Ohnishi, Masao &
Matsumoto, Kazuo
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Gentzen Method in Modal Calculi

By Masao OHNISHI and Kazuo MATSUMOTO

A decision procedure by Gentzen style has been given by H. B. Curry [4] only for $S4$. J. Ridder [16] intended to give the decision procedures for M, $S4$ and $S5$ by Gentzen style[1]. By using the different methods, the decision procedures for the various modal systems have been obtained by J. C. C. McKinsey [12], R. Carnap [3], G. H. von Wright [17], J. Ridder [15], Alan Ross Anderson [1], and M. Itoh [8].

The object of this paper is to give decision procedures by Gentzen style for modal sentential calculi $S2$, $S4$, $S5$ and M[2].

(I) Formulation

§1 Definitions of $S5^*$, $S4^*$, M^* and $Q2$.

1.1 Our formulation of the above systems is based upon "Sequenzenkalkül LK", which was constructed by G. Gentzen [6]. Namely:

- logical symbols:
  - $\cdot$ (and), $\sim$ (not), $\vee$ (or), $\supset$ (if$\cdots$, then)
- rules of inference:
  - structural rules
    - weakening, contraction, exchange and cut.
  - logical rules

$(\rightarrow \cdot)$ UES, $(\rightarrow \vee)$ OES, $(\rightarrow \sim)$ NES, $(\rightarrow \supset)$ FES,

$(\cdot \rightarrow)$ UEA, $(\vee \rightarrow)$ OEA, $(\sim \rightarrow)$ NEA, $(\supset \rightarrow)$ FEA.

Next, we add to LK two kinds of logical symbols:

$\diamond \cdots\cdots$ (possible),

$\Box \cdots\cdots$ (necessary),

and we define as follows: if α is a formula, then $\diamond\alpha$ and $\Box\alpha$ are also formulae.

Numbers in brackets refer to the bibliography at the end of this paper.

1) The authors have communicated to Prof. J. Ridder and he has admitted that his system was found to be unsatisfactory for the decision problem.

2) C. I. Lewis and C. H. Langford [9]. G. H. von Wright [17].

New rules for modality are:

$$\frac{\Gamma \to \Theta,\ \alpha}{\Gamma \to \Theta,\ \Diamond\alpha}\ (\to \Diamond) \qquad \Bigg| \qquad \frac{\alpha,\ \Theta \to \Gamma}{\Box\alpha,\ \Theta \to \Gamma}\ (\Box \to)$$

$$\frac{\alpha,\ \Diamond\Gamma \to \Diamond\Theta}{\Diamond\alpha,\ \Diamond\Gamma \to \Diamond\Theta}\ (\Diamond \to) \qquad \Bigg| \qquad \frac{\Box\Theta \to \Box\Gamma,\ \alpha}{\Box\Theta \to \Box\Gamma,\ \Box\alpha}\ (\to \Box)$$

By Γ, Θ we mean a series of formulae as in LK. $\Diamond\Gamma(\Box\Gamma$ or $\sim\Gamma)$ means the series of formulae which is formed by prefixing $\Diamond(\Box$ or $\sim)$ in front of each formulae of Γ.

Thus established sentential calculus which contains LK is $S5^*$.

$S4^*$ is the special case of $S5^*$, where Γ is empty in the rules $(\Diamond \to)$ and $(\to \Box)$[3].

M^* is defined by replacing the rules $(\Diamond \to)$ and $(\to \Box)$ in $S5^*$ by the following rules:

$$\frac{\alpha \to \Theta}{\Diamond\alpha \to \Diamond\Theta}\ (\Diamond \to) \qquad \Bigg| \qquad \frac{\Theta \to \alpha}{\Box\Theta \to \Box\alpha}\ (\to \Box)$$

The system $Q2$ is the special case of M^*, where Θ is non-empty in the above two rules.

The systems $S5^*$, $S4^*$, M^* and $Q2$ thus defined are distiguished from one another only by the rules $(\Diamond \to)$ and $(\to \Box)$, other rules including $(\to \Diamond)$ and $(\Box \to)$ being in common. Therefore the rules $(\Diamond \to)$, $(\to \Box)$ are put into explicit forms as follows:

$$\begin{array}{lcc}
S5^* & \dfrac{\alpha,\ \Diamond\Gamma \to \Diamond\Theta}{\Diamond\alpha,\ \Diamond\Gamma \to \Diamond\Theta} & \dfrac{\Box\Theta \to \Box\Gamma,\ \alpha}{\Box\Theta \to \Box\Gamma,\ \Box\alpha} \\[2ex]
S4^* & \dfrac{\alpha \to \Diamond\Theta}{\Diamond\alpha \to \Diamond\Theta} & \dfrac{\Box\Theta \to \alpha}{\Box\Theta \to \Box\alpha} \\[2ex]
M^* & \dfrac{\alpha \to \Theta}{\Diamond\alpha \to \Diamond\Theta} & \dfrac{\Theta \to \alpha}{\Box\Theta \to \Box\alpha} \\[2ex]
Q2 & \dfrac{\alpha \to \Theta}{\Diamond\alpha \to \Diamond\Theta} & \dfrac{\Theta \to \alpha}{\Box\Theta \to \Box\alpha}
\end{array}$$

Θ is non-empty in $Q2$.

1.2 Symmetry

In our systems, the symmetry (Spiegelbildlichkeit) of $\cdot$ and $\vee$ and of $\Box$ and $\Diamond$ are preserved. Namely considering $\Box\alpha$ is the abbreviation of $\sim\Diamond\sim\alpha$, we can easily verify that the rules of $\Box$ (at the right side)

3) This formulation has been already treated in R. Feys [5] and H. B. Curry [4]. See also §9.

can be derived from the corresponding rules of $\Diamond$ (at the left side) and vice versa. It results that, assuming $\Box\alpha = \sim\Diamond\sim\alpha$, we can do without one half of the above rules (right or left side).

§ 2 Equivalences of M^*, $S4^*$, $S5^*$ and M, $S4$, $S5$.

2.1[4] The definitions of M, $S4$ and $S5$, due to von Wright, are as follows:

I signs

Group A_α: The constants $\sim$, &, $\vee$, $\rightarrow$ and $\leftrightarrow$ of propositional logic.

Group A_β: The constants $\Diamond$ and $\Box$ of modal logic.

Group B : Sentence-variable a, b, c, $\cdots$.
(an unlimited multitude)

II Rules of Formation

RF-I : A sentence-variable is a formula.

RF-II : A formula preceded by $\sim$, by $\Diamond$ or by $\Box$ is a formula.

RF-III: Two formulae joined by &, $\vee$, $\rightarrow$ or $\leftrightarrow$ constitute a formula.

III Axiom

Group A: A set of axioms for propositional logic.

Group B: 1 $a \rightarrow \Diamond a$ (The axiom of Possibility)

2 $\Diamond(a \vee b) \leftrightarrow \Diamond a \vee \Diamond b$ (The axiom of Distribution)

Group C: 1 $\Diamond\Diamond a \rightarrow \Diamond a$ (The first axiom of Reduction)

2 $\Diamond\sim\Diamond a \rightarrow \sim\Diamond a$ (The second axiom of Reduction)

IV Definitions

If the axioms in Group A are so selected that not all the constants $\sim$, &, $\vee$, $\rightarrow$ and $\leftrightarrow$ occur in them, the missing constants have to be introduced by definition in the usual way.

The constant $\Box$ we introduce by the definition

$$\Box a = \sim\Diamond\sim a$$

V Rules of Transformation

Group A: The rules of transformation of propositional logic.

Group B: 1 If $f_1 \leftrightarrow f_2$ is provable, then $\Diamond f_1 \leftrightarrow \Diamond f_2$ is also provable. (The rule of Extentionality)

2 If f is provable, then $\Box f$ is also provable.
(The rule of Tautology)

If from the above description we omit the axiom of Group C, we

4) 2.1 is a quotation from von Wright [17], pp. 84-85.

obtain the system M. If to the system M we add the first or second Reduction axiom, we obtain the system $S4$ or $S5$ respectively.

2. 2 We prove here the equivalences of M^*, $S4^*$, $S5^*$ and M, $S4$, $S5$.

$M^* \Rightarrow M$

What we must prove are:

III Axiom Group B 1, 2.

V Rules of Transformation Group B 1, 2.

As to V,

1° $$\frac{\alpha \to \beta}{\Diamond\alpha \to \Diamond\beta}\ (\Diamond \to)$$

2° $$\cfrac{\cfrac{\cfrac{\to \alpha}{\sim\alpha \to}\ (\sim \to)}{\Diamond\sim\alpha \to}\ (\Diamond \to,\ \Theta \text{ being empty})}{\to \sim\Diamond\sim\alpha}\ (\to \sim)$$

As to III,

3° $$\frac{\alpha \to \alpha}{\alpha \to \Diamond\alpha}\ (\to \Diamond)$$

4° $$\cfrac{\cfrac{\cfrac{\alpha \to \alpha}{\alpha \to \alpha \vee \beta}\ (\to \vee)}{\Diamond\alpha \to \Diamond(\alpha \vee \beta)} \qquad \cfrac{\cfrac{\beta \to \beta}{\beta \to \alpha \vee \beta}\ (\to \vee)}{\Diamond\beta \leftarrow \Diamond(\alpha \vee \beta)}\ (\Diamond \to)}{\Diamond\alpha \vee \Diamond\beta \to \Diamond(\alpha \vee \beta)}\ (\Diamond \to)$$

5° $$\cfrac{\cfrac{\cfrac{\cfrac{\alpha \to \alpha}{\alpha \to \alpha,\ \beta} \qquad \cfrac{\beta \to \beta}{\beta \to \alpha,\ \beta}}{\alpha \vee \beta \to \alpha,\ \beta}\ (\vee \to)}{\Diamond(\alpha \vee \beta) \to \Diamond\alpha,\ \Diamond\beta}\ (\Diamond \to)}{\Diamond(\alpha \vee \beta) \to \Diamond\alpha \vee \Diamond\beta}$$

4° and 5° yield $\Diamond(\alpha \vee \beta) \leftrightarrow \Diamond\alpha \vee \Diamond\beta$

$M \Rightarrow M^*$

As we have the rule of Tautology in M, what we must prove essentially is the rule $(\Diamond \to)$ for non-empty Θ, namely

$$\frac{\alpha \to \beta,\ \gamma}{\Diamond\alpha \to \Diamond\beta,\ \Diamond\gamma}$$

In other words, we have only to prove that if $\alpha \supset \beta \vee \gamma$ then $\Diamond\alpha \supset \Diamond\beta \vee \Diamond\gamma$ in M.

If $\alpha \supset \beta \vee \gamma$ then $\Diamond\alpha \supset \Diamond(\beta \vee \gamma)$. (V. B. 1)

We have $\Diamond(\beta \vee \gamma) \supset \Diamond\beta \vee \Diamond\gamma$. (III, B. 2)

Accordingly, using modus ponens (V, A), we get $\Diamond\alpha \supset \Diamond\beta \vee \Diamond\gamma$.

$S4^* \Rightarrow S4$

III Group C, 1 is shown by

$$\frac{\Diamond\alpha \to \Diamond\alpha}{\Diamond\Diamond\alpha \to \Diamond\alpha}\ (\Diamond\to)$$

$S4 \Rightarrow S4^*$

Essentially, we must prove $\Diamond\alpha \supset \Diamond\beta \vee \Diamond\gamma$ from the assumption $\alpha \supset \Diamond\beta \vee \Diamond\gamma$. If $\alpha \supset \Diamond\beta \vee \Diamond\gamma$ then $\Diamond\alpha \supset \Diamond(\Diamond\beta \vee \Diamond\gamma)$.

On the other hand, we have $\Diamond(\Diamond\beta \vee \Diamond\gamma) \supset \Diamond\Diamond\beta \vee \Diamond\Diamond\gamma$, hence $\Diamond\alpha \supset \Diamond\beta \vee \Diamond\gamma$ by $\Diamond\Diamond\beta \supset \Diamond\beta$ and $\Diamond\Diamond\gamma \supset \Diamond\gamma$.

$S5^* \Rightarrow S5$

The only rule to be proved is III, Group C, 2, i.e.

$$\begin{array}{cl}
\Diamond\alpha \to \Diamond\alpha & \\ \hline
\sim\Diamond\alpha,\ \Diamond\alpha \to & (\sim\to) \\ \hline
\Diamond\sim\Diamond\alpha,\ \Diamond\alpha \to & (\Diamond\to) \\ \hline
\Diamond\sim\Diamond\alpha \to \sim\Diamond\alpha & (\to\sim)
\end{array}$$

$S5 \Rightarrow S5^*$

As has been proved we are able to use all rules of $S4^*$.
We must prove:

$$\frac{\alpha,\ \Diamond\beta \to \Diamond\gamma}{\Diamond\alpha,\ \Diamond\beta \to \Diamond\gamma}$$

The proof is:

$$\begin{array}{cl}
\alpha,\ \Diamond\beta \to \Diamond\gamma & \\ \hline
\Diamond\beta,\ \alpha \to \Diamond\gamma & \\ \hline
\alpha \to \Diamond\gamma,\ \sim\Diamond\beta & (\to\sim) \\ \hline
\alpha \to \Diamond\gamma,\ \Diamond\sim\Diamond\beta & (\to\Diamond) \\ \hline
\Diamond\alpha \to \Diamond\gamma,\ \Diamond\sim\Diamond\beta & (\Diamond\to \text{ in } S4^*) \\ \hline
\sim\Diamond\sim\Diamond\beta,\ \Diamond\alpha \to \Diamond\gamma & (\sim\to) \\ \hline\hline
\Diamond\beta,\ \Diamond\alpha \to \Diamond\gamma & (\text{characteristic formula of } S5) \\ \hline
\Diamond\alpha,\ \Diamond\beta \to \Diamond\gamma &
\end{array}$$

(II) Hauptsatz

§3 Hauptsatz for $Q2$, M^* and $S4$.[5]

We shall prove in this § the following Hauptsatz (elimination of cuts) for $Q2$, M^* and $S4^*$.

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5) The extentions of Lewis' systems to the functional calculi have been tried by R. Carnap [3] for $S5$, and R. C. Barcan [2] for $S4$. Our formulation can be extended to the functional calculi in a natural way, i.e., Gentzen's *AES*, *AEA*, *EES* and *EEA* are added to our systems. It is almost obvious that the establishment of the Hauptsatz for functional $Q2$, M^* and $S4^*$ are justified in these extended calculi.

Hauptsatz: *Any Q2 (or M* or S4*) proof-figure can be transformed into a Q2 (or M* or S4*) proof-figure with the same endsequent and without any cut as a rule of inference.*

Its proof is treated along the line of Gentzen.

We replace cut-rule by mix (Mischung)-rule as in Gentzen. Then, we have only to prove the following

Lemma: *Any proof-figure which has the mix-rule only as its lowest rule and does not include this rule elsewhere, can be transformed into the proof-figure which has the same endsequent and has no mix at all.*

Degree (Grad) and rank being the same as in *LK*, the proof of our lemma can be treated by the induction on rank ρ and degree γ.

The cases which are to be added to the proof for *LK* are the following:

(1) When $\rho=2$, and the outermost symbol of the mix formula $\mathfrak{M}$ is $\diamond$ (3.113.37).

(2) When $\rho>2$, right rank >1, and the upper sequent on the right side of mix is derived by the rule of $\diamond$ (3.121.223).

(3) When $\rho>2$, right rank $=1$, and the upper sequent on the left side of mix is derived by the rule of $\diamond$ (3.121.224).

3.1 Hauptsatz for *Q2*.

(1) When $\rho=2$, and the outermost symbol of the mix formula $\mathfrak{M}$ is $\diamond$, the mix has the following form:

$$\frac{\dfrac{\Gamma \rightarrow \Delta,\ \alpha}{\Gamma \rightarrow \Delta,\ \diamond\alpha} \qquad \dfrac{\alpha \rightarrow \Theta}{\diamond\alpha \rightarrow \diamond\Theta}\ (\Theta \text{ is non-empty})}{\Gamma \rightarrow \Delta,\ \diamond\Theta}\ (\text{mix of } \diamond\alpha)$$

(Δ does not contain $\diamond\alpha$)

We transform this into:

$$\cfrac{\cfrac{\cfrac{\Gamma \rightarrow \Delta,\ \alpha \qquad \alpha \rightarrow \Theta}{\Gamma \rightarrow \Delta^*,\ \Theta}\ (\text{mix of } \alpha)}{\Gamma \rightarrow \Delta,\ \Theta}}{\Gamma \rightarrow \Delta,\ \diamond\Theta}$$

This shows that we can omit the mix from the assumption of the induction, as the degree of the mix formula is decreased by 1.

(2) When $\rho>2$, and the right rank >1, and the upper sequent on the right side of mix is the lower sequent of the rules of $\diamond$, we have to treat only the following case:

$$\frac{\Pi \rightarrow \Sigma \qquad \dfrac{\Gamma \rightarrow \Delta,\ \alpha}{\Gamma \rightarrow \Delta,\ \diamond\alpha}\ (\rightarrow \diamond)}{\Pi,\ \Gamma^* \rightarrow \Sigma^*,\ \Delta,\ \diamond\alpha}\ (\mathfrak{M})$$

We transform this into:

$$\dfrac{\dfrac{\Pi \to \Sigma \qquad \Gamma \to \Delta,\ \alpha}{\Pi,\ \Gamma^* \to \Sigma^*,\ \Delta,\ \alpha}\,(\mathfrak{M})}{\Pi,\ \Gamma^* \to \Sigma^*,\ \Delta,\ \alpha}\,(\to \Diamond)$$

This shows that we can omit the mix from the assumption of the induction, as the rank of the mix formula is decreased by 1.

(3) When $\rho > 2$, and the right rank $=1$, and the upper sequent on the left side of mix is the lower sequent of the rules of $\Diamond$, we have to treat only the following:

$$\dfrac{\dfrac{\Gamma \to \Delta,\ \alpha}{\Gamma \to \Delta,\ \Diamond\alpha} \qquad \Pi \to \Sigma}{\Gamma,\ \Pi^* \to \Delta^*,\ (\Diamond\alpha)^*,\ \Sigma}\,(\mathfrak{M})$$

(Δ includes $\mathfrak{M}$)

We transform this into:

In case $\mathfrak{M} = \Diamond\alpha$,

$$\genfrac{}{}{3pt}{0}{\dfrac{\dfrac{\Gamma \to \Delta,\ \alpha \qquad \Pi \to \Sigma}{\Gamma,\ \Pi^* \to \Delta^*,\ \alpha,\ \Sigma}\,(\Diamond\alpha) \qquad \dfrac{\Pi \to \Sigma}{\alpha,\ \Pi \to \Sigma}}{\Gamma,\ \Pi^*,\ \Pi^* \to \Delta^*,\ \Sigma^*,\ \Sigma}\,(\Diamond\alpha)}{\Gamma,\ \Pi^* \to \Delta^*,\ \Sigma}$$

in case $\mathfrak{M} \neq \Diamond\alpha$,

$$\genfrac{}{}{3pt}{0}{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Delta,\ \alpha \qquad \Pi \to \Sigma}{\Gamma,\ \Pi^* \to \Delta^*,\ \alpha^*,\ \Sigma}\,(\mathfrak{M})}{\Gamma,\ \Pi^* \to \Delta^*,\ \alpha,\ \Sigma}}{\Gamma,\ \Pi^* \to \Delta^*,\ \Diamond\alpha,\ \Sigma} \qquad \Pi \to \Sigma}{\Gamma,\ \Pi^*,\ \Pi^* \to \Delta^{**},\ (\Diamond\alpha)^*,\ \Sigma^*,\ \Sigma}}{\Gamma,\ \Pi^* \to \Delta^*,\ (\Diamond\alpha)^*,\ \Sigma}$$

This shows that we can omit the mix from the assumption of the induction, as the rank of the mix formula is decreased by 1.

3.2 Hauptsatz for M^*.

The case (1) is the same with the case (1) in $Q2$, even if Θ is empty. The two cases (2) and (3) are quite the same as in $Q2$.

3.3 Hauptsatz for $S4^*$.

The case (1) does not occur. The case (2) is the same as before. In case (3) we have to consider the following mix in addition:

$$\dfrac{\dfrac{\alpha \to \Diamond\Theta}{\Diamond\alpha \to \Diamond\Theta} \qquad \Pi \to \Sigma}{\Diamond\alpha,\ \Pi^* \to (\Diamond\Theta)^*,\ \Sigma}\ (\text{mix of } \Diamond\delta)$$

As the right rank equals to 1 by the assumption, the only non-trivial case is the following form:

$$\dfrac{\dfrac{\alpha \to \Diamond\Theta}{\Diamond\alpha \to \Diamond\Theta} \qquad \Diamond\delta \to \Diamond\Gamma}{\Diamond\alpha \to (\Diamond\Theta)^*,\ (\Diamond\Gamma)^*}\ (\Diamond\delta)$$

We transform this into:

$$\dfrac{\dfrac{\alpha \to \Diamond\Theta \qquad \Diamond\delta \to \Diamond\Gamma}{\alpha \to (\Diamond\Theta)^*,\ (\Diamond\Gamma)^*}\ (\Diamond\delta)}{\Diamond\alpha \to (\Diamond\Theta)^*,\ (\Diamond\Gamma)^*}$$

Thus we have completed the proof of Hauptsatz in each case.

(III) Decision Procedures

§ 4 Decision procedure of $S2$.

4.1 Relations between $S2$ and $Q2$.

We shall prove the following four propositions:[6]

Proposition 1° $Q2 \vdash\ \to \alpha \Leftrightarrow Q2 \vdash p \strictif p \to \Box\alpha,$

Proposition 2° $Q2 \vdash\ \to \alpha \Leftrightarrow S2 \vdash \Box\alpha,$

Proposition 3° $S2 \vdash \alpha \Leftrightarrow Q2 \vdash p \strictif p \to \alpha,$

Proposition 4° $S2 \vdash \alpha \Leftrightarrow S2 \vdash p \strictif p : \strictif .\ \alpha.$[7]

Proof of Proposition 1°

$\Rightarrow$

$$\dfrac{\dfrac{\to \alpha}{p \supset p \to \alpha}\ \text{(weakening)}}{p \strictif p \to \Box\alpha}\ (\to \Box)$$

$\Leftarrow$

Let $p \strictif p \to \Box\alpha$ be provable in $Q2$.

Even if p appears in α, because of

$$\dfrac{q \strictif q \to p \strictif p \qquad p \strictif p \to \Box\alpha}{q \strictif q \to \Box\alpha}$$

for q which does not appear in α, we may without substantial loss of generality assume that p does not appear in α.

Then, since $p \strictif p \to \Box\alpha$ is not a beginning sequent, there exists a proof-figure with at least one inferential rule and without any cut.

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6) $S \vdash \alpha$ means that a formula α is provable in the system S. p denotes here a sentence variable.

7) Proposition 4° holds also in $S1$. See § 8.

Eliminating all formulae in form of $\Box\alpha$ appearing in this proof-figure, this proof-figure can no more rest as a $Q2$ proof-figure. For otherwise, $p \prec p \rightarrow$ is provable. But on the other hand, if we eliminate all $\Diamond$'s and $\Box$'s appearing in this proof-figure of $p \prec p \rightarrow$, we get a proof-figure of $p \supset p \rightarrow$, which is impossible.

Therefore there exists at least one rule—especially logical rules which has $\Box\alpha$ as a principal formula—which is not kept in $Q2$ by the elimination of $\Box\alpha$ mentioned above.

But in case that $\Box\alpha$ appears as a side formula or a parameter of one of these rules, the rule holds also after the elimination of $\Box\alpha$. Furthermore since $\Box\alpha$ is not $p \prec p$ by the assumption, $\Box\alpha$ never appears in the antecedent of any sequent.

Therefore

$$\frac{\Gamma \rightarrow \alpha}{\Box\Gamma \rightarrow \Box\alpha} \ (\rightarrow \Box)$$

must appear at least once in the proof-figure of $p \prec p \rightarrow \Box\alpha$. And here, $\Box\alpha$ appears in the succedent of each sequent appearing in the lower side of this rule. Then evidently any subformula of $\Box\alpha$ does not occur in $\Box\Gamma$.

For this reason, the formulae of Γ are all $p \prec p$. That is, the formulae of Γ are all $p \supset p$. Then operating cut with $\rightarrow p \supset p$ and $\Gamma \rightarrow \alpha$, we obtain that $\rightarrow \alpha$ is provable.

Proof of Proposition 2°.

$\Rightarrow$

Assume that $Q2$ proof-figure of $\rightarrow \alpha$ without cut is given.

We interpret each sequent in $Q2$ as follows:

$$\begin{array}{lll} \alpha,\ \beta \rightarrow \gamma,\ \delta & \cdots\cdots\cdots\cdots & \alpha \cdot \beta \prec \gamma \vee \delta, \\ \rightarrow \gamma & \cdots\cdots\cdots\cdots & \Box\gamma, \\ \beta \rightarrow & \cdots\cdots\cdots\cdots & \sim\Diamond\beta. \end{array}$$

(The sequent $\rightarrow$ does not appear)

It is sufficient to show that for each rule of inference in $Q2$ there exists an $S2$-proof from the $S2$-formula corresponding to the upper sequent to the $S2$-formula corresponding to the lower sequent.

We begin with the structural rules of inference.

Weakening in antecedent.

First we consider the case when both Γ and Θ are non-empty. We have here only to consider:

$$\frac{\alpha \rightarrow \beta}{\delta,\ \alpha \rightarrow \beta}$$

Therefore we must show in $S2$ that if $\alpha \prec \beta$ is provable, then so is $\delta \cdot \alpha \prec \beta$.

As we can easily have the rule Kettenschluss from both $\alpha \prec \beta \cdot \beta \prec \gamma : \prec . \alpha \prec \gamma$[8] (B6) and modus ponens, the above results from Kettenschluss and $\alpha\beta \prec \alpha$ (B2). If Θ is empty, the above rule can be issued from $\Diamond(\alpha\beta) \prec \Diamond\alpha$ [19.01], and if Γ is empty, from $\Box\alpha . \prec : \beta \prec \alpha$ [19.75]. Analogous considerations can be given to each of the rules cited above.

Namely, weakening in succedent can be obtained from $\alpha \prec \alpha \vee \beta$ [13.2], rule of Becker (if $\alpha \prec \beta$ is provable, then $\Diamond\alpha \prec \Diamond\beta$ is also), and $\sim\Diamond\alpha . \prec : \alpha \prec \beta$ [19.74]; contraction, from $\alpha \prec \alpha \cdot \alpha$ (B3), $\alpha \vee \alpha \prec \alpha$ [13.3], and the rule of substitution (a); exchange, from $\alpha\beta \prec \beta\alpha$ (B1), $\alpha \vee \beta \prec \beta \vee \alpha$ [13.1]. Cut can be omitted from our consideration.

Secondly, we treat the logical rules of inference. The necessary rules or formulae for respective logical rules are as follows:

$(\cdot \rightarrow)$ $\alpha\beta \prec \alpha$ (B2)
$\alpha \supset \beta : \prec : \alpha\gamma \supset \beta\gamma$ [16.11]
Rule of Becker
$\Diamond(\alpha\beta) \prec \Diamond\alpha$ [19.01]

$(\rightarrow \vee)$ $\alpha \prec \alpha \vee \beta$ [13.2]
$\alpha \supset \beta : \prec : \gamma \vee \alpha \supset \gamma \vee \beta$ [14.27]
Rule of Becker.

$(\rightarrow \cdot)$ $\alpha \vee (\beta\gamma) = (\alpha \vee \beta) \cdot (\alpha \vee \gamma)$ [16.73]

$(\vee \rightarrow)$ $\alpha(\beta \vee \gamma) = \alpha\beta \vee \alpha\gamma$ [16.72]

($\sim$-rules) Both formulae corresponding to the upper and lower sequents are identical, on any conditions of Γ and Θ.

Lastly, $(\Box \rightarrow)$ can be obtained from [14.27] and the rule of Becker, and $(\rightarrow \Box)$, from the additivity of $\Diamond$ [19.82] and the rule of Becker.

$\Leftarrow$

Suppose that $\Box\alpha$ is provable in $S2$, we can easily show that $p \prec p \rightarrow \Box\alpha$ is provable in $Q2$ by using the former half of Proposition 3° (which is deducible from only Proposition 1° as we show below). Then α is provable in $Q2$ using the latter half of Proposition 1°.

Proof of Proposition 3°

$\Rightarrow$

Suppose that α is provable in $S2$. As the outermost symbol of all

8) In the following, numbers in brackets ( ) and [ ] show the numbers of axioms or theorems in [9].

the axioms $\mathfrak{Ax}$ in $S2$ is $\prec$, we write simply $\mathfrak{Ax}^*$ the formulae which we get by replacing the outermost symbol $\prec$ of these axioms with $\supset$.

As $p \supset p \to \mathfrak{Ax}^*$ are all provable in $Q2$, $p \prec p \to \mathfrak{Ax}$ are also provable in $Q2$.

It is obvious that the following two rules:

$$\frac{p \prec p \to \alpha \qquad p \prec p \to \alpha \prec \beta}{p \prec p \to \beta}$$

$$\frac{p \prec p \to \alpha \qquad p \prec p \to \beta}{p \prec p \to \alpha \cdot \beta}$$

are admissible in $Q2$.

Finally on the rules of substitution,

$$\frac{p \prec p \to \alpha \prec \beta}{p \prec p \to \Diamond\alpha \prec \Diamond\beta}$$

should be admissible in $Q2$, because $\to \alpha \supset \beta$ is also provable in $Q2$ from Proposition 1°, assuming the sequent $p \prec p \to \alpha \prec \beta$ is provable in $Q2$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\to \alpha \supset \beta \qquad \cfrac{\alpha \to \alpha \qquad \beta \to \beta}{\alpha,\ \alpha \supset \beta \to \beta}}{\alpha \to \beta}}{\Diamond\alpha \to \Diamond\beta}}{\to \Diamond\alpha \supset \Diamond\beta}}{p \supset p \to \Diamond\alpha \supset \Diamond\beta}}{p \prec p \to \Diamond\alpha \prec \Diamond\beta}$$

$\Leftarrow$

If $p \prec p \to \alpha$ is provable in $Q2$, then $p \prec p : \prec . \Box\alpha$ is provable by Proposition 2° in $S2$. As we have $p \prec p$, $\Box\alpha$ is provable. Therefore α is provable in $S2$.

Proof of Proposition 4°

$\Rightarrow$

Suppose that α is provable in $S2$. As $p \prec p \to \alpha$ is provable in $Q2$ owing to Proposition 3°, we have $p \prec p . \prec . \alpha$ in $S2$ by using Proposition 2°.

The converse is obvious.

4. 2 Decision procedure of $S2$.

Proposition 3° in the previous section gives the decision procedure of $S2$. Namely, the provability of a formula α in $S2$ is reduced to the provability of a sequent $p \prec p \to \alpha$ in $Q2$, hence the decision procedure of the former has been given because of Hauptsatz in $Q2$ in (II).

§ 5 Decision procedures for M and $S4$.

Hauptsatz for M^* and for $S4^*$ are already proved in (II). Then, in an analogous way of Gentzen's procedure, we can give the decision procedures for M and $S4$.

§ 6 Decision procedure for $S5$.

We can not carry out to prove the Hauptsatz in $S5$ in an analogous way to $S4$ mentioned above. But the decision procedure of $S5$ has already been given in the previous paper [11], of which the main result is the following

Theorem: *γ is provable in S5, if and only if $\bigtriangledown\gamma$ is provable in S4, where $\bigtriangledown\gamma$ means $\sim\Diamond\sim\Diamond\sim\Diamond\sim\gamma$.*

In the above paper, this theorem has been proved by using McKinsey's result [13], which can be reduced essentially to the following formulae:

(1) $\bigtriangledown\alpha\cdot\bigtriangledown\beta\,.\mathrel{-\!\!\!3}.\,\bigtriangledown(\alpha\beta)$,

(2) $\bigtriangledown(\Diamond\alpha\mathrel{-\!\!\!3}\sim\Diamond\sim\Diamond\alpha)$

We notice that the above formulae (1), (2) can be proved without McKinsey's result, and that the above theorem can be proved as well without McKinsey's result. The proofs of (1), (2) in $S4^*$ are as follows:

Proof of (1):

$$\dfrac{\dfrac{\alpha\to\alpha}{\alpha,\ \beta\to\alpha}\qquad\dfrac{\beta\to\beta}{\alpha,\ \beta\to\beta}}{\alpha,\ \beta\to\alpha\cdot\beta}$$

$$\begin{array}{c}
\alpha,\ \beta\to\alpha\cdot\beta \\ \hline
\sim\Diamond\sim\alpha,\ \sim\Diamond\sim\beta\to\ \sim\Diamond\sim(\alpha\beta) \\ \hline
\sim\Diamond\sim\alpha\to\Diamond\sim\beta,\ \sim\Diamond\sim(\alpha\beta) \\ \hline
\sim\Diamond\sim\alpha\to\Diamond\sim\beta,\ \Diamond\sim\Diamond\sim(\alpha\beta) \\ \hline
\Diamond\sim\Diamond\sim\alpha\to\Diamond\sim\beta,\ \Diamond\sim\Diamond\sim(\alpha\beta) \\ \hline
\sim\Diamond\sim\beta\to\ \sim\Diamond\sim\Diamond\sim\alpha,\ \Diamond\sim\Diamond\sim(\alpha\beta) \\ \hline
\sim\Diamond\sim\beta\to\Diamond\sim\Diamond\sim\Diamond\sim\alpha,\ \Diamond\sim\Diamond\sim(\alpha\beta) \\ \hline
\Diamond\sim\Diamond\sim\beta\to\Diamond\sim\Diamond\sim\Diamond\sim\alpha,\ \Diamond\sim\Diamond\sim(\alpha\beta) \\ \hline
\sim\Diamond\cdot\sim\Diamond\sim\Diamond\sim\alpha,\ \Diamond\sim\Diamond\sim\beta\to\Diamond\sim\Diamond\sim(\alpha\beta) \\ \hline
\sim\Diamond\sim\cdot\sim\Diamond\sim\Diamond\sim\Diamond\sim\alpha,\ \sim\Diamond\sim\Diamond\sim\Diamond\sim\beta\to\ \sim\Diamond\sim\Diamond\sim\Diamond\sim(\alpha\beta)
\end{array}$$

This endsequent and the axiom C 10 lead us easily to

$$\bigtriangledown\alpha,\ \bigtriangledown\beta\to\bigtriangledown(\alpha\beta)$$

from which we get the following

$$\rightarrow \nabla\alpha \cdot \nabla\beta \mathrel{-\!\!3} \nabla(\alpha\beta)\,.$$

As the preparation for the proof of (2), we have

(i)
$$\begin{array}{c}
\Diamond\gamma \rightarrow \Diamond\sim\Diamond\gamma \\ \hline
\rightarrow \Diamond\sim\Diamond\gamma,\ \sim\Diamond\gamma \\ \hline
\rightarrow \Diamond\sim\Diamond\gamma,\ \Diamond\sim\Diamond\gamma \\ \hline
\rightarrow \Diamond\sim\Diamond\gamma
\end{array}$$

(ii)
$$\begin{array}{c}
\rightarrow \Diamond(\gamma \mathrel{-\!\!3} \delta) \\ \hline\hline
\rightarrow \Diamond\sim\Diamond\sim(\gamma \supset \delta) \\ \hline\hline
\rightarrow \Diamond\sim\Diamond\sim(\gamma \mathrel{-\!\!3} \delta) \\ \hline\hline
\rightarrow \nabla(\gamma \mathrel{-\!\!3} \delta)
\end{array}$$

in $S4$. By using the above two, we obtain the following

Proof of (2):

$$\begin{array}{c}
\Diamond\alpha \rightarrow \Diamond\alpha \\ \hline
\Diamond\alpha \cdot \Diamond\sim\Diamond\alpha \rightarrow \Diamond\alpha \\ \hline
\Diamond(\Diamond\alpha \cdot \Diamond\sim\Diamond\alpha) \rightarrow \Diamond\alpha \\ \hline
\Diamond\sim\Diamond\alpha \rightarrow \Diamond\sim\Diamond(\Diamond\alpha \cdot \Diamond\sim\Diamond\alpha) \\ \hline
\Diamond\alpha \cdot \Diamond\sim\Diamond\alpha \rightarrow \Diamond\sim\Diamond(\Diamond\alpha \cdot \Diamond\sim\Diamond\alpha) \\ \hline
\Diamond(\Diamond\alpha \cdot \Diamond\sim\Diamond\alpha) \rightarrow \Diamond\sim\Diamond(\Diamond\alpha \cdot \Diamond\sim\Diamond\alpha) \\ \hline\hline
\rightarrow \Diamond\sim\Diamond(\Diamond\alpha \cdot \Diamond\sim\Diamond\alpha) \qquad \text{(i)} \\ \hline\hline
\rightarrow \Diamond(\Diamond\alpha \mathrel{-\!\!3} \sim\Diamond\sim\Diamond\alpha) \\ \hline\hline
\rightarrow \nabla(\Diamond\alpha \mathrel{-\!\!3} \sim\Diamond\sim\Diamond\alpha) \qquad \text{(ii)}
\end{array}$$

In [11] we omitted the proof for γ which is the result from substitution. But this proof is reduced to the following:[9]

If $\dfrac{\alpha=\beta}{\Diamond\alpha=\Diamond\beta}$ is admissible in $S5$, then $\dfrac{\Diamond(\alpha=\beta)}{\Diamond(\Diamond\alpha=\Diamond\beta)}$ is admissible in $S4$. And we can easily show that its proof can be treated by characteristic formula of $S3$, $\alpha\,.\,\gamma \cdot \beta \mathrel{-\!\!3} \delta : \mathrel{-\!\!3} : \alpha\beta \mathrel{-\!\!3} \gamma\delta$ [19. 68], and the rule $\dfrac{\alpha=\beta}{\Diamond\alpha=\Diamond\beta}$.

The reduction theorem gives the decision procedure for $S5$.

As regards the sequent, we can show that if $\Gamma \rightarrow \Theta$ is provable in $S5$, then $\nabla\Gamma \rightarrow \nabla\Theta$ is provable in $S4$. But the inverse can not be kept in general. For example, $\nabla\alpha \rightarrow \nabla\sim\Diamond\sim\alpha$ is provable in $S4$, but $\alpha \rightarrow \sim\Diamond\sim\alpha$ is not so in $S5$.

9) A sign $=$ means a strict equivalence as in [9].

APPENDIX

We shall treat here some metatheorems.[10]

§ 7 We consider at first three following rules:

RT: *If α is provable, then $\Box\alpha$ is also provable.*

RE: *If $\alpha\supset\beta$ is provable, then $\Diamond\alpha\supset\Diamond\beta$ is also provable.*

RB: *If $\alpha\mathrel{-\!\!3}\beta$ is provable, then $\Diamond\alpha\mathrel{-\!\!3}\Diamond\beta$ is also provable.*

7.1 The rule of Tautology RT.

RT holds in $S5^*$, $S4^*$ and M^*, but does not hold in $S3$, $S2$ and $Q2$.

It is clear from the rule $(\to\Box)$ of our formulation that RT holds in $S5^*$, $S4^*$ and M^*. That RT does not hold in $S3$, $S2$ and $Q2$ is shown by the following results:

(i) A system which is obtained from $S3$ by the addition of RT is equivalent to $S4$. This is a result by Parry [14].

(ii) A system which is obtained from $S2$ by the addition of RT is equivalent to M.

Proof: Assume that $\alpha\supset\beta$ is provable. Then $\alpha\mathrel{-\!\!3}\beta$ is also provable by RT. As $\alpha\mathrel{-\!\!3}\beta : \mathrel{-\!\!3} : \Diamond\alpha\supset\Diamond\beta$ is provable in $S2$, we obtain $\Diamond\alpha\supset\Diamond\beta$.

(iii) A system which is obtained from $Q2$ by the addition of RT is equivalent to M.

Proof: $Q2$-proof of $\alpha\mathrel{-\!\!3}\beta\to\Diamond\alpha\supset\Diamond\beta$ is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha\to\alpha \qquad \beta\to\beta}{\alpha,\ \alpha\supset\beta\to\beta}}{\alpha\to\beta,\ \sim(\alpha\supset\beta)}}{\Diamond\alpha\to\Diamond\beta,\ \Diamond\sim(\alpha\supset\beta)}}{\Diamond\alpha,\ \sim\Diamond\sim(\alpha\supset\beta)\to\Diamond\beta}}{\sim\Diamond\sim(\alpha\supset\beta)\to\Diamond\alpha\supset\Diamond\beta}$$

Therefore we get the results analogously to (ii).

7.2 The rule of Extentionality RE.

RE holds in $S5^*$, $S4^*$ and $Q2$, but does not hold in $S3$ and $S2$.

(i) A system which is obtained from $S2$ by the addition of RE is equivalent to M.

(ii) A system which is obtained from $S3$ by the addition of RE is equivalent to $S4$.

10) It is not generally assured that a rule in a system S holds also in other system S' which is deductively equivalent to S.

Proof of (i) and (ii).

In our systems RT holds, for

$$\dfrac{\to \beta \mathrel{-\!\!\!3} \beta \qquad \dfrac{\dfrac{\to \alpha}{\beta \supset \beta \to \alpha}}{\beta \mathrel{-\!\!\!3} \beta \to \Box\alpha}}{\to \Box\alpha}$$

Therefore by the definition of M and $S4$, we get the results.

7.3 The rule of Becker RB.

RB holds in $S5^*$, $S4^*$, $S3$ and $S2$.

§8 Next we can prove Proposition 4° in (III), when we replace $S2$ by $S1$. i.e.,

$$S1 \vdash \alpha \Leftrightarrow S1 \vdash p \mathrel{-\!\!\!3} p : \mathrel{-\!\!\!3} . \alpha$$ [11][12]

For the preparation of the proof we show the following

Lemma: $\vdash \sim\Diamond\sim\alpha \Rightarrow \vdash p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \sim\Diamond\sim\alpha .$

Proof: As we have $\sim\Diamond\sim \alpha . \supset . \beta \mathrel{-\!\!\!3} \alpha$,[13] we have

$$\vdash \sim\Diamond\sim\alpha, \ \vdash \sim\Diamond\sim\beta \Rightarrow \vdash \alpha = \beta$$

Therefore if we assume $\vdash \sim\Diamond\sim\alpha$, we have $\vdash p \supset p . = . \alpha$. Using the rule of substitution (a), we have $\vdash p \mathrel{-\!\!\!3} p . = . \sim\Diamond\sim\alpha$.

Now a proof of proposition cited above is as follows:

We assume that α is provable. If α is an axiom, the proof is trivial by the lemma. Next assume that α is a result of adjunction. That is, we assume that $\vdash \alpha = \beta\gamma$ and β, γ are both provable. Then we have only to show that $p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \beta\gamma$ is provable assuming the provabilities of $p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \beta$ and $p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \gamma$. And this is trivial. Assume that α is a result of modus ponens. This case is analogous to before.

Lastly assume that α is a result of substitution (a). Essentially we have only to prove that $\vdash p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \Diamond\alpha = \Diamond\beta$, assuming the provability of $p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \alpha = \beta$. This proof is the following:

We have easily $\vdash \alpha = \beta$ from the assumption $p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \alpha = \beta$ and $\vdash p \mathrel{-\!\!\!3} p$. Therefore we have $\vdash \Diamond\alpha = \Diamond\beta$, i.e., $\vdash \Diamond\alpha \mathrel{-\!\!\!3} \Diamond\beta$, $\vdash \Diamond\beta \mathrel{-\!\!\!3} \Diamond\alpha$. Then, from the lemma we have $\vdash p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \Diamond\alpha \mathrel{-\!\!\!3} \Diamond\beta$, $\vdash p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \Diamond\beta \mathrel{-\!\!\!3} \Diamond\alpha$. So $\vdash p \mathrel{-\!\!\!3} p . \mathrel{-\!\!\!3} . \Diamond\alpha = \Diamond\beta$ is established.

$\Leftarrow$

This case is trivial because of $\vdash p \mathrel{-\!\!\!3} p$.

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11) We write simply $\vdash\alpha$ instead of $S1\vdash\alpha$ in this proof.

12) This results shows that a solution of decision problem for $S1$ is reduced to a solution of decision problem for formulae of the form $p \mathrel{-\!\!\!3} p : \mathrel{-\!\!\!3} . \alpha$.

13) S. Halldén [7].

As to RE and RB, it can be shown that both RE and RB are not admissible in $S1$. Because if RE is admissible in $S1$, then $q \supset q . \supset . p \prec p$ is obviously provable. Therefore $q \prec q . \supset . \Box(p \prec p)$ is also provable by our assumption. This leads us easily that $\Box(p \prec p)$ is provable. Hence $S1$ is equivalent to M using the result by Yonemitsu [18]. But this is impossible.

Next, if RB is admissible in $S1$, then $S1$-axiom $pq \prec q$ ($B2$) implies $\Diamond(pq) \prec \Diamond q$ which is a characteristic formula of $S2$. But this is impossible, too.

§ **9** We treat here the theorems on reductions of $S4$ to LJ and of $S5$ to LK in the domain of functional calculi.

In Maehara's paper[14] [10] on interpretation of intuitionistic calculus within an extended classical calculus, he obtained several results which are concerned with modal logic syntactically.

We can easily obtain $S4$ upon replacing "Bew" in his system BLK everywhere by a symbol $\Box$ for necessity. Then his main theorem can be rewritten by our symbols as follows:

If a symbol $\Box$ does not occur in Γ and Θ, then $\Gamma \rightarrow \Theta$ is LJ-provable if and only if $\varphi(\Gamma) \rightarrow \varphi(\Theta)$ is $S4$-provable, where for any formula α, a formula $\varphi(\alpha)$ is defined inductively in $S4$ as the formula which arises from α by replacing every subformula γ of α by $\Box\gamma$, and if Γ is a sequence of formulae $\alpha_1, \alpha_2, \cdots, \alpha_n$, then $\varphi(\Gamma)$ is $\varphi(\alpha_1), \varphi(\alpha_2), \cdots, \varphi(\alpha_n)$ for $n \geq 0$.

As an easy corollary, we have the following reduction theorem of $S4$ to LJ.

Theorem: *$LJ \vdash \rightarrow \alpha \Leftrightarrow S4 \vdash \rightarrow \varphi(\alpha)$, for an LJ-formula α.*

We shall prove here the following reduction of $S5$ to LK.

Theorem: *$LK \vdash \rightarrow \alpha \Leftrightarrow S5 \vdash \rightarrow \varphi(\alpha)$, for an LK-formula α.*

Proof:

$\Rightarrow$

As preparatory remarks, we have the following without proof:

(i) $\Box(\alpha \cdot \beta) = \Box\alpha \cdot \Box\beta$ [19. 81].

(ii) $\Box(\Box\alpha \vee \Box\beta) = \Box\alpha \vee \Box\beta$.

(iii) $\Box(\forall x \Box f(x)) = \Box \forall x\, f(x)$.

(iv) $\Box(\exists x \Box f(x)) = \exists x \Box f(x)$.

14) See also a review for [9] by one of the author, J. Symbolic, **22**, 79–80 (1957).

We substitute $\varphi(\Gamma)\rightarrow\varphi(\Theta)$ for each sequent $\Gamma\rightarrow\Theta$ appearing in an *LK*-proof-figure which possesses the endsequent $\rightarrow\alpha$. Then the uppermost sequent is $\varphi(\delta)\rightarrow\varphi(\delta)$ and this is again a beginning sequent.

Hence we have only to show the corresponding new rules are admissible in $S5$.

The proof for structural rules of inference is trivial.

Of the logical rules of inference, we take $(\rightarrow V)$ and $(\rightarrow\sim)$ for example:

As to $(\rightarrow V)$

$$\cfrac{\cfrac{\cfrac{\varphi(\Gamma)\rightarrow\varphi(f(a)),\ \varphi(\Theta)}{\varphi(\Gamma)\rightarrow Vx\varphi(f(x)),\ \varphi(\Theta)}\ (\rightarrow V)\ \left[\begin{array}{l}\text{There is no free variable}\\ a \text{ in the lower sequent}\end{array}\right]}{\varphi(\Gamma)\rightarrow\Box Vx\varphi(f(x)),\ \varphi(\Theta)}\ (\rightarrow\Box)\ \left[\begin{array}{l}\text{the outermost symbols of}\\ \varphi(\Gamma),\ \varphi(\Theta) \text{ are } \Box\end{array}\right]}{\varphi(\Gamma)\rightarrow\varphi(Vxf(x)),\ \varphi(\Theta)}$$

As to $(\rightarrow\sim)$

$$\cfrac{\cfrac{\cfrac{\varphi(\alpha),\ \varphi(\Gamma)\rightarrow\varphi(\Theta)}{\varphi(\Gamma)\rightarrow\varphi(\Theta),\ \sim\varphi(\alpha)}}{\varphi(\Gamma)\rightarrow\varphi(\Theta),\ \Box\sim\varphi(\alpha)}\ (\rightarrow\Box)\ \left[\begin{array}{l}\text{the outermost symbols of}\\ \varphi(\Gamma),\ \varphi(\Theta) \text{ are } \Box\end{array}\right]}{\varphi(\Gamma)\rightarrow\varphi(\Theta),\ \varphi(\sim\alpha)}$$

The other cases are easily verified from the above preparatory remarks.

$\Leftarrow$

We have only to show that each rule of $S5$ still holds for simultaneous elimination of all $\Box$'s which appear in each sequent of the rule. And this is trivial.

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Bibliography

[1] Alan Ross Anderson: Improved decision procedures for Lewis's calculus $S4$ and Von Wright's calculus M, J. Symbolic Logic **19,** 201–214, (1954).

[2] R. C. Barcan: A functional calculus of first order based on strict implication, J. Symbolic Logic **11,** 1–16, (1946).

[3] R. Carnap: Modalities and Quantification, J. Symbolic Logic **11,** 33–64, (1946).

[4] H. B. Curry: The Elimination Theorem when modality is present, J. Symbolic Logic **17,** 249–265 (1952).

[5] R. Feys: Les systèmes formalisée des modalités aristotéliciennes, Revue Phil. de Louvain **48,** 478–509 (1950).

[6] G. Gentzen: Untersuchungen über das logische Schliessen, I. II., Math. Z. **39,** 176–210, 405–431, (1935).

[7] S. Halldén: A note concerning the paradoxes of strict implication and Lewis's system $S1$, J. Symbolic Logic **13,** 138–139, (1948).

[8] M. Itoh: A lattice-theoretic research on modal propositional calculus and monadic predicate calculus (in Japanese) Kagaku-Kisoron-Kenkyu **1,** 142–145, 162–167, (1955), **2,** 258–265 (1956).

[9] C. I. Lewis and C. H. Langford: Symbolic Logic, New York 1932.

[10] S. Maehara: Eine Darstellung der Intuitionistischen Logik in der Klassischen, Nagoya Math. J. **7,** 45-64 (1954).

[11] K. Matsumoto: Reduction Theorem in Lewis' sentential calculi, Math. Japonicae **3,** 133–135 (1955).

[12] J. C. C. McKinsey: A solution of the decision problem for the Lewis system $S2$ and $S4$, with an application to topology, J. Symbolic Logic **6,** 117–134 (1941).

[13] J. C. C. McKinsey and A. Tarski: Some theorems about the sentential calculi of Lewis and Heyting, J. Symbolic Logic **13,** 1–15 (1948).

[14] W. T. Parry: Modalities in the Survey system of strict implication, J. Symbolic Logic **4,** 137–154 (1939).

[15] J. Ridder: Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen, I–IV, IV^{bis}, V, VI, VI^{bis}, Indag. Math. **14,** 213–223, 459-467 (1952), **15,** 1–11, 99–110 (1953), **16,** 117–128, 389–396 (1954).

[16] J. Ridder: Die Gentzenschen Schlussverfahren in modalen Aussagenlogiken, Indag. Math. **17,** 163–276 (1955).

[17] G. H. von Wright: An Essey in Modal Logic, Studies in Logic and the Foundation of Mathematics, Amsterdam 1951.

[18] N. Yonemitsu: A Note on Modal Systems, von Wright's M and Lewis's $S1$, Memoirs of the Osaka Univ. of the liberal arts and education, B. Natural Science, 45 (1955).