak{g} = \mathfrak{u}+\overline{\mathfrak{w}}+\mathfrak{n}^+(S) = \hat{\mathfrak{u}}+\mathfrak{n}^+(S)\,, \\
&\mathfrak{u} = \mathfrak{v}(S)+\mathfrak{h}_V+\mathfrak{w}+\mathfrak{n}^-(S)\,, \quad \hat{\mathfrak{u}} = \mathfrak{v}(S)+\mathfrak{h}(S)+\mathfrak{n}^-(S) \\
&\mathfrak{v}^c = \mathfrak{v}(S)+\mathfrak{h}_V, \qquad\qquad\qquad \hat{\mathfrak{v}}^c = \hat{\mathfrak{v}}(S)+\mathfrak{h}(S) \\
&\qquad\qquad\mathfrak{h}(S) = \mathfrak{h}_V+\mathfrak{w}+\overline{\mathfrak{w}}\,,
\end{aligned}
$$

where the notation is identical with the one adopted in Bott [7] except $\mathfrak{n}^+(S)$ and $\mathfrak{n}^-(S)$, which are defined as follows:

$$\mathfrak{n}^+(S) = \sum_{\alpha} C\{e_\alpha\}\ , \quad \mathfrak{n}^-(S) = \sum_{\alpha} C\{e_\alpha\}$$

where $\mathring{\alpha}$ runs through the positive roots not belonging to S and e_α denotes an element of $\mathfrak{g}$ such that $[h, e_\alpha]=\mathring{\alpha}(h)\cdot e_\alpha$ for $h\in\mathfrak{h}$.

We note that $\hat{X}$ is a flag manifold or a maximal C-manifold according as S is vacous or S consists of $l-1$ roots. The complex closed Lie subgroups generated by $\mathfrak{v}(S)$, $\mathfrak{n}^+(S)$, $\mathfrak{n}^-(S)$, $\mathfrak{h}$, $\mathfrak{h}(S)$, $\mathfrak{h}_V$, $\mathfrak{w}$, $\overline{\mathfrak{w}}$ and $\mathfrak{v}^c$ in G are denoted by $V(S)$, $N^+(S)$, $N^-(S)$, H, $H(S)$, H_V, W, $\overline{W}$ and V^C respectively. Recall that H, $H(S)$ and H_V are isomorphic to the direct product of some copies of the group C^*, that W, $\overline{W}$ are complex vector groups and that $\overline{W}$ is identified with the universal covering manifold of $\hat{U}/U$.

3. The main theorem of Bott.

Let (h, h') $(h,h'\in\mathfrak{h})$ be the non-degenerate bilinear form on $\mathfrak{h}$ defined by the Killing form of $\mathfrak{g}$. For any element $\mathring{\mu}$ of the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$, there exists an element $h'_\mu\in\mathfrak{h}$ such that $\mathring{\mu}(h)=(h'_\mu, h)$ for all $h\in\mathfrak{h}$. For $\mathring{\lambda}$, $\mathring{\mu}\in\mathfrak{h}^*$, let $(\mathring{\lambda}, \mathring{\mu})=(h'_\lambda, h'_\mu)$. Then $(\mathring{\lambda}, \mathring{\mu})$ is a non-degenerate bilinear form on $\mathfrak{h}^*$. For any $\mathring{\mu}\in\mathfrak{h}^*$, let h_μ be the element of $\mathfrak{h}$ such that $\mathring{\lambda}(h_\mu)=2(\mathring{\lambda}, \mathring{\mu})/(\mathring{\mu}, \mathring{\mu})$ for any $\mathring{\lambda}\in\mathfrak{h}^*$. Especially, for any fundamental root $\mathring{\alpha}_i$ $(1\leqq i\leqq l)$, set $h_{\alpha_i}=h_i$. An integral linear form or a weight $\mathring{\lambda}$ on $\mathfrak{h}$ is defined to be an element of $\mathfrak{h}^*$ such that $\mathring{\lambda}(h_i)$ $(1\leqq i\leqq l)$ are all integers. If $\mathring{\lambda}(h_i)$ are all non negative integers, $\mathring{\lambda}$ is called a dominant weight. The set of all weights on $\mathfrak{h}$ form a lattice. Let $\mathring{\Lambda}_1, \cdots, \mathring{\Lambda}_l$ be the dominant weights defined by $\mathring{\Lambda}_j(h_i)=\delta_{ij}$ for $1\leqq i,\ j\leqq l$. They form a base of the lattice of weights on $\mathfrak{h}$ and are called the fudamental dominant weights corresponding to the fundamental roots $\mathring{\alpha}_1, \cdots, \mathring{\alpha}_l$. We denote by $\mathfrak{h}^*_R$ the real l-dimensional subspace of $\mathfrak{h}^*$ consisting of all $\mathring{\lambda}\in\mathfrak{h}^*$ such that $\mathring{\lambda}(h_i)$ $(1\leqq i\leqq l)$ are real. The restriction to $\mathfrak{h}^*_R$ of the inner product $(\ ,\ )$ on $\mathfrak{h}^*$ is positive definite. The fundamental (Weyl) chamber $\mathfrak{P}$ of $\mathfrak{h}^*_R$ and its open kernel $\mathfrak{P}^0$ are the closed and open regions in $\mathfrak{h}^*_R$ defined respectively by $\mathfrak{P}=\{\mathring{\lambda}\in\mathfrak{h}^*_R \mid \mathring{\lambda}(h_i)\geqq 0$ for $1\leqq i\leqq l\}$ and $\mathfrak{P}^0=\{\mathring{\lambda}\in\mathfrak{h}^*_R \mid \mathring{\lambda}(h_i)>0$ for $1\leqq i\leqq l\}$. Now we define the *index* of a weight $\mathring{\lambda}$ (which will be denoted by Ind. $\mathring{\lambda}$) as the number of positive roots $\mathring{\alpha}$ such that

$$(\mathring{\lambda}+\mathring{\delta})(h_\alpha)<0\ ,$$

where $\mathring{\delta}$ is a weight defined by $\mathring{\delta}(h_i)=1$ for $1\leqq i\leqq l$ (i.e. $\mathring{\delta}=\mathring{\Lambda}_1+\mathring{\Lambda}_2+\cdots+\mathring{\Lambda}_l$). A weight $\mathring{\lambda}$ is called singular or regular according as there exists a root $\mathring{\alpha}$ such that $\mathring{\lambda}(h_\alpha)=0$ or not.

Let H be the Lie subgroup of G corresponding to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. H is called the Cartan subgroup of G. One-dimensional representations of H are called the characters of H. The differential $\mathring{\lambda}$ at the identity element of a character $\mathring{\lambda}$ is a weight of h. Conversely, every weight of $\mathfrak{h}$ can be obtained in this way, and this gives a one-to-one correspondence between the set of characters of H and the set of weights of $\mathfrak{h}$. Now let (ρ, F) be a holomorphic[1] irreducible representation of G in a complex vector space F. The restriction of ρ on H is denoted by ρ_H. The representation (ρ_H, F) of H is complety reducible. Among the weights which appear as the irreducible components of (ρ_H, F), there is only one dominant weight $\mathring{\lambda}$ which is the highest among these weights, the linear order being given by the fundamental root system $\{\mathring{\alpha}_1, \cdots, \mathring{\alpha}_l\}$. The weight $\mathring{\lambda}$ is called the highest weight of the representation (ρ, F) of G and the corresponding character λ is called the highest character of (ρ, F).

Let $X=G/U$ be a C-manifold and (ρ, F) a representation of U. We denote by $E(\rho, F)$ the homogeneous vector bundle over X defined by (ρ, F). In particular, when (ρ, F) is a one-dimensional representation λ (i.e. $\lambda \in \mathrm{Hom}\,(U, C^*)$), we simply write the line bundle $E(\rho, F)$ as E_λ, or sometimes $E_{\mathring{\lambda}}$, where $\mathring{\lambda}$ is the differential of λ : $\mathring{\lambda} \in \mathrm{Hom}\,(\mathfrak{u}, C)$.

Lemma 1[2]. *If a representation ρ of U is completely reducible, then the restriction of ρ on the closed normal subgroup $N^-(S)$ of U is trivial. Conversely, if $\rho(N^-(S))$ consists of the identity element alone, then ρ is completely reducible provided that X is kählerian.*

Proof. Let ρ' be any one of irreducible components of ρ. The radical of $\mathfrak{u}$ being $\mathfrak{h}_V+\mathfrak{w}+\mathfrak{n}^-(S)$, its element is represented via ρ by triangular matrices, and so the elements of $\mathfrak{n}^-(S)$ by nilpotent matrices, since $\mathfrak{n}^-(S)$ is the derived algebra of the radical. Therefore by the theorem of Engel and the irreducibility of ρ' we see that $\rho'(\mathfrak{n}^-(S))=\{0\}$. Conversely, for any kählerian X, we have $U=V^C\cdot N^-(S)$ and so $\rho(U)=\rho(V^c)\cdot\rho(N^-(S))$. Since V is compact, $\rho(U)$ is completely reducible if $\rho(N^-(S))=\{1\}$.

Let $E=E(\rho, F)$ be a homogeneous vector bundle over $X=G/U$ defined by a representation (ρ, F) of U. The *p-th induced representation* $(\rho^{\#(p)}, H^p(X, \boldsymbol{E}))$ of (ρ, F) is defined by R. Bott, where $H^p(X, \boldsymbol{E})$ denotes the p-dimensional cohomology group over X with coefficient in the sheaf $\boldsymbol{E}$ of germs of holomorphic section of E (cf. Bott [7]) ; this represen-

1) Hereafter we shall often omit for brevity the adjective, *holomorphic or complex analytic* for bundles, homomorphisms etc. as far as there are no fear of misunderstandings.

2) This lemma is partially stated in the proof of Theorem W_2, Corollary 3 in [7], but we state here for completeness' sake.

tation is the one induced on $H^p(X, \boldsymbol{E})$ from the natural action of G on $E=G\times_U F$ as the bundle automorphisms. We denote the highest character of $\rho^{\#(p)}$ by $\lambda^{\#(p)}$ when $\rho^{\#(p)}$ is irreducible. Thus we recall the main result of Bott;

Theorem of Bott. *Let $X=G/U$ be a kählerian C-manifold and let (ρ, F) be an irreducible representation of U. Let $E=E(\rho, F)$. Then the cohomology groups $H^p(X, \boldsymbol{E})$ $(p\geqq 0)$ vanish except for at most one p. More precisely, we have the following results: By Lemma 1, we can consider (ρ, F) as a representation of the complex reductive Lie group $V^c=V(S)\cdot H(S)$. H being a Cartan subgroup of V^c, let $\mathring{\lambda}$ be the highest weight of the representation (ρ, F) of V^c. Then,*

(i) *if $\mathring{\lambda}+\mathring{\delta}$ is singular, $H^p(X, \boldsymbol{E})=\{0\}$ for all $p\geqq 0$;*

(ii) *if $\mathring{\lambda}+\mathring{\delta}$ is regular and Ind. $(\mathring{\lambda}+\mathring{\delta})=p$, then $H^q(X, \boldsymbol{E})=\{0\}$ for $q\neq p$. Moreover the p-th induced representation $(\rho^{\#(p)}, H^p(X, \boldsymbol{E}))$ is irreducible and its highest weight $\mathring{\lambda}^{\#(p)}$ is determined by $\mathring{\lambda}$ as follows:*

$$\mathring{\lambda}^{\#(p)}+\mathring{\delta} = \sigma_p(\mathring{\lambda}+\mathring{\delta}),$$

where σ_p is an element of the Weyl group uniquely determined by the condition $\sigma_p(\mathring{\lambda}+\mathring{\delta})\in \mathfrak{P}$.

4. Line bundles over a C-manifold.

Let $X=G/U$ be a C-manifold and assume that G is *simply connetcted.* Denote by Hom (U, C^*) the abelian group of all holomorphic homomorphisms of U into C^*. We can define a homomorphism η of Hom (U, C^*) into the group $H^1(X, \boldsymbol{C}^*)$[3] of the equivalent classes of complex line bundles over X by assigning to every element $\lambda\in$ Hom (U, C^*) the corresponding homogeneous line bundle E_λ (in reality, the homogenous C^*-bundle P_λ).

Now we introduce the exact sequence of sheaves over X:

$$0\to Z\to \boldsymbol{C}\xrightarrow{\mathcal{E}}\boldsymbol{C}^*\to 0,$$

where $\mathcal{E}$ is, as usual, defined by $\mathcal{E}(f)=\exp 2\pi\sqrt{-1}f$ for every holomorphic function germ f on X. Let

$$(2)\qquad H^1(X, Z)\to H^1(X, \boldsymbol{C})\xrightarrow{\mathcal{E}} H^1(X, \boldsymbol{C}^*)\xrightarrow{\delta^*} H^2(X, Z)\to H^2(X, \boldsymbol{C})$$

be the corresponding cohomology exact sequence, where we denote again by $\mathcal{E}$ the homomorphism induced by $\mathcal{E}: \boldsymbol{C}\to\boldsymbol{C}^*$.

3) For a given complex Lie group A and a complex manifold X, we denote from now on by $\boldsymbol{A}$ the sheaf (of groups) of germs of holomorphic mappings of X into A.

The following theorem has been already obtained by Murakami (cf. [18], Théorème 3), but it is convenient for the later discussions to give here a new proof[4].

Theorem 1. (i) *The homomorphism* $\eta:\mathrm{Hom}\,(U, C^*)\to H^1(X, \boldsymbol{C}^*)$ *is bijective. In particular, every line bundle over a C-manifold is homogeneous.*

(ii) $$H^1(X, \boldsymbol{C}^*)=\phi^*H^1(\hat{X}, \boldsymbol{C}^*)+\varepsilon H^1(X, \boldsymbol{C}),$$

where $\phi: X\to\hat{X}$ *is the fundamental fibering.*

Proof. First we shall prove the theorem when X is kählerian (and so $X=\hat{X}$). Let E_λ be the homogenous line bundle over $\hat{X}$ defined by $\lambda\in\mathrm{Hom}\,(\hat{U}, C^*)$ and let $P_\lambda\in H^1(\hat{X}, \boldsymbol{C}^*)$ be the corresponding C^*-bundle. Then $\delta^*(P_\lambda)$ is the Chern class $c(E_\lambda)$ of E_λ. On the other hand we define the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}\,(\hat{U}, C^*) &\cong \mathrm{Hom}\,(H(S), C^*)\cong \mathrm{Hom}_{\infty}(T(S), T^1)\\ &\cong H^1(T(S), Z)\cong H^1(\hat{V}, Z),\end{aligned}$$

where $T(S)$ (resp. T^1) is the maximal toral subgroup of $H(S)$ (resp. C^*) and $\mathrm{Hom}_{\infty}(T(S), T^1)$ is the group of all differentiable homomorphisms of $T(S)$ into T^1. The first (resp. the fourth) isomorphism is obtained by the fact that $\hat{U}$ (resp. $\hat{V}$) is the product of its commutator subgroup and of $H(S)$ (resp. $T(S)$); the second exists because $H(S)$ is the complexification of the maximal compact subgroup $T(S)$; the third is defined by $\mathrm{Hom}_{\infty}(T(S), T^1)\cong\mathrm{Hom}\,(\pi_1(T(S), Z)\cong H^1(T(S),Z)$. Let $\zeta:\mathrm{Hom}\,(\hat{U}, C^*)\to H^1(\hat{V}, Z)$ be the isomorphism which is the composition of the above isomorphisms. Denoting by τ the transgression in the differentiable principal bundle $K(\hat{X}, \hat{X})$, it follows from a result of Borel-Hirzebruch ([5], Theorem 10.3) that the following diagram is commutative.

$$(3)\qquad \begin{array}{ccc}\mathrm{Hom}\,(\hat{U}, C^*) & \xrightarrow{\zeta} & H^1(\hat{V}, Z)\\ \eta\downarrow & & \downarrow\tau\\ H^1(\hat{X}, \boldsymbol{C}^*) & \xrightarrow{\delta^*} & H^2(\hat{X}, Z)\end{array}$$

Since K is simply connected and semi-simple, we see that τ is bijective (even if X is non-kählerian). On the other hand, it is known that $H^1(\hat{X}, \boldsymbol{C})=H^2(\hat{X},\boldsymbol{C})=\{0\}$ ([5], [7]). It follows then from the exact sequence (2) that δ^* is bijective. Therefore we conclude by (3) that η is bijective. This prove (i) in this case.

4) The present proof might be essentially analogous to that of Murakami, but our intention lies in clarifying the relation between the line bundles over a (non-kählerian) C-manifold and those over the associated kählerian C-manifold, which will become clear in the course of the proof.

To prove the theorem in the general case, we show that η is injective. Let $\lambda \in \operatorname{Hom}(U, C^*)$ be in the kernel of η. Then E_λ is trivial. We shall see later that λ is the restriction on U of a homomorphism $\tilde{\lambda}: G \to C^*$ (Lemma 3 in II). Since G is semi-simple, $\tilde{\lambda}(G) = \{1\}$ and hence λ is the identity element of $\operatorname{Hom}(U, C^*)$. Thus η is injective.

We prove that η is surjective. Let $X(\hat{X}, \hat{U}/U, \phi)$ be the fundamental fibering of X. The exact sequences (2) for X and $\hat{X}$ imply the commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 \to H^1(\hat{X}, \boldsymbol{C}^*) & \xrightarrow{\delta^*} & H^2(\hat{X}, Z) \to 0 \\ (4) & & & & \downarrow \phi^* & & \downarrow \phi^* \\ & 0 \to H^1(X, \boldsymbol{C}) & \xrightarrow{\varepsilon} & & H^1(X, \boldsymbol{C}^*) & \xrightarrow{\delta^*} & H^2(X, Z) \to 0\,. \end{array}$$

We show that ϕ^* in the right hand side is surjective. In fact, let $\gamma: H^1(\hat{V}, Z) \to H^1(V, Z)$ be the homomorphism induced by the inclusion $V \subset \hat{V}$. Then it holds the commutative diagram:

$$\begin{array}{ccc} H^1(\hat{V}, Z) & \xrightarrow{\tau} & H^2(\hat{X}, Z) \\ \gamma \downarrow & & \downarrow \phi^* \\ H^1(V, Z) & \xrightarrow{\tau} & H^2(X, Z) \end{array}$$

As we have seen that the transgressions τ are bijective, it is sufficient to see that γ is surjective. While, using the isomorphisms $H^1(\hat{V}, Z) \cong H^1(T(S), Z)$, $H^1(V, Z) \cong H^1(T_V, Z)$ (where T_V is the maximal toral subgroup of H_V) and $T_V \subset T(S)$, we infer immediately that γ is surjective. Then, the diagram (4) implies that

$$H^1(X, \boldsymbol{C}^*) = \phi^* H^1(\hat{X}, \boldsymbol{C}^*) + \varepsilon H^1(X, \boldsymbol{C})\,.$$

The theorem being proved for $\hat{X}$, it is easy to see that $\eta(\operatorname{Hom}(U, C^*)) \supset \phi^* H^1(\hat{X}, \boldsymbol{C}_*)$. Therefore, in order to show that η is surjective, it is sufficient to prove that $\varepsilon H^1(X, \boldsymbol{C})$ consists also of homogeneous bundles. Because $\varepsilon H^1(X, \boldsymbol{C})$ is the group of C^*-bundles which are ε-extensions of C-bundles, this will be established if the homomorphism $\eta': \operatorname{Hom}(U, C) \to H^1(X, \boldsymbol{C})$, which assigns to $\sigma \in \operatorname{Hom}(U, C)$ the homogeneous C-bundle P_σ defined by σ, is bijective. We show that η' is injective. In fact, if P_σ is the trivial C-bundle, so is the C^*-bundle $\varepsilon(P_\sigma)$. Since the bundle $\varepsilon(P_\sigma)$ is defined by $\varepsilon \circ \sigma \in \operatorname{Hom}(U, C^*)$ and since η is injective, this implies $\varepsilon \circ \sigma = 1$. Then it follows at once that σ is the identity element of $\operatorname{Hom}(U, C)$. To see that η' is surjective, we remark that η' is a complex linear mapping of the complex vector space $\operatorname{Hom}(U, C)$ into $H^1(X, \boldsymbol{C})$; then we need only to show that $\dim \operatorname{Hom}(U, C) = \dim H^1(X, \boldsymbol{C})$. To compute $\dim H^1(X, \boldsymbol{C})$ we shall use the Leray's spectral sequence $\{E_r\}$ associated with the fundamental fibering $X(\hat{X}, \hat{U}/U, \phi)$ and the sheaf $\boldsymbol{C}$ over X. We need the term E_2^1 and the final term E_∞^1 associated to $H^1(X, \boldsymbol{C})$. The C-module $E_2^1 = E_2^{1,0} + E_2^{0,1}$ is given by

$$E_2^{1,0} = H^1(\hat{X}, \phi^0(\boldsymbol{C}))\,, \quad \text{and} \quad E_2^{0,1} = H^0(\hat{X}, \phi^1(\boldsymbol{C}))\,,$$

where $\phi^p(\boldsymbol{C})$ ($p=0, 1$) is the p-dimensional direct image sheaf over $\hat{X}$ defined by the pre-sheaf $\phi^p(\boldsymbol{C})_N = H^p(\phi^{-1}(N), \boldsymbol{C})$ (for every open set $N \subset \hat{X}$). We can see, by an easy argument, $\phi^p(\boldsymbol{C}) = \boldsymbol{C} \otimes H^p(\hat{U}/U, C)$ ($p=0, 1$). Therefore, by using $H^1(\hat{X}, \boldsymbol{C}) = \{0\}$, we have

$$E_2^{1,0} = \{0\}\,, \quad E_2^{0,1} = H^1(\hat{U}/U, \boldsymbol{C})\,.$$

While the coboundary operator d_2 sends $E_2^{0,1}$ into $E_2^{2,0} = H^2(\hat{X}, \phi^1(\boldsymbol{C})) = H^2(\hat{X}, \boldsymbol{C}) = \{0\}$, so $E_2^{0,1} = E_\infty^{0,1}$. Hence we infer that

$$H^1(X, \boldsymbol{C}) \cong E_\infty^1 = E^{0,1} \cong H^1(\hat{U}/U, \boldsymbol{C}) \cong H^{0,1}(\hat{U}/U, C)\,.$$

As $\hat{U}/U$ is a complex torus, the above-obtained module is isomorphic to $\bar{\mathfrak{w}}$. On the other hand, $\mathrm{Hom}\,(U, C) \cong \mathrm{Hom}\,(H_V W, C) \cong \mathrm{Hom}\,(W, C) \cong \mathrm{Hom}\,(\mathfrak{w}, C)$. Therefore $\dim \mathrm{Hom}\,(U, C) = \dim H^1(X, \boldsymbol{C})$. This completes the proof.

We add here that Murakami's théorème 3 in [18] can be proved in the same manner as above. That is,

Theorem 1′ (MURAKAMI). *Every principal holomorphic fiber bundle over a C-manifold X with a connected complex abelian Lie group A as structure group is always homogeneous, and more precisely we have* $\mathrm{Hom}\,(U, A) \cong H^1(X, \boldsymbol{A})$.

From the proof of Theorem 1, we can derive several corollaries.

Corollary 1. *A C-manifold X is kählerian if and only if $H^1(X, \boldsymbol{C}) = \{0\}$.*

Proof. As is shown in the course of the above proof, $\dim H^1(X, \boldsymbol{C}) = \dim \mathfrak{w}$. On the other hand X is kählerian if and only if $\mathfrak{w} = \{0\}$.

Making use of a theorem of Borel-Weil [6], we derive the following theorem of Bott ([7], Theorem V).

Corollary 2 (BOTT). *Let E_λ be a line bundle over a C-manifold X defined by $\lambda \in \mathrm{Hom}\,(U, C^*)$. Then $H^0(X, \boldsymbol{E}_\lambda) \neq \{0\}$ if and only if there exists $\hat{\lambda} \in \mathrm{Hom}\,(\hat{U}, C^*)$ whose restriction to U is λ and whose differential $\mathring{\lambda}$, considered as a linear from on $\mathfrak{h}(S)$ and so on $\mathfrak{h}$, belongs to the fundamental chamber $\mathfrak{P}$. In this case, the induced representation of G on*

$H^0(X, \boldsymbol{E}_\lambda)$ is the irreducible representation with the highest character $\hat{\lambda}$.

Proof. Let $X(\hat{X}, Y, \phi)$ be the fundamential fibering of X. We show first that if $H^0(X, \boldsymbol{E}_\lambda) \neq \{0\}$ then E_λ is induced by ϕ from a line bundle over $\hat{X}$. Let $E_\lambda | Y$ be the restriction of E_λ on the fibre $\hat{U}/U = Y$. Then $H^0(X, \boldsymbol{E}_\lambda) \neq \{0\}$ implies $H^0(Y, \boldsymbol{E}_\lambda | Y) \neq \{0\}$, because E_λ is homogeneous (see, the proof of Lemma 2 in II). On the other hand, we see easily that $E_\lambda | Y$ is a homogeneous line bundle over the complex torus Y. Then, by a result of Matsushima ([13], Proposition 3. 6), $E_\lambda | Y$ is trivial. The bundle E_λ is, therefore, induced from a line bundle $E_{\hat{\lambda}}$ over $\hat{X}$ by ϕ, since λ is the restriction of a homomorphism $\hat{\lambda} \in \mathrm{Hom}(\hat{U}, C^*)$ by Lemma 3 in II. Moreover, the fibre Y being compact connected, we see easily that the natural homomorphism $H^0(\hat{X}, \boldsymbol{E}_{\hat{\lambda}}) \to H^0(X, \boldsymbol{E}_\lambda)$ is bijective. The isomorphism so obtained clearly preserves the actions of G *via* the induced representations. Our corollary now follows from the theorem of Borel-Weil concerning the induced representation of G on $H^0(\hat{X}, \boldsymbol{E}_{\hat{\lambda}})$.

Corollary 3. *The divisor class group of a C-manifold X coincides with $\phi^* H^1(\hat{X}, \boldsymbol{C}^*)$.*

Proof. As $\hat{X}$ is an algebraic manifold (cf. [6], [9] or § 6, § 7), we know that any line bundle $\hat{E}$ over $\hat{X}$ is defined by a divisor $\hat{D}$. Then the line bundle over X induced from $\hat{E}$ by $\phi: X \to \hat{X}$ is defined by the divisor induced from $\hat{D}$ by ϕ. This proves that the divisor class group of X contains $\phi^* H^1(\hat{X}, \boldsymbol{C}^*)$. Conversely, let D be a divisor on X, and let E the line bundle defined by D. We know that there are positive divisors D^+, D^- such that $D = D^+ - D^-$. If E^+ (resp. E^-) are the line bundle defined by D^+ (resp. D^-), then $E = E^+ \otimes (E^-)^*$ Since D^+ and D^- are positive, $H^0(X, \boldsymbol{E}^+) \neq \{0\}$ and $H^0(X, \boldsymbol{E}^-) \neq \{0\}$. Then, by Corollary 2, there exist line bundles $\hat{E}^+$, $\hat{E}^-$ over $\hat{X}$ such that $E^+ = \phi^* \hat{E}^+$, $E^- = \phi^* \hat{E}^-$. Thus, $E = \phi^*(\hat{E}^+ \otimes (\hat{E}^-)^*) \in \phi^* H^1(\hat{X}, \boldsymbol{C}^*)$. This completes the proof.

II. The Classification Theorem for Homogeneous Vector Bundles.

The complex analytic analogue of the topological classification theorem of fibre bundles is valid for vector bundles, which has been already formulated and proved by S. Nakano, K. Kodaira and J. P. Serre (see [3], [19]). In this chapter we shall state a sharpened form of this theorem for homogeneous vector bundles over C-manifolds and add a few applications.

5. Homogeneous vector bundles with sufficiently many sections. The imbedding theorem.

Let $X=G/U$ be a C-manifold, and $E=E(\rho, F)$ the homogeneous vector bundle over X defined by a representation (ρ, F) of dimensions m. We can identify the 0-dimensional cohomology group $H^0(X, \boldsymbol{E})$ with the complex vector space of all the holomorphic mappings s of G into F such that $s(gu)=\rho(u^{-1})s(g)$ for any $g\in G$ and $u\in U$, and the 0-th induced representation $\rho^{\#(0)}$ is defined, under this identification, by $(\rho^{\#(0)}(g)s)(g')=s(g^{-1}\cdot g')$ for any $g, g'\in G$, (see [7]). Now we define a homomorphism ν of $H^0(X, \boldsymbol{E})$ into F by

$$\nu(s) = s(e)\,, \qquad e = \text{the unit element of } G\,.$$

Then ν is compatible with the U-module structures on $H^0(X, \boldsymbol{E})$ and F. Let F' be the kernel of ν. Then F' is invariant under $\rho^{\#(0)}(U)$ and we obtain an exact sequence of U-modules:

$$0 \to F' \to H^0(X, \boldsymbol{E}) \xrightarrow{\nu} F\,.$$

When ν is surjective, we say that the vector bundle E *has sufficiently many sections*, and in addition, if $H^p(X, \boldsymbol{E})=\{0\}$ $(p\geqq 1)$, we say that E is *ample*.

Lemma 2. *If a homogeneous vector bundle E over a C-manifold X is defined by an irreducible representation (ρ, F) of U (in particular, if E is a line bundle), then $H^0(X, \boldsymbol{E})\neq\{0\}$ implies that E has sufficiently many sections and that E is ample if X is kählerian.*

Proof. If $H^0(X, \boldsymbol{E})\neq\{0\}$, then there exists a cross-section $s\in H^0(X, \boldsymbol{E})$ such that $s(e)\neq 0$, since $s(g)=(\rho^{\#(0)}(g^{-1})s)(e)$. Therefore $\nu H^0(X, \boldsymbol{E})\neq\{0\}$ and it is $\rho(U)$-invariant, which implies that ν is surjective, since (ρ, F) is irreducible. If X is kählerian, $H^p(X, \boldsymbol{E})=\{0\}$ $(p\geqq 1)$ by the theorem of Bott.

Note that there are no ample homogeneous vector bundles over any non-kählerian C-manifold X, because, for such a vector bundle E, the Euler characteristic $\chi(X, \boldsymbol{E})=0$ (Theorem III, Corollary 1 in [7]).

Lemma 3. *A homogeneous vector bundle $E(\rho, F)$ over a C-manifold X is trivial if and only if ν is bijective, or ρ is the restriction of a representation of G.*

Proof. If ρ is the restriction of a representation $\rho^{\#}$ of G, then the cross-section h of the associated principal bundle $P=G\times_U GL(F)$ is defined by $h(gU)=(g, \rho^{\#}(g^{-1}))$, $(g\in G)$ and so E is trivial. Conversely if E is trivial, then ν is of course an isomorphism of U-modules and ρ can be considered as the restriction of the induced representation $\rho^{\#(0)}$ of ρ.

In the sequel we assume that E has sufficiently many sections and set $\rho^{\#(0)}=\rho^{\#}$. Then we have

$$(1) \qquad 0 \to F' \to H^0(X, \boldsymbol{E}) \to F \to 0\,,$$

$$(1^*) \qquad 0 \to F^* \to H^0(X, \boldsymbol{E})^* \to F'^* \to 0\,.$$

Set dim $H^0(X, \boldsymbol{E})=n$ $(\geqq m)$. Then these exact sequences give rise to the exact sequences of homogeneous vector bundles:

$$(2) \qquad 0 \to E' \to I^n \to E \to 0\,,$$

$$(2^*) \qquad 0 \to E^* \to I^{*n} \to E'^* \to 0\,,$$

where $E'=E(\rho^{\#}, F')$, $E^*=(\check{\rho}, F^*)$, $E'^*=E(\check{\rho}^{\#}, F'^*)$, $I^n=E(\rho^{\#}, H^0(X, \boldsymbol{E}))$ and $I^{*n}=E(\check{\rho}^{\#}, H^0(X, \boldsymbol{E})^*)$ ($\check{\rho}^{\#}$ denotes the contragredient representation of $\rho^{\#}$). The last two vector bundles are trivial by Lemma 3.

Next we shall define the classifying manifold and the universal bundles. By $GL(n, m; C)$ we mean the subgroup of $GL(n, C)$ consisting of the matrices of the form:

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad \begin{array}{l} A\in GL(n-m, C),\ B\in GL(m, C), \\ C = \text{a matrix of } (n-m)\text{-rows and } m\text{-columns.} \end{array}$$

Then, the coset spaces $GL(n, C)/GL(n, m; C)$ and $GL(n, C)/GL(n, n-m; C)$ represent the complex Grassmann manifold $G(n, m)$ and its dual Grassmann manifold $G(n, n-m)$ respectively. The group $G(n, m; C)$ (resp. $GL(n, n-m; C)$) leaves an $(n-m)$-dimensional subspace C^{n-m} of C^n(resp. an m-dimensional subspace $(C^m)^*$ of $(C^n)^*$) invariant, and so we have the exact sequences of $GL(n, m: C)$-(resp. $GL(n, n-m; C)$-) modules:

$$(3) \qquad 0 \to C^{n-m} \to C^n \to C^m \to 0\,, \quad C^m = C^n/C^{n-m}\,,$$

$$(3^*) \qquad 0 \to (C^m)^* \to (C^n)^* \to (C^{n-m})^* \to 0\,.$$

We obtain the corresponding exact sequences of homogeneous vector bundles:

$$(4) \qquad 0 \to W_S \to I^n \to W_Q \to 0$$

$$(4^*) \qquad 0 \to W_Q^* \to I^{*n} \to W_S^* \to 0$$

over $G(n, m)$ (resp. $G(n, n-m)$) in the same manner as above. In fact, if we denote by σ_m^n the m-dimensional representation of $GL(n, m; C)$ defined by

$$\sigma_m^n\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = B\,,$$

then σ_m^n defines the homogeneous vector bundle W_Q over $GL(n, C)/GL(n, m; C)$ for which $H^0(G(n, m), \boldsymbol{W}_Q)$ is of dimension n. Identifying $H^0(G(n, m), \boldsymbol{W}_Q)$ with C^n, the induced representation $(\sigma_m^n)^\#$ of σ_m^n is nothing but the identity one of $GL(n, C)$ as is easily checked. Thus, the exact sequences (3), (3*) (4) and (4*) are the special cases of (1), (1*), (2) and (2*) respectively. The bundle W_S (resp. W_Q^*) is called *the universal subbundle* and W_Q (resp. W^*) *the universal quotient bundle over the classifying manifold* $G(n, m)$ (resp. $G(n, n-m)$). We know that $H^*(G(n, m), \boldsymbol{W}_S) = H^*(G(n, n-m), \boldsymbol{W}_Q^*) = \{0\}$ and that both W_Q and W_S^* are ample (see [7], Proposotion 14. 3).

Now we take a basis $\{\xi_1, \cdots, \xi_{n-m}, \cdots, \xi_n\}$ of the complex vector space $H^0(X, \boldsymbol{E})$ whose first $(n-m)$-vectors $\{\xi_1, \cdots, \xi_{n-m}\}$ belong to F', and identify the two exact sequences (1) and (3). Then we have the holomorphic homomorphism $\rho^\#$ of G into $GL(n, C)$ such that $\rho^\#(U) \subset GL(n, m; C)$, which induces a holomorphic mapping f_ρ of X into $G(n, m)$ defined by $f_\rho(gU) = \rho^\#(g) \cdot GL(n, m; C)$. As is easily seen, the exact sequence (2) is induced from the exact sequence (4):

$$E = f_\rho^*(W_Q), \quad E' = f_\rho^*(W_S).$$

We note that this mapping f_ρ coincides with the classifying mapping f_E associated to E defined by Nakano and Serre (cf. [3], [19]). In fact, the isomorphism η of homogeneous vector bundle $G \times_U H^0(X, \boldsymbol{E})$ onto $I^n = X \times C^n$ in (4) is defined by $\eta[g, s] = (gU, \rho^\#(g)s)$ for every $g \in G$ and $s \in H^0(X, \boldsymbol{E})$, hence the fibre E'_x of E' over a point $x = gU \in X$ corresponds *via* η to the $(n-m)$-dimensional subspace $\rho^\#(g) \cdot F'$ of $H^0(X, \boldsymbol{E}) = C^n$ and $\rho^\#(g) \cdot F'$ represents the point $\rho^\#(g) \cdot GL(n, m: C)$ in our coset space form of $G(n, m)$. Therefore $f_\rho = f_E$.

These arguments run quite similarly for the vector bundle E'^* and the contragredient representation $\check{\rho}^\#$. Thus we obtain

Theorem 2. *Let $E(\rho, F)$ be a homogeneous vector bundle with sufficiently many sections over a C-manifold X. Then the induced representation $\rho^\#$ defines a classifying holomorphic mapping f_ρ of X into the classifying manifold $G(n, m)$ and E is induced from the universal quotient bundle W_Q by the mapping f_ρ. Similarly $\check{\rho}^\#$ defines the mapping $\check{f}_\rho$ of X into $G(n, n-m)$ and E is induced from W_Q by $\check{f}_\rho$.*

This theorem is usually called *the imbedding theorem* (cf. [19]).

Proposition 2. *For any kählerian C-manifold $X = G/U$, the classifying mapping f_ρ in the imbedding theorem is biregular if and only if (ρ, F) can not be extended to a representation of any C-subgroup U' of G containing U.*

Proof. Set $U'=\{g\in G\mid \rho^{\#}(g)F'=F'\}$. Then U' is a closed complex subgroup of G such that $U'\supset U$ and the compact homogeneous space $X'=G/U'$, can be biregularly imbedded in $G(n, m)$ by the mapping f' defined by $f'(gU')=\rho^{\#}(g)F'$ for any $g\in G$. Now let K be a maximal compact subgroup of G which acts transitively on X. Then K also acts on X' transitively and X' is represented as K/V' $(V'=U'\cap K)$. The canonical kählerian metric on $G(n, m)$ which is invariant under the actions of $U(n)$ induces a K-invariant kählerian metric on X'. Therefore $X'=K/V'$ is simply connected since K is semi-simple (cf. [4]). Hence $X'=G/U'$ is a kählerian C-manifold and U' is a (connected) C-subgroup. Moreover the representation (ρ, F) of U is extendable to a representation (ρ', F) of U' using the exact sequence (1). Conversely if (ρ, F) is the restriction of a representation (ρ', F) of a C-subgroup U' such that $U'\supsetneqq U$, then $f_{\rho}=f_{\rho'}\circ\psi$ where ψ is the canonical projection of G/U onto G/U'. Hence f_{ρ} is not biregular.

6. The case of tangential bundles.

For a compact complex manifold X, the condition that X is homogeneous is equivalent to the condition that the tangential vector bundle Θ of X has sufficiently many sections. Bott [7] proved that Θ is ample for any kählerian C-manifold X. So the imbedding theorem can be applied to the tangential bundle of a C-manifold X. Let $X=G/U$ be a C-manifold with $G=A^0(X)$. Then the exact sequence of U-modules (under the adjoint actions):

$$0\to\mathfrak{u}\to\mathfrak{g}\to\mathfrak{g}/\mathfrak{u}\to 0$$

gives rise to Atiyah's exact sequence for the principal bundle $G(X, U, \pi)$ (cf. [2] and [7], p. 232):

$$0\to L(G)\to Q(G)\xrightarrow{\pi}\Theta\to 0,\quad \Theta=E(\mathrm{Ad}, \mathfrak{g}/\mathfrak{u}).$$

We see that these exact sequences are nothing but the exact sequences (1) and (2) for Θ. In fact, we can identify $\mathfrak{g}$ with $H^0(X, \Theta)$; for every $x\in\mathfrak{g}$ we define $s_x\in H^0(X, \Theta)$ by setting $s_x(g)=\pi(\mathrm{Ad}(g^{-1})x)$. Then $\nu(s_x)=s_x(e)=\pi(x)$, moreover we have $(\rho^{\#}(g)s_x)(g')=s_x(g^{-1}\cdot g')=\pi(\mathrm{Ad}(g'^{-1}g)x)=\pi(\mathrm{Ad}(g'^{-1})\mathrm{Ad}(g)\cdot x)$. This means, under the above identification $\mathfrak{g}=H^0(X, \Theta)$, that $\rho^{\#}(g)=\mathrm{Ad}g$; therefore we have the classifying mapping f_{Ad} of X into $G(n, m)$ such that

$$f_{\mathrm{Ad}}(gU)=\mathrm{Ad}g\cdot\mathfrak{u}\in G(n, m),$$

where $n=\dim G$, $m=\dim X$ and we may regard $G(n, m)$ as the set of all $(n-m)$-dimensional subspaces of $\mathfrak{g}$. If X is kählerian, then G is semi-simple and we can easily check by (1) in I that the normalizer of $\mathfrak{u}$ coincides with $\mathfrak{u}$ itself. Hence, by Proposition 2, f_{Ad} is a biregular imbedding of X into $G(n, m)$.

The above consideration is the main part of Gotô's preceding studies [9]. In fact, he proved moreover that $f_{\mathrm{Ad}}(X)$ has the structure of a rationol variety.

7. The case of line bundles.

Let E_λ be a (homogeneous) line bundle over a C-manifold such that $\dim H^0(X, E_\lambda)=n(\geqq 1)$ (cf. Lemma 2). Then the classifying manifold $G(n, 1)$ is nothing but the $(n-1)$-dimensional complex projective space P^{n-1}, and the universal quotient bundle W_Q is the line bundle of hyperplanes of P^{n-1}. The induced representation $\rho^\#$ of λ is irreducible and λ is the restriction of the highest character $\lambda^\#$ of $\rho^\#$ (for the precise meaning, see Theorem 1, Corollary 2). Now let X be kählerian. Then, the character λ being a representation of $H(S)$, $\mathring{\lambda}$ is expressed as $\mathring{\lambda}=\sum_{\mathring{\alpha}_i \notin S} p_i \mathring{\Lambda}_i$, $p_i \geqq 0$. If λ is extendable to a representation of a C-subgroup U' corresponding to a subset S' of fundamental roots containing S, then, in the above expression of $\mathring{\lambda}$, we have $p_i=0$ for $\mathring{\alpha}_i \in S'$. Therefore, by Proposition 2, the classifying mapping f_λ is a biregular projective imbedding if and only if $\mathring{\lambda}=\sum_{\mathring{\alpha}_i \notin S} p_i \mathring{\Lambda}_i$ with $p_i>0$ (cf. [5]). These results are due to Borel-Weil [6], and so we call the biregular imbedding f_λ *a Borel-Weil's imbedding* of a kählerian C-manifold X. For such an imbedding f_λ, the dimension of P^{n-1} is computable from the well-known Weyl's formula. This dimension attains its minimum by the character λ such that $\mathring{\lambda}=\sum_{\mathring{\alpha}_i \notin S} \mathring{\Lambda}_i$, and the corresponding projective imbedding f_λ will be called *the canonical imbedding* of X. For example, the Plücker coordinates of the complex Grassmann manifold $G(n, m)$ and the Segre representation of a multiply complex projective space amount to the canonical imbeddings of these manifolds, as is easily verified.

8. A generalization of Severi's theorem.

In this section we shall discuss an application of our imbedding theorem.

Let X be an algebraic manifold with the biregular projective imbedding f into the complex projective space P, and identify X with the projective manifold $f(X)$. We shall discuss here *when every positive divisor D of X can be obtained as a hypersurface section $X \cdot S$ of X* (the dot $\cdot$ means the intersection product and S denotes a hypersurface of P). It has been proved by Severi that this is the case if X is a complex Grassmann manifold and f its projective imbedding by the Plücker coordinates (cf. for example, [13]). If the above property is satisfied for the couple (X, f), then we say that *Severi type's theorem is valid for it.* In order to consider this problem, we first recall some concepts related with the divisors over an algebraic manifold Y (cf. [12]). Let D denote a positive divisor on Y. We denote by $L(D)$ the complex vector space of all meromorphic functions on Y which define the divisors multiple of $-D$, and by $|D|$ the complete linear system consisting of all positive divisors which are linearly equivalent to D. If we denote by $[D]$ the line bundle corresponding to D, then we can identify $H^0(Y, [\boldsymbol{D}])$ with $L(D)$ canonically and $|D|$ with the associated projective space of $L(D)$. Recall that every positive divisor of the complex projective space P is nothing but a hypersurface of some degree. Now assume that D is a positive divisor of X of the form $X \cdot S$ for some hypersurface S. Then $[D]$ coincides with the induced bundle $f^*[S]$ from the definition of intersection product. Therefore there exists a natural homomorphism γ of $H^0(P, [\boldsymbol{S}])$ into $H^0(X, [\boldsymbol{D}])$, or what is the same, of $L(S)$ into $L(D)$, which induces naturally the mapping $\bar{\gamma}$ of $|S|$ into $|D|$. Hence the linear system $\bar{\gamma}|S|$ is complete if and only if the homomorphism γ is surjective. Next we have a commutative diagram :

$$
\begin{array}{ccccc}
 & H^1(X, \boldsymbol{C}^*) & \xrightarrow{c} & H^2_{1.1}(X, Z) \to 0 & \\
(5) & f^* \uparrow & & \uparrow f^* & \\
 & 0 \to H^1(P, \boldsymbol{C}^*) & \xrightarrow{c} & H^2_{1.1}(P, Z) = H^2(P, Z) \to 0\,, &
\end{array}
$$

where $H^2_{1.1}(X, Z)$ denotes the subgroup of $H^2(X, Z)$ which consists of 2-cohomology class containing a closed form of type (1. 1) and c denotes the characteristic homomorphism. We note that $H^1(P, \boldsymbol{C}^*) \cong H^2(P, Z) \cong Z$ and that $H^2_{1.1}(X, Z) \neq \{0\}$.

Assume that Severi type's theorem is valid for the couple (X, f). From the above considerations we see the following facts. First f^* in the left hand side in (5) is bijective and so we have $H^1(X, \boldsymbol{C}^*) \cong H^2_{1.1}(X, Z) \cong Z$ and the Picard variety of X is trivial. Moreover if we denote by W the line bundle of hyperplanes of P, $f^*(W)$ is the generator of $H^1(X, \boldsymbol{C}^*)$ which has sufficiently many sections, since so is W of $H^1(P, \boldsymbol{C}^*)$. Therefore the imbedding f is determined uniquely up to the equivalence in the sense of the general classification theorem as far as X is given. Next, for the line bundle W^r (= the r-copies tensor product of W) corresponding to the hypersurface S^r of degree r, we write the mapping γ as $\gamma^{(r)}$ in this case :

$$(6) \qquad \gamma^{(r)} : H^0(P, \boldsymbol{W}^r) \to H^0(X, f^*(\boldsymbol{W}^r))\,.$$

Then $\gamma^{(r)}$ must be surjective for every positive integer r. Conversely the properties derived above are sufficient for the validity of Severi type's theorem for (X, f), as is easily seen.

Now we consider the case of a kählerian C-manifold X. Let E_λ be an ample line bundle over X and let $\dim H^0(X, \boldsymbol{E}_\lambda) = N(\geqq 1)$. Then $E_\lambda = f_\lambda(W)$, where f_λ is the Borel-Weil's imbedding of X associated to E_λ and W is the line bundle of hyperplanes in the ambient projective space P^{N-1} of the algebraic manifold $f_\lambda(X)$. Consider the mapping $\gamma^{(r)}$ for W^r and $E_\lambda^r = f_\lambda^*(W^r)$:

$$\gamma^{(r)}: H^0(P^{N-1}, \boldsymbol{W}^r) \to H^0(X, \boldsymbol{E}_\lambda^r),$$

where W^r and E_λ^r are defined by $(\sigma_1^N)^r$ and λ^r respectively. If we denote by $\sigma_{(r)}^\#$ and $\rho_{(r)}^\#$ the induced representations of $(\sigma_1^N)^r$ and λ^r, then the linear homomorphism $\gamma^{(r)}$ is given by

$$\gamma^{(r)}(s)(g) = s(\rho_{(1)}^\#(g)), \quad \text{for} \quad s \in H^0(P^{N-1}, \boldsymbol{W}^r), \quad g \in G$$

under the usual expression for cross-sections. The group G acts on $H^0(X, E_\lambda^r)$ *via* $\rho_{(r)}^\#$ and similarly $GL(N, C)$ acts on $H^0(P^{N-1}, \boldsymbol{W}^r)$ *via* $\sigma_{(r)}^\#$ so that G acts also on $H^0(P^{N-1}, \boldsymbol{W}^r)$ *via* $\sigma_{(r)}^\# \circ \rho_{(1)}^\#$. Now we see that these actions of G and $\gamma^{(r)}$ are compatible:

$$\rho_{(r)}^\#(g) \cdot \gamma^{(r)}(s) = \gamma^{(r)}((\sigma_{(r)}^\# \circ \rho_{(1)}^\#(g))s).$$

Therefore $\gamma^{(r)}$ is surjective if the image of $\gamma^{(r)}$ does not vanish, as the representation $\rho_{(r)}^\#$ is irreducible. On the other hand the image of $\gamma^{(r)}$ does not vanish, because there exists at least one hypersurface S of degree $r > 0$ passing through each point of X. Hence *all the hypersurface sections of a given positive degree r constitute a complete linear system.* Next, we know that $H^2(X, Z) = H_{1,1}^2(X, Z)$ for any kählerian C-manifold X (cf. [5]). Therefore $H^1(X, \boldsymbol{C}^*) \cong H_{1,1}^2(X, Z) \cong Z$ if and only if the second Betti number $b^2(X)$ of X equals 1, and such a X is nothing but a maximal C-manifold in our terminology.

We prove now the following

Theorem 3. *Let X be a maximal C-manifold and let f be the canonical imbedding of X into the complex projective space P^{N-1}. Then every positive divisor D of X can be represented as a hypersurface section in P^{N-1}, i.e. there exists a hypersurface S_r of degree r such that*

$$D = f^{-1}(f(X) \cdot S_r),$$

where r is the degree of D (= the Chern class of $[D]$).

Proof. X being a maximal C-manifold, the group $H^1(X, \boldsymbol{C}^*)$ of all line bundles is a cyclic group with the generator $E_{\mathring{\Lambda}}$ where $\mathring{\Lambda}$ is the fundamental weight defined by $\mathring{\Lambda}(h_i)=0$ for all fundamental roots $\mathring{\alpha}_i \in S$. The line bundle $E_{\mathring{\Lambda}}$ realizes the canonical imbedding $f=f_{\mathring{\Lambda}}$. We note here that for the projective space P^{N-1} our $E_{\mathring{\Lambda}}$ is nothing but W. Therefore the correspondence $E_{\mathring{\Lambda}}^r \leftrightarrow W^r$ gives rise to the canonical isomorphism $f^* : H^1(P^{N-1}, \boldsymbol{C}^*) \cong H^1(X, \boldsymbol{C}^*)$. All the other requirements have already been satisfied.

Corollary (Severi). *Every positive divisor on $G(n, m)$ is the complete intersection of $G(n, m)$ and a hypersurface of the complex projective space P^{N-1} into which $G(n, m)$ is imbedded by using the Plücker-coordinates, where* $N=\binom{n}{m}$ (cf. [13]).

9. The classification theorem[5)]

To state the so-called classification theorem, it is necessary to formulate the condition of equivalence between two homogeneous vector bundles in terms of the representation.

Let $E_1(\rho_1, F_1)$ and $E_2(\rho_2, F_2)$ be two m-dimensional homogeneous vector bundles with sufficiently many sections over a C-manifold X and we assume $\dim H^0(X, \boldsymbol{E}_1)=\dim H^0(X, \boldsymbol{E}_2)=n$. We have then the exact sequences with the same meaning as (1), (2):

$$0 \to F_i' \to H^0(X, \boldsymbol{E}_i) \to F_i \to 0\,,$$
$$0 \to E_i' \to I^n \to E_i \to 0\,, \qquad (i = 1, 2)\,.$$

Proposion 3. *Two vector bundles E_1 and E_2 are equivalent (isomorphic as vector bundles) if and only if there exists a linear isomorphism φ of $H^0(X, \boldsymbol{E}_1)$ onto $H^0(X, \boldsymbol{E}_2)$ such that*

$$\varphi(\rho_1^\sharp(g)F_1') = \rho_2^\sharp(g)F_2'$$

for every element $g \in G$.

Proof. First we recall that $E_1 \cong E_2$ as vector bundles if and only if there exists a holomorphic mapping h of G into $\operatorname{Hom}(F_1, F_2)$ such that each $h(g)$ $(g \in G)$ is an isomorphism and $h(gu)=\rho_2(u^{-1})h(g)\rho_1(u)$ for any $g \in G$ and $u \in U$. Assume that such a mapping h exists. Then we can define a linear isomorphism $\varphi=\varphi(h)$ of $H^0(X, \boldsymbol{E}_1)$ onto $H^0(X, \boldsymbol{E}_2)$ by setting $(\varphi(s))(g)=h(g)\cdot s(g)$ under the identifications:

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5) This section is not new in its contents, but is added to the present chapter only for completeness' sake, since there are no explicit statements in the literature concerning the classification theorem (see [3]). Not necessarily homogemeous case can be treated quite analogously.

$$H^0(X, \boldsymbol{E}_i) = \{s: G \to F_i \,|\, s(gu) = \rho_i(u^{-1})s(g)\,,\\ \text{for any } g \in G, u \in U\}, \qquad (i = 1, 2)\,.$$

Then $s(e)=0$ implies $(\varphi(s))\cdot(e)=h(e)\cdot s(e)=0$, which means $\varphi(F_1')=F_2'$. Moreover for every $g \in G$ and $s \in F_1'$, we have $(\rho_2^\sharp(g^{-1})\varphi\cdot\rho_1^\sharp(g)\cdot s)\cdot(e)=0$ and so $\varphi(\rho_1^\sharp(g)F_1')=\rho_2^\sharp(g)F_2'$.

Conversely if there exists such a linear isomorphism φ, we can difine a holomorphic mapping h of G into $\mathrm{Hom}\,(F_1, F_2)$ by setting $h(g)\xi=\varphi(s)\cdot g$ for $g \in G$, $\xi \in F_1$ and $s \in H^0(X, \boldsymbol{E}_1)$ such that $s(g)=\xi$. In fact, $t(g)=0$ $(t \in H^0(X, \boldsymbol{E}_1))$ means $(\rho_2^\sharp(g^{-1})t)(e)=0$ and so $\rho_1^\sharp(g^{-1})t=t' \in F'$, which implies $\varphi(t)=\varphi(\rho_1^\sharp(g)t') \in \rho_2^\sharp(g)F_2'$. This shows that $\varphi(t)\cdot g=0$. On the other hand $h(gu)\xi=\varphi(s)\cdot(gu)=\rho_2(u^{-1})(\varphi(s)g)=\rho_2(u^{-1})h(g)s(g)=\rho_2(u^{-1})h(g)\rho_1(u)s(gu)$ for $s(gu)=\xi$. This furnishes the proof.

Now let $\mathfrak{H}_m^n(X)$ be the set of all equivalent classes of m-dimensional homogeneous vector bundles E over a C-manifold X which have sufficiently many sections with $\dim H^0(X, \boldsymbol{E})=n$, and let $\mathfrak{M}_m^n(X)$ denote the set of all holomorphic mappings $f_{\tilde\rho}$ of X into $G(n, m)$ which are induced by the homomorphisms ρ of G into $GL(n, C)$ such that $\rho(U) \subset GL(n, m; C)$. Let $\mathfrak{H}_m^S(X)=\bigcup_{n \geqq m} \mathfrak{H}_m^n(X)$ and $\mathfrak{M}_m(X)=\bigcup_{n \geqq m} \mathfrak{M}_m^n(X)$. By Theorem 2, to every element $E=E(\rho, F) \in \mathfrak{H}_m^n(X)$ corresponds an element $f_E=f_{\tilde\rho} \in \mathfrak{M}_m^n(X)$. Conversely let $f_{\tilde\rho} \in \mathfrak{M}_m^n(X)$ and let $E=f_{\tilde\rho}^* W_Q$ be the bundle induced from the universal quotient bundle W_Q by the mapping $f_{\tilde\rho}$. Then E is an m-dimensional homogeneous vector bundle over X; in fact,

$$E = E(\rho,F),\ \rho = \sigma_m^n \circ \tilde\rho\ ;\ U \to GL(F) = GL(m, C)\,,$$

where $W_Q=E(\sigma_m^n, F)$. Then we have the commutative diagram:

$$\begin{array}{ccccc}
H^0(G(n, m), \boldsymbol{W}_Q) & \xrightarrow{\nu_m^n} & F = C^m & \longrightarrow & 0 \\
\gamma\downarrow & & \| & & \\
H^0(X, \boldsymbol{E}) & \xrightarrow{\nu} & F & &
\end{array}$$

where γ is the linear mapping defined by $\gamma(s)\cdot(g)=s(\rho(g))$ for every $g \in G$ and $s \in H^0(G(n, m), \boldsymbol{W}_Q)$ (the verification of $\gamma(s) \in H^0(X, \boldsymbol{E})$ is trivial!), and where ν_m^n is the mapping ν for the ample vector bundle W_Q. The commutativity of the above diagram is checked as follows; for any element $s \in H^0(G(n, m), \boldsymbol{W}_Q)$, $\nu(\gamma(s))=\gamma(s)\tilde\rho(e)=s(\tilde\rho(e))=s(e_n)=\nu_m^n(s)$ (e_n is the unit matrix in $GL(n, C)$). This shows that ν is surjective and hence E has sufficiently many sections, that is $E \in \mathfrak{H}_m^S(X)$. It is noted that the processes thus obtained: $E(\rho, F) \to f_\rho=f_{\rho^\sharp}$ and $f_{\tilde\rho} \to E(\rho, F)$ are not reversible in general, since for $f_{\tilde\rho} \in \mathfrak{M}_m^n(X)$ it does not necessarily hold $E(\rho, F)$

$\in \mathfrak{H}^{n}_{m}(X)$, and that γ is not surjective in general (cf. the case of Theorem 3). We shall now introduce in $\mathfrak{M}_m(X)$ *an equivalence relation.* Let $f_i = f_{\tilde{\rho}_i} \in \mathfrak{M}^{n_i}_{m}(X)$ $(i=1, 2)$ and let $E_i = f^*_i(W_i) = E(\rho_i, F_i)$, where W_i $(i=1, 2)$ are the universal quotient bundles over $G(n_i, m)$, $\rho_i = \sigma^{n_i}_{m} \circ \tilde{\rho}_i$ and $F_i \cong C^m \cong C^{n_i}/C^{n_i-m}$ $(i=1, 2)$. We say that f_1 *and* f_2 *are equivalent* if the following conditions are satisfied: *there exists a suitable complex Grassmann manifold* $G(n, m)$ *and two holomorphic mappings* φ_i $(i=1, 2)$ *of* $f_i(X)$ *into* $G(n, m)$ *such that* (i) $\varphi_1 \circ f_1 = \varphi_2 \circ f_2$ *and* (ii) $\varphi^*_i(W) \cong W_i$ *on* $f_i(X)$ $(i=1, 2)$, *where* W *is the universal quotient bundle of* $G(n, m)$. Denote by $\tilde{\mathfrak{M}}_m(X)$ the set of all equivalent classes in $\mathfrak{M}_m(X)$. We have then the so-called *classification theorem*:

Theorem 4. *The correspondence* $E(\rho, F) \to f_\rho$ *given in Theorem* 2 *defines a one-to-one correspondence between* $\mathfrak{H}^{S}_{m}(X)$ *and* $\tilde{\mathfrak{M}}_m(X)$.

Proof. The above correspondence is clearly injective; in fact, take two vector bundles $E_i \in \mathfrak{H}^{n_i}_{m}(X)$ $(i=1,2)$ and let f_i be the canonical mappings f_{ρ_i} associated to E_i in Theorem 2. Suppose that f_1 and f_2 are equivalent. Then $\varphi^*_i(W) \cong W_i$ on $f_i(X)$ $(i=1, 2)$ and $\varphi_1 \circ f_1 = \varphi_2 \circ f_2$. Since $E_i \cong f^*_i(W_i)$ by Theorem 2, we obtain $E_1 \cong E_2$.

We shall now show that our correspondence is surjective. Take a mapping $f_1 \in M^{n_1}_{m}(X)$ which is induced by a homomorphism ρ_1 of G into $GL(n_1, C)$ such that $\rho_1(U) \subset GL(n_1, m; C)$, and let $E_1 = f^*_1(W_1)$ be the induced bundle of the universal quotient bundle W_1 of $G(n_1, m)$ by f_1. Assuming that $\dim H^0(X, \boldsymbol{E}_1) = n (\geqq m)$, we take the complex Grassmann manifold $G(n, m)$ and its universal quotient bundle W. The bundle E_1 is defined by $\rho_1 = \sigma^{n_1}_{m} \circ \tilde{\rho}_1$. We define the holomorphic homomorphisms τ_1 of $\rho_1(G)$ into $GL(n, C)$ and the induced holomorphic mapping φ_1 of $f_1(X)$ into $G(n, m)$ by setting:

$$\tau_1(\tilde{\rho}_1(g)) = \rho^\sharp_1(g)$$
$$\varphi_1(\tilde{\rho}_1(g) \cdot GL(n_1, m; C)) = \rho^\sharp_1(g) \cdot GL(n, m; C),$$

where $\rho^\sharp_1$ means the induced representation of ρ_1. We shall show that these definitions are well defined. To see this, we have only to verify that $\tilde{\rho}_1(g) = e_1$ implies $\rho^\sharp_1(g) = e$ (e_1 and e denote the unit elements of $GL(n_1, C)$ and $GL(n, C)$ respectively), and that $\tilde{\rho}_1(g) \in GL(n_1, m; C)$ implies $\rho^\sharp_1(g) \in GL(n, m; C)$. Let $G' = \{g \in G \mid \tilde{\rho}_1(g) = e_1\}$ and $U' = \{g \in G \mid \tilde{\rho}_1(g) \in GL(n_1, m; C)\} \supset G'$. Then $\tilde{\rho}_1$ induces the natural *biregular* mapping f'_1 of G/U' into $G(n_1, m; C)$ such that f_1; $G/U \to G/U' \to G(n_1, m; C)$ and ρ_1 is extendable to a homomorphism of U' into $GL(m, C)$ for which the induced representation coincides with $\rho^\sharp$ itself, therefore $\rho^\sharp_1(U') \subset GL(n, m;$

C). Moreover we can identify $G/U'=G/G'/U'/G'$. Now we may regard $\tilde{\rho}$ as an isomorphism of G/G' into $GL(n_1, C)$, and so ρ as a homomorphism of U'/G' into $GL(m, C)$. Hence the induced representation of ρ can be considered as a representation of G/G', and therefore $\rho^{\#}(G')=\{e\}$. We have $\varphi_1\circ f_1=f_{\rho_1}$ and $\varphi_1^*(W)=W_1$ on $f_1(X)$ by the definition of φ_1. Let $E(\rho, F)\in \mathfrak{H}_m^n(X)$ and let $E\cong E_1$. By Proposition 3 there is a biregular transformation φ of $G(n, m)$ such that $\varphi(\rho_1^{\#}(g)GL(n, m; C))=\rho^{\#}(g)GL(n, m; C)$. We have then $\varphi\circ f_{\rho_1}=f_\rho$. Therefore $(\varphi\circ\varphi_1)\circ f_1=f_\rho$ and $(\varphi\circ\varphi_1)^*(W)\cong W_1$ on $f_1(X)$. The mapping $\varphi\circ\varphi_1$ is the required one and the proof is completed.

REMARK. The imbedding theorem and the classification theorem are formulated only for vector bundles with sufficiently many sections, in particular for ample vector bundles. The relation between the general vector bundles and these ones is given by the fundamental theorem of J. P. Serre [20] if the base manifold is algebraic. Kähleian C-manifolds being algebraic, Serre's theorem is applicable. For non-kählerian C-manifolds, the analogue of Théorème A in [20] is not true in general. In fact, let E be a homogeneous vector bundle over a non-kählerian C-manifold X with the associated fundamental fibering $X(\hat{X}, Y, \phi)$, where $Y=\hat{U}/U$ is a complex torus. Then the restriction E_Y of E on Y is also homogeneous, so it decomposes into the direct sum of indecomposable homogeneous vector bundles E_i, $E_Y=E_1\oplus\cdots\oplus E_k$, $E_i=E_i'\otimes E_i''$ $(1\leqq i\leqq k)$, where E_i' is a homogeneous line bundle and E_i'' a homogeneous indecomposable vector bundle obtained by a successive extension by trivial line bundles (see, for detail, Matsushima [16]). Moreover $H^0(Y, \boldsymbol{E}_Y)=H^0(Y, \boldsymbol{E}_1)+\cdots+H^0(Y, \boldsymbol{E}_k)$, and $H^0(Y, \boldsymbol{E}_i)\neq\{0\}$ if and only if E_i' is trivial ([16], Lemma 5, 2). If E has sufficiently many sections, the same is true for $E_i (1\leqq i\leqq k)$, which implies that all $E_i' (1\leqq i\leqq k)$ are trivial line bundles. Now let $E'=\phi^*\hat{E}\oplus E_0$, where $\hat{E}$ and E_0 are non-trivial line bundles over $\hat{X}$ and X respectively such that E_0 is not the induced one from a line bundle over $\hat{X}$ and that the Chern class of E_0 is zero. Since any line bundle over X is homogeneous by Theorem 1, E' is homogeneous. Suppose now that $E=E'\otimes F$ has sufficiently many sections for a suitable line bundle F. Then E is also homogeneous and it follows from the above consideration that $E_Y=I\oplus I$, $I=(\phi^*\hat{E})_Y\otimes F_Y$, $I=(E_0)_Y\otimes F_Y$, I denoting a trivial line bundle over Y. Clearly $(\phi^*\hat{E})_Y=I$ and hence $F_Y=I$. Therefore $(E_0)_Y=I$ and this is a contradiction. Thus $E'\otimes F$ can not have sufficiently many sections for any line bundle F.

III. On a Conjecture of A. Grothendieck.

Let $\mathfrak{E}_m(X)$ be the set of equivalent classes of m-dimensional vector bundles over X, and let $\mathfrak{T}_m(X)$ be the subset of $\mathfrak{E}_m(X)$ of the classes containing a vector bundle whose structure group is reducible to the group $\Delta(m, C)$ consisting of triangular matrices with coefficients 0 under the diagonal. Let $\mathfrak{S}_m(X)$ be the subset of $\mathfrak{E}_m(X)$ of the classes containing a vector bundle which is the direct sum of m line bundles. Obviously we have

$$\mathfrak{E}_m(X) \supset \mathfrak{T}_m(X) \supset \mathfrak{S}_m(X)\,, \qquad (m \geqq 1)\,.$$

Moreover let $\mathfrak{H}_m(X)$ denotes the subset of $\mathfrak{E}_m(X)$ of the classes containing a homogeneous vector bundle. Theorem 1 implies

$$\mathfrak{H}_m(X) \supset \mathfrak{S}_m(X) \qquad (m \geqq 1)\,,$$

since the direct sum of homogeneous vector bundles is again homogeneous. A. Grothendieck [10] proved that if X is a complex projective line P^1, then every vector bundle is decomposed into the direct sum of line bundles, that is

$$\mathfrak{E}_m(P^1) = \mathfrak{S}_m(P^1)\,, \qquad (m \geqq 1)\,.$$

Therefore

$$\mathfrak{E}_m(P^1) = \mathfrak{H}_m(P^1) = \mathfrak{T}_m(P^1) = \mathfrak{S}_m(P^1)\,, \qquad (m \geqq 1)\,.$$

We shall study here mutual relations between $\mathfrak{T}_m(X)$, $\mathfrak{S}_m(X)$ and $\mathfrak{H}_m(X)$ for higher dimensional C-manifolds. Note that $\dim X > 1$ is equivalent to $\dim G > 3$ or $l = \operatorname{rank} G > 1$.

10. Vector bundles over a flag manifold

Proposition 4. *If X is a flag manifold whose dimension is greater than* 1, *then we have*

$$\mathfrak{T}_m(X) \supsetneqq \mathfrak{H}_m(X) \supsetneqq \mathfrak{S}_m(X)$$

for any positive integer $m \geqq 2$.

Proof. Since $X = G/U$ is a flag manifold, U is a maximal solvable subgroup of G. For any m-dimensional holomorphic representation ρ of U, the image $\rho(U)$ is considered to be contained in $\Delta(m, C)$ by Lie's theorem and this implies $\mathfrak{H}_m(X) \subset \mathfrak{T}_m(X)$. To prove that $\mathfrak{T}_m(X) \supsetneqq \mathfrak{H}_m(X) \supsetneqq \mathfrak{S}_m(X)$, it is sufficient to show that $\mathfrak{T}_2(X) \supsetneqq \mathfrak{H}_2(X) \supsetneqq \mathfrak{S}_2(X)$, because if there exist $E \in \mathfrak{T}_2(X)$ and $E' \in \mathfrak{H}_2(X)$ such that $E \notin \mathfrak{H}_2(X)$ and $E' \notin \mathfrak{S}_2(X)$ respectively then $E \oplus I^{m-2} \in \mathfrak{T}_m(X)$, $\notin \mathfrak{H}_m(X)$ and $E' \oplus I^{m-2} \in \mathfrak{H}_m(X)$, $\notin \mathfrak{S}_m(X)$

by Théorème 2 in [15] and the uniqueness theorem for the direct sum decomposition in [1].

Every vector bundle $E \in \mathfrak{T}_2(X)$ is obtained by an extension of a line bundle by another one and conversely. We consider here the extension Ξ of the trivial line bundle I by a line bundle E_λ:

$$\Xi \, : 0 \to E_\lambda \to E \to I \to 0 \,, \qquad E \in \mathfrak{T}_2(X) \,.$$

From the exact sequence Ξ, we derive the following three exact sequences.

$$\begin{aligned} \Xi^* &: \ 0 \to I \to E^* \to E_\lambda^* \to 0 \,, \\ \Xi^* \otimes E_\lambda &: \ 0 \to E_\lambda \to \operatorname{Hom}(E, E_\lambda) \to I \to 0 \,, \\ E^* \otimes \Xi &: \ 0 \to \operatorname{Hom}(E, E_\lambda) \to \operatorname{Hom}(E, E) \to E^* \to 0 \,. \end{aligned}$$

The second exact sequence $\Xi^* \otimes E_\lambda$ induces the cohomology exact sequence:

$$\begin{aligned} 0 \to H^0(X, \boldsymbol{E}_\lambda) &\to H^0(X, \operatorname{Hom}(\boldsymbol{E}, \boldsymbol{E}_\lambda)) \to H^0(X, \boldsymbol{C}) \\ &\xrightarrow{\delta^*} H^1(X, \boldsymbol{E}_\lambda) \to H^1(X, \operatorname{Hom}(\boldsymbol{E}, \boldsymbol{E}_\lambda)) \to H^1(X, \boldsymbol{C}) \,, \end{aligned}$$

where $H^0(X, \boldsymbol{C}) = C$, $H^1(X, \boldsymbol{C}) = \{0\}$ and $\delta^*(1) \in H^1(X, \boldsymbol{E}_\lambda)$ represents the obstruction class of Ξ. As is well known, there is a one-to-one correspondence between the set of equivalent classes of extensions of type Ξ and the group $H^1(X, \boldsymbol{E}_\lambda)$ given by $\{E\} \leftrightarrow \delta^*(1)$, and Ξ is the trivial extension if and only if $\delta^*(1) \neq 0$ (cf. [2]). Therefore if $H^1(X, \boldsymbol{E}_\lambda) \neq \{0\}$, there is a non-trivial extension Ξ for which $\delta^*(1) \neq 0$ and in this case δ^* is injective. Since $H^0(X, \boldsymbol{E}_\lambda) = \{0\}$ by Bott's thorem, $H^0(X, \operatorname{Hom}(\boldsymbol{E}, \boldsymbol{E}_\lambda)) = \{0\}$. Thus we obtain:

$$0 \to H^0(X, \boldsymbol{C}) \xrightarrow{\delta^*} H^1(X, \boldsymbol{E}_\lambda) \to H^1(X, \operatorname{Hom}(\boldsymbol{E}, \boldsymbol{E}_\lambda)) \to 0 \,.$$

Next, from $E^* \otimes \Xi$ we get:

$$0 \to H^0(X, \operatorname{Hom}(\boldsymbol{E}, \boldsymbol{E})) \to H^0(X, \boldsymbol{E}^*) \xrightarrow{\delta^*} H^1(X, \operatorname{Hom}(\boldsymbol{E}, \boldsymbol{E}_\lambda)) \,.$$

Suppose that E is decomposable and let $E = E_1 \oplus E_2$, $E_i \in E_1(X)$. Then there are two linearly independent elements $s_1, s_2 \in H^0(X, \operatorname{Hom}(\boldsymbol{E}, \boldsymbol{E}))$, which are defined by the properties that $s_1(x)$ (resp. $s_2(x)$) is the identical automorphism on $(E_1)_x$ (resp. on $(E_2)_x$) and the zero homomorphism on $(E_2)_x$ (resp. on $(E_1)_x$). Therefore $\dim H^0(X, \operatorname{Hom}(\boldsymbol{E}, \boldsymbol{E})) \geqq 2$ (cf. IV, § 14) and we have $\dim H^0(X, \boldsymbol{E}^*) \geqq 2$ by the above exact sequence. While, from Ξ^* we have

$$0 \to H^0(X, \boldsymbol{C}) \to H^0(X, \boldsymbol{E}^*) \to H^0(X, \boldsymbol{E}_\lambda^*) \to 0 \,.$$

It follows that $\dim H^0(X, \boldsymbol{E}_\lambda^*) \geqq 1$. From these arguments and Bott's theorem, we have

Lemma 4. *Let* λ *be a character satisfying the following conditions*: i) $\mathring{\lambda}+\mathring{\delta}$ *is regular and* Ind. $(\mathring{\lambda}+\mathring{\delta})=1$, ii) $-\mathring{\lambda}+\mathring{\delta}$ *is singular or regular and* Ind. $(-\mathring{\lambda}+\mathring{\delta})>0$. *Then, for any non-trivial extension* Ξ, E *is indecomposable and hence* $E \notin \mathfrak{S}_2(X)$, $E \in \mathfrak{T}_2(X)$.

Next we shall consider the condition that the 2-dimensional vector bundles defined as above should be homogeneous.

Lemma 5. *Assume that* Ξ *is a non-trivial extension. Then the vector bundle* E *is homogeneous if* $\dim H^1(X, \boldsymbol{E}_\lambda)=1$.

Proof. Let $\dim H^1(X, \boldsymbol{E}_\lambda)=1$. The structure group of E is reducible to the group $\Delta_1(2, C)=\left\{\begin{pmatrix} * & * \\ 0 & 1\end{pmatrix} \in GL(2, C)\right\}$. The associated principal $\Delta_1(2, C)$-bundle of E will be denoted by $P(X, \Delta_1(2, C), \varpi)$. For the bundle P, we consider Atiyah's exact sequence and the corresponding cohomology exact sequence:

$$0 \to L(P) \to Q(P) \xrightarrow{\varpi} \Theta \to 0\,,$$

$$0 \to H^0(X, \boldsymbol{L}(\boldsymbol{P})) \to \mathfrak{f}(P) \xrightarrow{\varpi} \mathfrak{a}(X) \to H^1(X, \boldsymbol{L}(\boldsymbol{P}))\,.$$

Now we note that $L(P)=P\times_{\Delta_1}\delta_1(2, C)$, where $\delta_1(2, C)=\left\{\begin{pmatrix} * & * \\ 0 & 0\end{pmatrix} \in \mathfrak{gl}(2, C)\right\}$ is the Lie algebra of $\Delta_1(2, C)$ and the action of $\Delta_1(2, C)$ on $\delta_1(2, C)$ is the adjoint one. For any element $u=\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix} \in \Delta_1(2, C)$, $\mathrm{Ad}\,u=\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}$ in a suitable basis of $\delta_1(2, C)$, so that we have $L(P)\cong E$. The corresponding cohomology exact sequence of Ξ is

$$0 \to H^0(X, \boldsymbol{E}) \to H^0(X, \boldsymbol{C}) \xrightarrow{\delta^*} H^1(X, \boldsymbol{E}_\lambda) \to H^1(X, \boldsymbol{E}) \to 0\,.$$

Since Ξ is non trivial, the image of δ^* does not vanish (cf. [3] Lemma 13), and so we have $H^0(X, \boldsymbol{E})=H^1(X, \boldsymbol{E})=\{0\}$, and hence $H^0(X, \boldsymbol{L}(\boldsymbol{P}))=H^1(X, \boldsymbol{L}(\boldsymbol{P}))=\{0\}$. Therefore from the above exact sequence we have $\varpi: \mathfrak{f}(P)\cong \mathfrak{a}(X)$, which implies that P is homogeneous. The bundle E is thus homogeneous.

Now we shall construct vector bundles satisfying the conditions of Lemma 4 and Lemma 5 respectively.

(a). Let $\bar{p}_k (1 \leqq k \leqq l)$ denotes the maximum of p_k which is the k-th coefficient in any positive root $\mathring{\alpha}=\sum_{i=1}^{l} p_i\mathring{\alpha}_i$, and let q_k be an arbitrary integer not less than $3\bar{p}_k$. We define the weights $\mathring{\lambda}_{(k)}$ by setting:

$$\mathring{\lambda}_{(k)} = -2\mathring{\Lambda}_k + q_k(\mathring{\Lambda}_1 + \cdots + \mathring{\Lambda}_{k-1} + \mathring{\Lambda}_{k+1} + \cdots + \mathring{\Lambda}_l)\,.$$

Then, for any positive root $\mathring{\alpha} = \sum_{i=1}^{l} p_i \mathring{\alpha}_i$ different from $\mathring{\alpha}_k$, we have

$$\begin{aligned}(\mathring{\lambda}_{(k)} + \mathring{\delta})(h_\alpha) &= \sum_{i=1}^{l} \frac{(\mathring{\alpha}_i, \mathring{\alpha}_i)}{(\mathring{\alpha}, \mathring{\alpha})} p_i (\mathring{\lambda}_{(k)} + \mathring{\delta})(h_i) \\ &= \frac{(\mathring{\alpha}_k, \mathring{\alpha}_k)}{(\mathring{\alpha}, \mathring{\alpha})} \left\{ -p_k + \sum_{i \neq k} \frac{(\mathring{\alpha}_i, \mathring{\alpha}_i)}{(\mathring{\alpha}_k, \mathring{\alpha}_k)} p_i (q_k + 1) \right\} > 0\,,\end{aligned}$$

and $(\mathring{\lambda}_{(k)} + \mathring{\delta})(h_k) = -2 + 1 < 0$. Therefore Ind. $(\mathring{\lambda}_{(k)} + \mathring{\delta}) = 1$. On the other hand, $(-\mathring{\lambda}_{(k)} + \mathring{\delta})(h_i) = 1 - q_k \leqq 0$ for any $i \neq k$. This means that $-\mathring{\lambda}_{(k)} + \mathring{\delta}$ is singular or regular and Ind. $(-\mathring{\lambda}_{(k)} + \mathring{\delta}) > 0$. Thus the character $\mathring{\lambda}_{(k)}$ satisfies the conditions i) and ii) in Lemma 4.

(b). Let $\mathring{\mu}_{(k)} = -\mathring{\alpha}_k$ $(1 \leqq k \leqq l)$. Then, $\mathring{\mu}_{(k)} + \mathring{\delta} = \sigma_k(\mathring{\delta})$, where σ_k is the element of the Weyl group $\mathfrak{W}$ corresponding to $\mathring{\alpha}_k$, and we have $(\mathring{\mu}_{(k)} + \mathring{\delta}, \mathring{\alpha}) = (\mathring{\delta}, \sigma_k(\mathring{\alpha})) > 0$ for any $\mathring{\alpha} \in \sum^+$, $\mathring{\alpha} \neq \mathring{\alpha}_k$, and $(\mathring{\mu}_{(k)} + \mathring{\delta}, \mathring{\alpha}_k) = (\mathring{\delta}, -\mathring{\alpha}_k) < 0$. Moreover $\sigma_k(\mathring{\delta})$ is clearly regular and on the other hand, $-\mathring{\mu}_{(k)} + \mathring{\delta} = \mathring{\alpha}_k + \mathring{\delta}$ is singular, or regular and of positive index. Thus i) and ii) in Lemma 4 are satisfied for the character $\mu_{(k)}$. Now, by the theorem of Bott in I, $\mu_{(k)}^{\#(1)} = 1$, so that we have $\dim H^1(X, \boldsymbol{E}_{\mu(k)}) = 1$, which is the condition of Lemma 5.

Now we shall return to the proof of Proposition 4. First, $\mathfrak{H}_2(X) \supsetneqq \mathfrak{S}_2(X)$ is obvious from Lemma 4, Lemma 5 and the examples in (b). We show that any vector bundle E given by Lemma 4 and the examples in (a) is not homogeneous. We have then $\mathfrak{T}_2(X) \supsetneqq \mathfrak{H}_2(X)$ and the proof will be complete.

Assume that E is a homogeneous vector bundle $E(\rho, F)$. Then there exists an exact sequence of U-modules : $0 \to (\lambda_1, C^1) \to (\rho, F) \to (\lambda_2, C^1) \to 0$, and the corresponding one of homogeneous vector bundles: $0 \to E_{\lambda_1} \to E \to E_{\lambda_2} \to 0$. The last extension being not splittabe by $E \notin \mathfrak{S}_2(X)$, the coboundary operator $\delta^* : H^0(X, \boldsymbol{I}) \to H^1(X, \boldsymbol{E}_{\lambda_1 \cdot \lambda_2^{-1}})$ is injective as G-modules under the induced representations, and moreover the obstruction class $\delta^*(1)$ is G-invariant. Therefore, by Bott's theorem, we have $\dim H^1(X, \boldsymbol{E}_{\lambda_1 \cdot \lambda_2^{-1}}) = 1$. There exists then a fundamental root $\mathring{\alpha}_i (1 \leqq i \leqq l)$ such that $\sigma_i(\mathring{\lambda}_1 - \mathring{\lambda}_2 + \mathring{\delta}) = \mathring{\delta}$ or $\mathring{\lambda}_1 - \mathring{\lambda}_2 = -\mathring{\alpha}_i$. On the other hand, considering the 1st Chern class of E, we have $\mathring{\lambda}_{(k)} = \mathring{\lambda}_1 + \mathring{\lambda}_2$ (cf. (3) in I). Now, from the exact sequence of homogeneous vector bundles : $0 \to E^*_{\lambda_2} \to E^* \to E^*_{\lambda_1} \to 0$, we have the cohomology exact sequence :

$$0 \to H^0(X, \boldsymbol{E}^*_{\lambda_2}) \to H^0(X, \boldsymbol{E}^*) \to H^0(X, \boldsymbol{E}^*_{\lambda_1}) \to\,,$$

which is compatible with G-module structures. While we see that $\dim H^0(X, \boldsymbol{E}^*) = 1$ since $H^0(X, \boldsymbol{E}^*_{\lambda(k)}) = \{0\}$, therefore it must be

$\dim H^0(X, \boldsymbol{E}^*_{\lambda_1})=1$ or $\dim H^0(X, \boldsymbol{E}^*_{\lambda_2})=1$, which means $\mathring{\lambda}_1=0$ or $\mathring{\lambda}_2=0$ by Lemma 3 and Theorem 1 (or by Bott's theorem). Thus we obtain $\mathring{\lambda}_{(k)}=\pm\mathring{\alpha}_i$. This is a contradiction, since $q_i>3$ $(i\neq k)$ and the absolute values of the Cartan integers are at most 3. Our Proposition 4 is now completely established.

REMARK. In Theorem 1′ we have proved that every holomorphic principal bundle over a C-manifold with a complex abelian group as structure group is homogeneous. On the other hand, Proposition 3 in [17] and Proposition 1 implies that *every holomorphic principal bundle over a kählerian C-manifold with a complex nilpotent group as structure group is also homogeneous.* (We do not know whether this result is still valid for non-kählerian C-manifolds or not). Thus, Matsushima has raised the question if the above result is true for bundles over kählerian C-manifolds with a complex solvable group as structure group. Now Proposition 4 above gives a negative answer to this question. In fact, let $\boldsymbol{GL}(m, \boldsymbol{C})$ (resp. $\boldsymbol{\Delta}(m, \boldsymbol{C})$) be the sheaf of germs of holomorphic mappings of a flag manifold X $(\neq P^1)$ into $GL(m, C)$ (resp. $\Delta(m, C)$). The injection $j: \Delta(m, C)\rightarrow GL(m, C)$ induces the mapping $j: H^1(X, \boldsymbol{\Delta}(m, \boldsymbol{C}))\rightarrow H^1(X, \boldsymbol{GL}(m, \boldsymbol{C})$. The image of j coincides with $\mathfrak{T}_m(X)$. If $H^1(X, \boldsymbol{\Delta}(m, \boldsymbol{C}))$ consists only of homogeneous $\Delta(m, C)$-bundles, then $\mathfrak{T}_m(X)$ does so, but the latter contradicts Proposition 4 for $m\geqq 2$.

11. Vector bundles over a maximal C-manifold

Proposition 5. *If X is a maximal C-manifold whose dimension is greater than* 1, *then we have*

$$\mathfrak{T}_m(X) = \mathfrak{S}_m(X)$$

for any positive integer m, and

$$\mathfrak{H}_m(X) \supsetneqq \mathfrak{S}_m(X)$$

for any $m\geqq\dim X$.

For the proof of the Proposition, we need the

Lemma 6. *Let $\mathfrak{g}$ be a complex simple Lie algebra with the rank greater than* 1, *$\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $\{\mathring{\alpha}_1, \mathring{\alpha}_2, \cdots, \mathring{\alpha}_l\}$ $(l\geqq 2)$ a fundamental root system. Let $\mathring{\lambda}$ be a weight on $\mathfrak{h}$ such that $\mathring{\lambda}(h_i)=0$ for all fundamental roots $\mathring{\alpha}_i$ except for a single one $\mathring{\alpha}_k$. Then $\mathring{\lambda}+\mathring{\delta}$ is singular or, $\mathring{\lambda}+\mathring{\delta}$ is regular and* Ind. $(\mathring{\lambda}+\mathring{\delta})\neq 1$.

Proof. We set $\mathring{\lambda}+\mathring{\delta}=\mathring{\Lambda}$ and $\mathring{\lambda}(h_k)=c$ (c=integer). If $c\geqq 0$, then $\mathring{\Lambda}(h_k)=c+1>0$ and so $\mathring{\Lambda}(h_\alpha)>0$ for any positive root $\mathring{\alpha}$. Therefore $\mathring{\Lambda}$

is regular and Ind. $\mathring{\Lambda}=0$. If $c=-1$, then $\mathring{\Lambda}(h_k)=0$ and so $\mathring{\Lambda}$ is singular. Finally let $c\leqq -2$. Note that, for any fundamental root $\mathring{\alpha}_i$ different from $\mathring{\alpha}_k$ and for a positive integer p, we have

$$\mathring{\Lambda}(h_\alpha) = \frac{1}{(\mathring{\alpha},\mathring{\alpha})}\{(\mathring{\alpha}_i,\mathring{\alpha}_i)\mathring{\Lambda}(h_i)+p(\mathring{\alpha}_k,\mathring{\alpha}_k)\mathring{\Lambda}(h_k)\}$$
$$\leqq \frac{1}{(\mathring{\alpha},\mathring{\alpha})}\{(\mathring{\alpha}_i,\mathring{\alpha}_i)-p(\mathring{\alpha}_k,\mathring{\alpha}_k)\},$$

where $\mathring{\alpha}=\mathring{\alpha}_1+p\mathring{\alpha}_k$ is a positive root.

If $\mathfrak{g}$ is one of the type $A_l(l\leqq 1)$, $D_l(l\geqq 4)$ or $B_l(l=6, 7, 8)$, then there exists a positive root of the form $\mathring{\alpha}_i+\mathring{\alpha}_k$ for any given $\mathring{\alpha}_k$ such that $(\mathring{\alpha}_i,\mathring{\alpha}_i)=(\mathring{\alpha}_k,\mathring{\alpha}_k)$. So that for such a $\mathring{\alpha}=\mathring{\alpha}_i+\mathring{\alpha}_k$ we have $\mathring{\Lambda}(h_\alpha)\leqq 0$. For the other types of $\mathfrak{g}$, this argument is still valid except for the case $\mathring{\alpha}_k=\mathring{\alpha}_l$ in the type $B_l(l\geqq 2)$, the case $\mathring{\alpha}_k=\mathring{\alpha}_{l-1}$ in the type C_l $(l\geqq 3)$, the case $\mathring{\alpha}_k=\mathring{\alpha}_2$ in the type G_2 and finally the case $\mathring{\alpha}_k=\mathring{\alpha}_3$ in the type F_4. However for these exceptional cases we may take the positive root $\mathring{\alpha}$ such that $\mathring{\Lambda}(h_\alpha)\leqq 0$ as follows (the corresponding figures are the Schläfli diagrams)[6];

$$B_l(l\geqq 2),\quad \mathring{\alpha}=\mathring{\alpha}_{l-1}+\mathring{\alpha}_l;\quad \overset{\mathring{\alpha}_1}{\bigcirc}-\overset{\mathring{\alpha}_2}{\bigcirc}-\cdots-\overset{\mathring{\alpha}_{l-1}}{\bigcirc}\Rrightarrow\overset{\mathring{\alpha}_l}{\bigcirc}$$

$$C_l(l\geqq 3),\quad \mathring{\alpha}=\mathring{\alpha}_{l-1}+\mathring{\alpha}_{l+1};\quad \overset{\mathring{\alpha}_1}{\bigcirc}-\overset{\mathring{\alpha}_2}{\bigcirc}-\cdots-\overset{\mathring{\alpha}_{l-1}}{\bigcirc}\Leftarrow\overset{\mathring{\alpha}_l}{\bigcirc}$$

$$G_2,\quad \mathring{\alpha}=\mathring{\alpha}_1+3\mathring{\alpha}_2;\quad \overset{\mathring{\alpha}_1}{\bigcirc}\Lleftarrow\overset{\mathring{\alpha}_2}{\bigcirc}$$

$$F_4,\quad \mathring{\alpha}=\mathring{\alpha}_4+\mathring{\alpha}_3;\quad \overset{\mathring{\alpha}_1}{\bigcirc}-\overset{\mathring{\alpha}_2}{\bigcirc}\Leftarrow\overset{\mathring{\alpha}_3}{\bigcirc}-\overset{\mathring{\alpha}_4}{\bigcirc}.$$

Therefore in any case, $\mathring{\Lambda}$ is singular or $\mathring{\Lambda}$ is regular and Ind. $\mathring{\Lambda}\geqq 2$. The lemma is thus proved.

Proof of Proposition 5. Let $X=G/U$ be a maximal C-manfold, and let $S=\{\mathring{\alpha}_1, \mathring{\alpha}_2, \cdots, \mathring{\alpha}_{k-1}, \mathring{\alpha}_{k+1}, \cdots, \mathring{\alpha}_l\}$. Any line bundle E_λ over X can be defined by a character λ whose differential $\mathring{\lambda}$ satisfies the assumption of Lemma 6. Therefore by Bott's theorem, we have $H^1(X, \boldsymbol{E}_\lambda)=\{0\}$.

We shall prove $\mathfrak{T}_m(X)=\mathfrak{S}_m(X)$ by induction on m. Suppose that $\mathfrak{T}_{m-1}(X)=\mathfrak{S}_{m-1}(X)$. By the definition of $\mathfrak{T}_m(X)$, every vector bundle $E\in\mathfrak{T}_m(X)$ is an extension of a vector bundle $E'\in\mathfrak{T}_{m-1}(X)$: $0\to E'\to E\to E_m\to 0$, where E_m is a line bundle and E' is isomorphic to the direct sum of $(m-1)$-line bundles $E_1, \cdots, E_{m-1}$. The obstruction class corresponding to

6) For the roots of complex simple Lie algebras, we refer to the book of N. Iwahori, "Theory of Lie groups II" (in Japanese).

the above extension is an element of the cohomology group $H^1(X, \text{Hom}(\boldsymbol{E}_m, \boldsymbol{E}')) \cong \sum_{n=1}^{m-1} H^1(X, \text{Hom}(\boldsymbol{E}_m, \boldsymbol{E}_i))$, where $\text{Hom}(E_m, E_i)$ $(1 \leqq i \leqq m-1)$ are line bundles. Therefore $H^1(X, \text{Hom}(\boldsymbol{E}_0, \boldsymbol{E}')) = \{0\}$ and $E \cong E' \oplus E_m = E_1 \oplus \cdots \oplus E_m \in \mathfrak{S}_m(X)$. The proof of $\mathfrak{T}_m(X) = \mathfrak{S}_m(X)$ is now complete.

Next we consider the tangential vector bundle Θ of X. This bundle is homogeneous but is not contained in $\mathfrak{S}_n(X)$ $(n = \dim X)$. In fact if $\Theta \in \mathfrak{S}_n(X)$ then $\Theta^* \in \mathfrak{S}_n(X)$ also, so that $H^1(X, \Theta^*)$ has to vanish. On the other hand $H^1(X, \Theta^*) = H^1(X, \Omega^1) \cong H^{1,1}(X, C) = H^2(X, C) \neq \{0\}$ which is a contradiction. Our proposition is now completely established.

12. A characterization of the complex projective line.

Theorem 5. *A C-manifold X is a complex projective line P^1 if and only if the following property is satisfied:*

$$\mathfrak{E}_m(X) = \mathfrak{S}_m(X)\,, \quad \textit{for any} \quad m \geqq 1\,.$$

For the proof, the following lemma is essential (cf. [10]).

Lemma 7. *Let ϕ be a holomorphic mapping of a complex manifold X onto another complex manifold X' such that $\phi^{-1}(x')$ is a compact connected complex submanifold of X for every point $x' \in X'$, and let E be a (complex analytic) vector bundle over X'. Then E is indecomposable if and only if the induced bundle ϕ^*E is indecomposable.*

Proof of Theorem 5. Let X be a C-manifold such that $\mathfrak{E}_m(X) = \mathfrak{S}_m(X)$ for $m \geqq 1$. Suppose $X \geqq 2$ and we shall derive a contradiction from this. Let $X(\hat{X}, Y, \phi)$ be the fundamental fibering of X. Then $\dim \hat{X} \geqq 2$, since there is essentially only one C-subgroup of $SL(2, C)$. Moreover we have $\mathfrak{E}_m(\hat{X}) = \mathfrak{S}_m(\hat{X})$ by Lemma 7. Therefore we may assume from the biginning that X is kählerian. If X is a maximal C-manifold, we have $\mathfrak{H}_m(X) \supsetneqq \mathfrak{S}_m(X)$ for $m \geqq \dim X$ by Proposition 5[7] and so $\mathfrak{E}_m(X) \neq \mathfrak{S}_m(X)$ for such m. Hence X can not be a maximal C-manifold. Consider the fibering $X(X_m, U_m/U, \psi_m)$. Suppose that X is not a product of several complex projective lines. Then we can choose X_m so that $\dim X_m \geqq 2$. By Lemma 7 and by our assumption $\mathfrak{E}_n(X) = \mathfrak{S}_n(X)$ $(n \geqq 1)$, we get $\mathfrak{E}_n(X_m) = \mathfrak{S}_n(X_m)$ for $n \geqq 1$. This is a contradiction as we have shown above. Therefore X must be a product of complex projective lines. Then X is a flag manifold. It follows from Proposition 4 that $\mathfrak{E}_m(X) \neq \mathfrak{S}_m(X)$ for $m \geqq 2$, since we have supposed $\dim X \geqq 2$. These contradictions show

7) If we are only concerned with the proof of Theorem 5, we can apply Theorem 6 in IV instead of Proposition 5.

that $\dim X=1$. Then we see clearly that $X=P^1$. Theorem 5 is thus proved.

REMARK. As we have mentioned in the Introduction, Theorem 5 gives a partial answer to a problem posed by Grothendieck [10].

IV. Tangential Vector Bundles.

13. Certain cohomology groups over a non-kählerian C-manifold.

Let $X=G/U$ be a non-kählerian C-manifold and let $X(\hat{X}, \hat{U}/U, \phi)$ be the fundamental fibering of X. Consider Atiyah's exact sequence over $\hat{X}$ associated with the fundamental fibering:

$$(1) \qquad 0 \to L(X) \to Q(X) \to \hat{\Theta} \to 0$$

(cf. [2] and [17], p. 165), where $\hat{\Theta}$ denotes the tangential bundle of $\hat{X}$. As is readily seen, this is nothing but the exact sequence of homogeneous vector bundles defined from the exact sequence of $\hat{U}$-modules under the adjoint actions:

$$(2) \qquad 0 \to \hat{\mathfrak{u}}/\mathfrak{u} \to \mathfrak{g}/\mathfrak{u} \to \mathfrak{g}/\hat{\mathfrak{u}} \to 0 .$$

The vector bundle $L(X)$ is trivial, since $\mathfrak{u}$ is an ideal of $\hat{\mathfrak{u}}$ and so the structure group Ad U acts trivially on $\hat{\mathfrak{u}}/\mathfrak{u}$. Moreover we remark that the standard fibre of $L(X)$ may be regarded as $\overline{\mathfrak{w}} \cong \hat{\mathfrak{u}}/\mathfrak{u}$ and that $H^0(\hat{X}, \boldsymbol{L}(\boldsymbol{X}))$ is identified with $\overline{\mathfrak{w}}$. Recall that $H^0(\hat{X}, \boldsymbol{\Theta})$ is the complex Lie algebra $\mathfrak{a}(\hat{X})$ and that $H^0(\hat{X}, \boldsymbol{Q}(\boldsymbol{X}))$ is identified with the complex Lie algebra $\mathfrak{k}(X)$ of all infinitesimal bundle automorphisms of $X(\hat{X}, \hat{U}/U, \phi)$ (i.e. holomorphic vector fields over X which is invariant under the right translations of the structure group, cf. [17]). Thus corresponding to the exact sequence (1), we obtain an extension of complex Lie algebras:

$$0 \to \overline{\mathfrak{w}} \to \mathfrak{k}(X) \xrightarrow{\phi} \mathfrak{a}(\hat{X}) \to H^1(\hat{X}, \boldsymbol{L}(\boldsymbol{X})) = \{0\} .$$

Here $\overline{\mathfrak{w}}$ is not only an abelian ideal of $\mathfrak{k}(X)$ but also is contained in the centre of $\mathfrak{k}(X)$. For $\overline{\mathfrak{w}}$ is the Lie algebra of the abelian structure group $\hat{U}/U$, and every vector field in $\mathfrak{k}(X)$ is invariant by the actions of the elements of $\hat{U}/U$. Since $\mathfrak{a}(\hat{X})$ is known to be a complex semi-simple Lie algebra by Matsushima [15], it follows that $\mathfrak{k}(X)$ is isomorphic to the direct sum of $\overline{\mathfrak{w}}$ and $\mathfrak{a}(\hat{X})$.

On the other hand, the exact sequence of U-modules (2) induces the exact sequence of homogeneous vector bundle ($\mathfrak{u}$ is an ideal of $\hat{\mathfrak{u}}$!):

$$(3) \qquad 0 \to I^r \to \Theta \to \phi^*\hat{\Theta} \to 0\,,$$

where $\Theta = E(\mathrm{Ad}, \mathfrak{g}/\mathfrak{u})$, $\hat{\Theta} = E(\mathrm{Ad}, \mathfrak{g}/\hat{\mathfrak{u}})$ (over $G/\hat{U}$) and $I^r = E(\mathrm{Ad}, \hat{\mathfrak{u}}/\mathfrak{u})$ is a trivial vector bundle over X of r-dimension ($r = \dim \hat{\mathfrak{u}}/\mathfrak{u}$). This exact sequence is clearly the one induced by ϕ from Atiyah's exact sequence (1). Now we consider the cohomology exact sequence corresponding to (3):

$$(4) \qquad \begin{array}{l} 0 \to H^0(X, \boldsymbol{C}^r) \to H^0(X, \Theta) \to H^0(X, \phi^*\hat{\Theta}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \to H^i(X, \boldsymbol{C}^r) \to H^i(X, \Theta) \to H^i(X, \phi^*\hat{\Theta}) \to \end{array}$$

where $H^0(X, \Theta) = \mathfrak{a}(X) \simeq H^0(\hat{X}, \boldsymbol{Q}(\boldsymbol{X}))$, $H^0(X, \phi^*\hat{\Theta}) \simeq H^0(\hat{X}, \hat{\Theta})$ and $H^0(X, \boldsymbol{C}^r) \simeq H^0(\hat{X}, \boldsymbol{L}(\boldsymbol{X}))$. Therefore we conclude that $\mathfrak{f}(X) = \mathfrak{a}(X)$. Hence we have the following result which sharpens a result of H.C. Wang ([22], Theorem 3).

Proposition 6. *Let X be a (non-kählerian) C-manifold with the fundamental fibering $X(\hat{X}, \hat{U}/U, \phi)$. Then $\mathfrak{a}(X)$ is isomorphic to the direct sum of $\bar{\mathfrak{w}}$ and $\mathfrak{a}(\hat{X})$; in particular $A^0(X)$ is a complex reductive Lie group whose connected centre is isomorphic to $\hat{U}/U$.*

Proof. We have only to prove the last statement. For any element $\hat{u} \in U$, we define a bundle automorphism $f_{\hat{u}}$ of the principal fibering $X(\hat{X}, \hat{U}/U, \phi)$ by setting $f_{\hat{u}}(gU) = g\hat{u}U$ for every $gU \in X = G/U$, which is well-defined since U is a normal subgroup of $\hat{U}$. Then the kernel of the homomorphism : $\hat{u} \to f_{\hat{u}}$ is clearly U. Moreover $f_{\hat{u}}(\phi^{-1}(\hat{x})) = \phi^{-1}(\hat{x})$ for every $\hat{x} \in \hat{X}$. Therefore the group $\{f_{\hat{u}} \mid \hat{u} \in U\}$ coincides with the subgroup of $A^0(X)$ generated by $\bar{\mathfrak{w}}$ and is isomorphic to $\hat{U}/U$. This completes the proof.

Corollary. *A C-manifold X is kählerian if and only if the connected automorphism group $A^0(X)$ is semi-simple.*

EXAMPLE. As an illustration of the above proposition, we take Calabi-Eckmann's example (cf. [8]). Namely, let X be the complex coset space G/U, where $G = GL(m+1, C) \times GL(l+1, C)$ $(k, l \geqq 1)$ and U is the connected complex closed subgroup of G consisting of the matrices $A \times B$: $A \in GL(k+1, C)$, $B \in GL(l+1, C)$ such that,

$$A = \left(\begin{array}{c|c} * & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline 0, \cdots, 0 & e^z \end{array}\right) \qquad B = \left(\begin{array}{c|c} * & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline 0, \cdots, 0 & e^{\sqrt{-1}z} \end{array}\right),$$

where z is any complex number. Then a maximal compact subgroup

K of G and $V=K\cap U$ are given by $K=U(k+1)\times U(l+1)$, $V=U(k)\times U(l)$. And as is easily seen, K acts transitively on X and so $X=G/U=K/V$ $=U(k+1)/U(k)\times U(l+1)/U(l)=S^{2k+1}\times S^{2l+1}$. Moreover let $\hat{U}=GL(k+1,1;C)\times GL(l+1,1;C)\supset U$. Then $\hat{X}=G/\hat{U}=P^k\times P^l$ and $\hat{U}/U\cong T^1$. Thus the fundamental fibering $X(\hat{X}, T^1, \phi)$ coincides with the well-known fibering $(S^{2k+1}\times S^{2l+1})(P^k\times P^l, T^1, \phi)$. Note that G acts on X not effectively but the quotient group $G'=G/N$ acts *effectively* and transitively, where N is a normal subgroup of G consisting of the matrices $A\times B$; $A\in GL(k+1, C)$, $B\in GL(l+1, C)$ such that,

$$A=\begin{pmatrix} e^z & & 0 \\ & \ddots & \\ 0 & & e^z \end{pmatrix}, \quad \cdot \quad B=\begin{pmatrix} e^{\sqrt{-1}z} & & 0 \\ & \ddots & \\ 0 & & e^{\sqrt{-1}z} \end{pmatrix},$$

where z runs all the complex numbers. Therefore we have $G'\subset A^0(X)$. On the other hand, $A^0(\hat{X})\cong SL(k+1, C)\times SL(l+1, C)$ and dim $G'=\dim A^0(X)+1$. Hence we conclude that $G'=A^0(X)$ by Proposition 6.

Next we shall compute the cohomology groups $H^i(X, \mathfrak{F})$ $(i\geqq 1)$ which have an important meaning in connection with Kodaira-Spencer's deformation theory of complex structures. (The following results shall be needed on another occasion).

For this sake, we shall show that in the exact sequence (4) there hold the following extensions:

$$(5)\qquad 0\to H^i(X, \boldsymbol{C}^r)\to H^i(X, \Theta)\to H^i(X, \phi^*\hat{\Theta})\to 0\,, \qquad (i\geqq 0)\,.$$

Lemma 8. *Let $E=E(\rho, F)$ be a homogeneous vector bundle over $\hat{X}=G/\hat{U}$ such that $H^i(\hat{X}, \boldsymbol{E})=\{0\}$ for $i\geqq 1$. Then $H^i(X, \phi^*\boldsymbol{E})$ is isomorphic to $H^0(\hat{X}, \boldsymbol{E})\otimes H^i(\hat{U}/U, \boldsymbol{C})$ $(i\geqq 0)$ in a natural manner.*

Proof. Take a spectral sequence $\{E_r\}$ such that E_∞ is associated to $H^*(X, \phi^*\boldsymbol{E})$ and $E_2^{p,q}=H^p(\hat{X}, \phi^q(\phi^*\boldsymbol{E}))$ (for the notations, see the proof of Theorem 1 in I). Then the analytic sheaf $\phi^q(\phi^*\boldsymbol{E})$ is by Bott ([7], Theorem VI) the sheaf of germs of holomorphic sections of the homogeneous vector bundle $G\times_{\hat{U}}H^q(\hat{U}/U, \boldsymbol{E}_Y)$, where E_Y denotes the homogeneous vector bundle $\hat{U}\times_U F$ (the restriction of ϕ^*E on $\hat{U}/U$) and the action of $\hat{U}$ on $H^q(\hat{U}/U, \boldsymbol{E}_Y)$ is the induced one from the defining representation of E_Y. Now we can identify E_Y with the trivial vector bundle $\hat{U}/U\times F$ *via* the correspondence $[\hat{u}, \xi]\leftrightarrow \hat{u}U\times\rho(\hat{u})\xi$ for $\hat{u}\in\hat{U}$ and $\xi\in F$. Therefore we can also identify $H^q(\hat{U}/U, \boldsymbol{E}_Y)$ with $H^q(\hat{U}/U, \boldsymbol{C})\otimes F$. The action of $\hat{U}$ on the last module, under this identification, is the composition of those on $H^q(\hat{U}/U, \boldsymbol{C})$ and F; the first one is trivial since it is the one induced from the left translations of $\hat{U}$ and $\hat{U}/U$ is kählerian, and the second is clearly that of ρ. This implies that the homogeneous vector bundle $G\times_{\hat{U}}H^q(\hat{U}/U, \boldsymbol{E}_Y)$ is $\hat{\boldsymbol{E}}\otimes H^q(\hat{U}/U, \boldsymbol{C})$. Therefore we have $E_2^{p,q}=\{0\}$ for $p\geqq 1$ by the assumption of the lemma and so $E_2^i=E_2^{0,i}=H^0(\hat{X}, \boldsymbol{E}\otimes H^q(\hat{U}/U, \boldsymbol{C})$. On the other other hand, d_2: $E_2^{0,i}\to E_2^{2,i-1}=\{0\}$. Hence $H^i(X, \phi^*\hat{\boldsymbol{E}})\cong E_\infty^i\cong E_2^i=H^0(\hat{X}, \hat{\boldsymbol{E}})\otimes H^i(\hat{U}/U, \boldsymbol{C})$.

Now recall that $H^i(\hat{X}, \boldsymbol{C})=H^i(\hat{X}, \hat{\Theta})=\{0\}$ $(i\geqq 1)$ (cf. [7], Theorem VII). Hence $H^i(\hat{X}, \boldsymbol{Q}(\boldsymbol{X}))=\{0\}$ $(i\geqq 1)$. The vector bundles of (1), therefore, satisfy the assumptions of Lemma 8, and we have the following commutative diagram:

$$
\begin{array}{lccccc}
 & \to H^i(X, \boldsymbol{C}^r) & \to & H^i(X, \Theta) & \to & H^i(X, \phi^*\hat{\Theta}) \to \\
 & \| & & \| & & \| \\
(5') & H^0(\hat{X}, \boldsymbol{L}(\boldsymbol{X}))\otimes A^i & \to & H^0(\hat{X}, \boldsymbol{Q}(\boldsymbol{X}))\otimes A^i & \to & H^0(\hat{X}, \hat{\Theta})\otimes A^i,
\end{array}
$$

where $A^i=H^i(\hat{U}/U, \boldsymbol{C})$ and the exact sequence (5′) coincides with the one induced from (1), in accompany with the module A^i. Thus we have from the earlier discussions the following:

$$
\begin{array}{ccccccccc}
0 \to & H^i(X, \boldsymbol{C}^r) & \to & H^i(X, \Theta) & \to & H^i(X, \phi^*\hat{\Theta}) & \to 0 \\
 & \| & & \| & & \| & \\
0 \to & \overline{\mathfrak{w}}\otimes A^i & \to & \mathfrak{a}(X)\otimes A^i & \to & \mathfrak{a}(\hat{X})\otimes A^i & \to 0,
\end{array}
$$

where we note that $A^i=\{0\}$ if $i>r$ and that $\dim A^i=\binom{r}{i}$ if $i\leqq r$.

Proposition7.[8)] *For a C-manifold X with the fundamental fibering* $X(\hat{X}, \hat{U}/U, \phi)$, *we have*

$$
H^i(X, \boldsymbol{C}) = H^i(X, \Theta) = \{0\}, \quad \text{if} \quad i>r
$$

$$
\left.\begin{array}{l}
\dim H^i(X, \boldsymbol{C}) = \binom{r}{i} \\[2ex]
\dim H^i(X, \Theta) = \binom{r}{i} \dim \mathfrak{a}(X)
\end{array}\right\} \text{if } 0\leqq i\leqq r.
$$

8) In Propositions 6 and 7 we have showed that $H^i(X, \Theta)=H^i(X, C^r)+H^i(X, \phi^*\hat{\Theta})$ for every $i\geqq 0$. However we remark here that the extension (3) is not splittable. In fact, if (3) is splittable, so is (1). This can be seen from the following commutative diagram and the argument of obstruction classes of extensions:

$$
\begin{array}{ccc}
H^0(X, \mathrm{Hom}\,(I^r, I^r)) & \xrightarrow{\delta^*} & H^1(X, I^r\otimes\phi^*\hat{\Theta}^*) \\
\uparrow \phi^* & & \uparrow \phi^* \\
H^0(\hat{X}, \mathrm{Hom}\,(L(X), L(X))) & \xrightarrow{\phi^*} & H^1(X, L(X)\otimes\hat{\Theta}^*)
\end{array}
$$

where ϕ^* in the right side is proved to be bijective using the spectral sequences. This implies by Atiyah [2] that the fundamental fibering $X(\hat{X}, T^r, \phi)$ has a holomorphic connection. Then we see by a result of Murakami ([18], Théorème 5) that our fibering must be trivial. But this clearly contradicts to the simply-connectedness of X.

14. The indecomposability of the tangential vector bundle of a kählerian C-manifold.

Here we shall discuss the indecomposability of the tangential vector bundle of a kählerian C-manifold. We recall at first a criterion of indecomposability of a general vector bundle due to Atiyah (cf. [2], Proposition 16). Let E be an m-dimensional vector bundle over a compact complex manifold X and consider the vector bundle Hom $(E, E) = E^* \otimes E$ whose fibre Hom $(E, E)_x$ at a point $x \in X$ consists of the endomorphisms of the fibre E_x of E at x; Hom $(E, E)_x =$ Hom (E_x, E_x). Now the 0-dimensional cohomology group $H^0(X, \mathrm{Hom}\,(\boldsymbol{E}, \boldsymbol{E}))$ has the natural algebra structure; for s, $t \in H^0(X, \mathrm{Hom}\,(\boldsymbol{E}, \boldsymbol{E}))$ the product $s \circ t$ is the section defined by $(s \circ t)(x) = s(x) \cdot t(x)$ (in the product of Hom (E_x, E_x)) at every point $x \in X$. For every $x \in X$ we define the mapping ν_x of $H^0(X, \mathrm{Hom}\,(\boldsymbol{E}, \boldsymbol{E}))$ into Hom (E_x, E_x) by setting $\nu_x(S) = s(x)$ for every section s. The linear mapping ν_x is obviously an algebra homomorphism and the image of ν_x is a linear subalgebra of Hom (E_x, E_x), which we denote by $S(E)_x$. A result of Atiyah [2] asserts that E is indecomposable if and only if the algebra $S(E)_x$ is written as

$$S(E)_x = C \cdot I_x + N(E)_x \quad \text{for every } x \in X,$$

where I_x denote the identity element in Hom (E_x, E_x), $N(E)_x$ a subalgebra of $S(E)_x$ consisting of matrices, in suitable basis of E_x, of the form:

$$\left.\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}\right\} m$$

In particular, if $H^0(X, \mathrm{Hom}\,(\boldsymbol{E}, \boldsymbol{E}))$ is of dimension 1, then E is indecomposable.

When E is the tangential vector bundle Θ of X, the cohomology group $H^0(X, \mathrm{Hom}\,(\Theta, \Theta))$ is the linear space of all holomorphic tensor fields of type (1. 1). While we know the following result (cf. Yano-Bochner [21]).

Lemma 9 (BOCHNER). *If X is a compact complex manifold with a kählerian, Einstein metric, every holomorphic tensor field of type (p, q) $(p \geqq 1,\ q \geqq 1)$ is parallel.*

Therefore, if X is an irreducible compact kählerian Einstein manifold (i.e. whose restricted homogeneous holomony group Ψ_x at a point $x \in X$ is irreducible), then every element of Ψ_x commutes with that of $\nu_x(H^0(X, \mathrm{Hom}\,(\Theta, \Theta)) = S(\Theta)_x$ for any $x \in X$. So that, by Schur's lemma,

$S(\Theta)_x$ has to be spanned by I_x, or what is the same, $H^0(X, \mathrm{Hom}\,(\Theta, \Theta))$ is of dimension 1. Hence Θ is indecomposable.

On the other hand, a kählerian C-manifold X is decompsed into the direct product of a certain number of irreducible kählerian C-manifolds X_i $(1 \leqq i \leqq k)$; therefore for a reducible kählerian C-manifold X, Θ is decomposable. However an irreducible kählerian C-manifold is necessarily an Einstein manifold (cf. [14]). Here, recall that X is irreducible if and only if G is (complex) simple (cf. [11]). Hence we have

Theorem 6[9]. *The tangential vector bundle of a kählerian C-manifold $X=G/U$ (for any almost effective Klein form) is indecomposable if and only if G is simple.*

REMARKS. As mentioned in the Introduction, the problem of characterizing the indecomposable homogeneous vector bundles in terms of their defining representations seems to be rather interesting. The defining representation of an indecomposable homogeneous vector bundle is not necessarily irreducible as Theorem 6 shows (Note that a C-manifold $X=G/U$ is an irreducible hermitian symmetric space if and only if the linear isotropic representation of U is irreducible), and moreover such a vector bundle is not necessarily isomorphic to the one defined by an irreducible representation; in fact a homogeneous vector bundle over a flag manifold is a line bundle if its defining representation is irreducible, but there are indecomposable two dimensional homogeneous vector bundles over it (III. Proposition 4). However it seems to us very plausible that *any homogeneous vector bundle over a kählerian C-manifold defined by an irreducible representation of the isotropy subgroup is indecomposable.* On the other hand, from the view-point of the classification problem of homogeneous vector bundles over a given kählerian C-manifold, it is desirable to obtain a condition of equivalence between two homogeneous vector bundles which is more precise than Theorem 4. For example, *if two homogeneous vector bundle $E_1(\rho_1\, F_1)$ and $E_2(\rho_2, F_2)$ are equivalent and if both (ρ_1, F_1) and (ρ_2, F_2) are irreducible, then are these representations equivalent?* We add that this problem has a negative answer for the differentiable homogeneous vector bundles.

(Received July 1, 1960)

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9) The original proof of this theorem is valid only for hermitian symmetric spaces, and depends entirely upon the theorem of Bott. The present proof is due to Matsushima.

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