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Stonian Spaces and the Second Conjugate Spaces of AM Spaces

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Let X be a compact space and let $C(X)$ be the set of all real-valued continuous functions on X. If any non-void subset of $C(X)$ with an upper bound has a least upper bound in $C(X)$, such a compact space X is called a stonian space.[1] Stone [10] has shown that a compact space X is stonian if and only if it is extremally disconnected, that is to say, if for any open set U in X its closure $\bar{U}$ is open. While, Kelley [9] has proved that if for any Banach space F containing a Banach space E there exists a projection of F on E whose norm is 1, E is isometric to $C(X)$, where X is stonian. Also Dixmier [4] has considered a compact space X such that $C(X)$ is isomorphic to an $L^\infty(R, \mu)$ as Banach algebras, where R is a locally compact space and μ is a positive measure on R. He called such a space X a hyperstonian space. A hyperstonian space is stonian. We shall see that a compact space X is hyperstonian if and only if $C(X)$ is lattice-isomorphic and isometric to a conjugate space of an AL space.[2]

In §1 we state some general properties of stonian spaces, and in §2 we consider an AM space $C(X)$ which is the second conjugate space of an AM space. Such a space X is hyperstonian, and if the character of X is countable, then X is the space βN_0, where N_0 is a discrete space whose cardinal number is at most countable (cf. Theorem 3, Corollary).

§1. Stonian spaces

For a completely regular space X, let βX denote the Čech compactification of X. (cf. Čech [2]). Dixmier [4] has shown that $\beta U = X$ for any open dense set U in a stonian space X. Therefore we obtain easily the following:

1) See Stone [10] and Dixmier [4]. Numbers in bracket refer to the reference cited at the end of the paper.

2) See Kakutani [7] and [8].

(i) X is stonian if and only if for any open set U in X, $\beta U \subset X$.
(ii) If X is a stonian space and if X has an open dense subspace which satisfies the 1st axiom of countability, X is a space βN, where N is a discrete space.
(iii) If X is a stonian space and if a point x of X is not isolated, then $\{x\}$ is not a G_δ set.

We shall prove here the following theorem.

Theorem 1. *Let X be a stonian space. If X is a product $R \times R$ of a compact space R, then X is finite.*[3]

Proof. Suppose that a stonian X be of the form $R \times R$, where R is a compact space with infinite points. Then R is stonian. Let U be a dense open set in R. Then $R \times R = \beta U \times \beta U \supset U \times U$ and $U \times U$ is a dense open set in $R \times R$. Since $R \times R$ is stonian, $\beta(U \times U) = R \times R = \beta U \times BU$.

Now if for a fully normal space[4] S $\beta(S \times S) = \beta S \times \beta S$, then S is compact. (Ishiwata [6]). Therefore it is sufficient to prove that there exists an open dense subset U in R which is not compact but fully normal. We shall construct such an open set U. Since R is an infinite set, there exists a countable family $\{U_i\}_{i=1}^{\infty}$ of mutually disjoint non void sets which are both open and closed. Let $V = R - \overline{\sum_{i=1}^{\infty} U_i}$ and let $U = V \cup \sum_{i=1}^{\infty} U_i$. Since the set U is the union of a countable family of open and compact sets in R which are mutually disjoint, U is fully normal. Clearly, U is not compact and is dense in R.

REMARK. Theorem 1 shows that there exists no stonian space S with infinite points of the form $R \times R$. But we can find easily a totally disconnected compact space S which is a product space $R \times R$, where R is compact and infinite.

§ 2. Second conjugate spaces of AM spaces.

Let X be a stonian space and let $M(X)$ be the set of all measures on X. A positive measure μ on X is called a *normal measure* if for any nowhere dense set A, $\mu(A) = 0$. A real measure μ on X is called *normal* if its positive part and its negative part are both normal. Let $M'(X)$

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3) Henrikson and Isbell announced the following theorem (Bull. Amer. Soc. Vol. 63. 1957 Abstract): if X and Y are infinite completely regular spaces such that $\beta(X \times Y) = \beta X \times \beta Y$, then $X \times Y$ is pseudo-compact, that is, any continuous function on $X \times Y$ is always bounded. If we make use of this theorem, we obtain moreover that if a stonian space X is a product $R \times S$ of compact spaces R and S, then either R or S is finite.

4) See Tukey [11], p. 53.

denote the set of all normal measures on X. A stonian space X is called *hyperstonian* if it has positive normal measures, the union of whose carriers is dense in X. We shall see that a compact space X is hyperstonian if and only if $C(X)$ is lattice-isomorphic and isometric to the conjugate space of an *AL* space. Let E be an *AM* space. Then the second conjugate space of E is lattice-isomorphic and isometric to $C(X)$, where X is hyperstonian, and the conjugate space of E is lattice-isomorphic and isometric to $M'(X)$. $M'(X)$ is also lattice-isomorphic and isometric to an $L^1(\Omega, \mu)$, where Ω is an open dense set in X and μ is a suitable positive measure on Ω. (cf. [4]). We consider now an *AM* space $C(X)$ which is the second conjugate space of an *AM* space. Let E be a Banach space and E^*, E^{**} denote the conjugate space of E and the second conjugate space of E respectively. For any closed linear subspace V in E^* we define its *characteristic* r by $r = \inf_{x \in E} \sup_{f \in V \cap S} \frac{|f(x)|}{\| x \|}$, where S is a unit sphere in E^*. A closed linear subspace V in E^* is called *minimally weakly dense* if it is weakly dense in E^* and if any other closed subspace in V is not weakly dense in E^*.

The following lemma was proved by Dixmier [3].

Lemma. (i) *Let E be a Banach space. Then E is a minimally weakly dense svbspace in E^{**} which is characteristic one.*

(ii) *If V is a minimally weakly dense subspace in E^* which is characteristic one, then $E^{**} = E \oplus V^+$ and $\| x \| \leqslant \| x+z \|$ for $x \in E$, $z \in V^+$, where V^+ denotes the set $\{z \mid z \in E^{**}, z(f) = 0$ for any $f \in V\}$.*

Let K be an open set in a hyperstonian space. Then the *character* of K is said to be countable if any family of non-void open and closed sets in K which are mutually disjoint is at most countable.

We can prove the following theorem. Hereafter X denotes a hyperstonian space.

Theorem 2. *$C(X)$ is lattice-isomorphic and isometric to the second conjugate space of an AM space with a unit*[2)] *if and only if there exists a lattice-closed and (topologically) closed linear subspace V in $C(X)$ which has constant functions such that*

(I) *for any $f \in C(X)$ and for any open and closed set K (in X) whose character is countable, there exists a sequence $\{f_n\}$ in V such that f_n pointwise converges to f on K except a nowhere dense set.*

(II) *V is a minimal closed linear space which has the property* (I): *any other closed subspace in V does not satisfy* (I).

Proof. (a) Let $C(X)$ be lattice-isomorphic and isometric to E^{**}

and let E be an AM space with a unit. Then, by Lemma, there exists a minimally weakly dense subspace V in $C(X)$ which is lattice-closed. We see here that E and E^* are lattice-isomorphic and isometric to V and $M'(X)$ respectively. Since E has a unit, we can assume that V has constant functions in $C(X)$. In order to prove (I) and (II), we are only to prove the equivalence of (I) and that V is weakly dense. Now if the property (I) is satisfied, then we see easily that V is weakly dense. Conversely, if V is weakly dense, then we see easily that for any $f \in C(X)$ and for any open and closed set K (in X) whose character is countable, there exist $f_n \in V$ such that

$$\int_{K \cap \Omega} |f(x)-f_n(x)|\, d\mu(x) < \frac{1}{n} \quad (n=1, 2, \cdots),$$

where Ω is an open dense set in X and μ is a suitable positive measure on Ω. Therefore, as is well known, a subsequence f_{n_i} of $\{f_n\}$ pointwise converges to f almost everywhere on $K \cap \Omega$.[5] Since any set of measure null on $K \cap \Omega$ is nowhere dense, f_{n_i} pointwise converges to f on K except a nowhere dense set.

(b) If properties (I) and (II) are satisfied, we see easily that V is a minimally weakly dense subspace in $C(X)$. (cf. (a)). We shall prove that V is of characteristic one. For any $u \in M'(X)$, let A and B be carriers of the positive part u^+ of u and of the negative part u^- of u respectively and let the function f take the value 1 on A and the value -1 on B. Then, by (I), there exists a sequence $\{f_n\}$ in V such that f_n pointwise converges to f on $A \cup B$ except a nowhere dense set. We may assume here that for any n, $\| f_n \|_\infty \leqslant 1$, since V has constant functions.[6] Since $u(f_n)$ converges to $u(f) = \| u \|$, V is of characteristic one. By Lemma, $M(X) = M'(X) \oplus V^+$ and $\| u \| \leqslant \| u+z \|$ for $u \in M'(X)$, $z \in V^+$. Therefore if F is a linear functional on V, then there exists $u \in M'(X)$ such that $F(f) = u(f)$ for any $f \in V$ and $\| F \| = \| u \|$, that is, $M'(X)$ is lattice-isomorphic and isometric to V^*, and $C(X)$ is lattice-isomorphic and isometric to V^{**}. This concludes the proof.

We consider next an AL space with an F-unit. Let l^1 be the set of all sequences $\{\xi_i\}$ of real numbers with convergent $\sum_{i=1}^{\infty} |\xi_i|$. l^1 is a Banach space where the norm of $x = \{\xi_i\} \in l^1$ is $\sum_{i=1}^{\infty} |\xi_i|$. (cf. Banach [1]).

Theorem 3. *If an AL space E with an F-unit*[2] *is lattice-isomorphic*

5) See, for example, Halmos [5].

6) $\| f \|_\infty$ denotes the uniform norm: $\| f \|_\infty = \sup_{x \in X} |f(x)|$.

and isometric to a conjugate space of an AM space, E is lattice-isomorphic and isometric to l^1.

Proof. Let E be of the form $L^1(\Omega, \mu)$, where is an open set in a hyperstonian space. Since E has an F-unit, the character of Ω is countable. If $L^1(\Omega, \mu)$ is the conjugate space of an AM space F and if F is of the form of $C(Y, y_\alpha, y'_\alpha, \lambda_\alpha) = \{f \mid f \in C(Y),\ f(y_\alpha) = \lambda_\alpha f(y'_\alpha),\ 0 \leqq \lambda_\alpha < 1,\ \alpha \in \mathfrak{m}\}$[7], then function g_α in $L^1(\Omega, \mu)$ which correspond to $\mu_{y'_\alpha} \in F^*$ are mutually distinct, where $\mu_{y'_\alpha}$ is a dirac measure, that is to say $\mu_{y'_\alpha}(f) = f(y'_\alpha)$ for any $f \in F$. We see easily that the carrier of function g_α[8] is a one-point set x_α, and therefore, x_α is an isolated point in Ω. Since the character of Ω is countable, the cardinal number of $Z_0 = \{y'_\alpha\}_{\alpha \in \mathfrak{m}}$ is at most countable. Since $C(Y) \supset F$, any linear functional ξ on F can be extended to a linear functional ξ' on $C(Y)$. ξ' is a measure on Y and for any $f \in F$, $\xi(f) = \xi'(f) = \int_Y f(x) d\xi'(x)$. Since the cardinal number of Y_0 is countable, we can put $Z_0 = \{z_1, z_2, \cdots\}$. For any n, let Y_n denote the set of y_β with $y'_\beta = z_n$. Then we have $\xi(f) = \sum_{i=1}^{\infty} \int_{Y_n} f(y) d\xi'(y) = \sum_{n=1}^{\infty} \left(\int_{Y_n \ni y_\beta} \lambda_\beta d\xi'(y_\beta) \right) f(z_n)$. If we put $p_n = \int_{Y_n \ni y_\beta} \lambda_\beta d\xi'(y_\beta)$, we obtain that $\xi(f) = \sum_{n=1}^{\infty} p_n f(z_n)$. We see easily that if ξ is positive, any p_n is non-negative and $\| \xi \| = \sum_{n=1}^{\infty} p_n$.

Corollary. *If $C(X)$ is lattice-isomorphic and isometric to the second conjugate space of an AM space and if the character of X is countable, then X is the space βN_0, where N_0 is a discrete space whose cardinal number is at most countable.*

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7) See Kakutani [8], Theorem 1.

8) We may assume that g_α is a continuous function on Ω. See [4].

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