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On Intuitionistic Functional Calculus

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This paper is separated into two parts; the first one is devoted to logical calculus, and the second one to the intuitionistic theory of real numbers based on Kuroda's treatment[1].

The next schema is due to A. Heyting[2]:

1. $\forall xF(x) \rightarrow \neg\neg \forall xF(x) \rightarrow \forall x \neg\neg F(x) \rightleftarrows \neg\neg \forall x \neg\neg F(x) \rightleftarrows \neg \exists x \neg F(x)$,
2. $\exists xF(x) \rightarrow \exists x \neg\neg F(x) \rightarrow \neg\neg \exists xF(x) \rightleftarrows \neg\neg \exists x \neg\neg F(x) \rightleftarrows \neg \forall x \neg F(x)$,
3. $\forall x \neg F(x) \rightleftarrows \neg\neg \forall x \neg F(x) \rightleftarrows \neg \exists x \neg\neg F(x) \rightleftarrows \neg \exists xF(x)$,
4. $\exists x \neg F(x) \rightarrow \neg\neg \exists x \neg F(x) \rightleftarrows \neg \forall x \neg\neg F(x) \rightarrow \neg \forall xF(x)$.

Explanations for the symbols used here: $\forall x$ and $\exists x$ are universal and existential quantifiers with respect to the individual variable x respectively. $F(*)$ is functional variable with certain (finite) number of arguments. $\rightarrow$ is one-way implication and $\rightleftarrows$ is (logical) equivalence of ante- and succedent formulae. (Hence these two are meta-logical symbols.) $\neg$ is negation and $\neg\neg$ is double negation of the remaining sub-formula after it.

1. Iterated quantifications.

Starting from the above schema by Heyting let us consider the case of iterated quantifications, where we shall be mainly concerned with the implicative relations between such formulae as follows: $\forall x\, F(x)$, $\exists x\, F(x)$, $\forall x\, \exists y\, F(xy)$, $\exists x\, \forall y\, F(xy)$, $\forall x\, \exists y\, \forall z\, F(xyz)$, $\exists x\, \forall y\, \exists z\, F(xyz)$ and their weakened forms to which $\neg\neg$'s are attached.

1. 1. Formulae with one quantifier. In this case the implicative relations are:

(1) $$\forall x\, F(x) \rightarrow \neg\neg \forall x\, F(x) \rightarrow \forall x \neg\neg F(x) \rightleftarrows \neg\neg \forall x \neg\neg F(x),$$

(2) $$\exists x\, F(x) \rightarrow \exists x \neg\neg F(x) \rightarrow \neg\neg \exists x\, F(x) \rightleftarrows \neg\neg \exists x \neg\neg F(x).$$

Hereafter some conventions will be used. The attached symbol $\neg\neg$

1) Kuroda (3).
2) Heyting (2).

indicating weakened modality of succeeding formula or quantifier[3], we might denote the above relations as follows:

1.11. $$\forall(++)\rightarrow(-+)\rightarrow(+-)\rightleftarrows(--),$$

1.12. $$\exists(++)\rightarrow(+-)\rightarrow(-+)\rightleftarrows(--).$$

1.2. Formulae with two quantifiers.

1.21. **$\forall\exists$-schema.** We shall find the implicative relations between the following eight formulae: $\forall x\,\exists y\,F(xy)$, $\forall x\,\exists y\,\neg\neg\,F(xy)$, $\forall x\,\neg\neg\,\exists y\,F(xy)$, $\forall x\,\neg\neg\,\exists y\,\neg\neg\,F(xy)$, and the four to each of which $\neg\neg$ is prefixed; using the above denotations these are: $\forall\exists(+++)$, $(++-)$, $(+-+)$, $(+--)$, $(-++)$, $(-+-)$, $(--+)$, $(---)$.
First we write the $\forall$-schema:

$$\forall(++)\rightarrow(-+)\rightarrow\left\{\begin{matrix}(+-)\\(--)\end{matrix}\right\}$$

(here the last two are equivalent), then adjoining the symbol $+$ and $-$ *after* each member of it, we get the following two schemata:

$$(+++)\rightarrow(-++)\rightarrow\left\{\begin{matrix}(+-+)\\(--+)\end{matrix}\right\},$$

and

$$(++-)\rightarrow(-+-)\rightarrow\left\{\begin{matrix}(+--)\\(---)\end{matrix}\right\}.$$

Next beginning from $\exists$-schema we get the following two by adjoining $+$ and $-$ *before* each member:

$$(+++)\rightarrow(++-)\rightarrow\left\{\begin{matrix}(+-+)\\(+--)\end{matrix}\right\},$$

$$(-++)\rightarrow(-+-)\rightarrow\left\{\begin{matrix}(--+)\\(---)\end{matrix}\right\}.$$

It is almost evident that we are now able to establish the full $\forall\exists$-schema which comes out from the above four schemata, namely,

$$\forall\exists(+++)\begin{matrix}\nearrow(-++)\searrow\\ \searrow(++-)\nearrow\end{matrix}(-+-)\rightarrow\left\{\begin{matrix} & (--+) & \\ (+-+) & & (---)\\ & (+--) & \end{matrix}\right\}.$$

Herein the last four formulae within a bracket { } are found to be equivalent. It might be useful to rewrite this schema in its original form:

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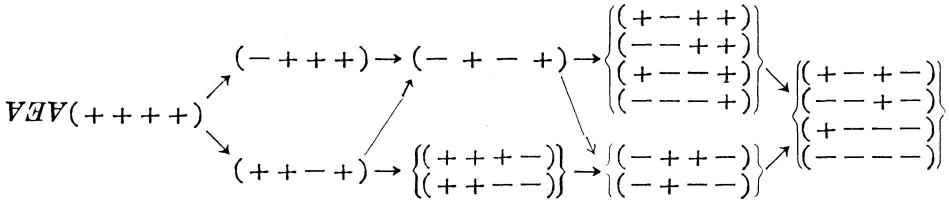
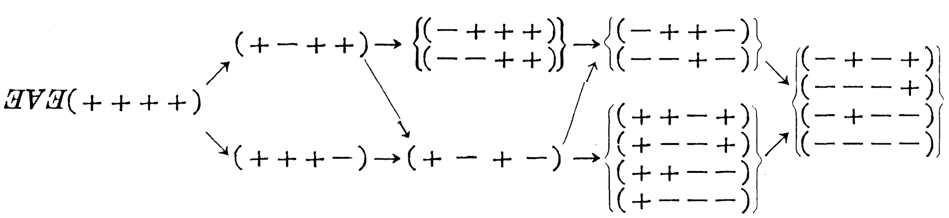
3) Cf. Kuroda (4).



1. 31. **$\exists\forall\exists$-schema.** The resultant schema is:



2. Intuitionistic theory of real numbers.

2. 1. Let us define a real number α by a fundamental sequence $\{a_r\}$ of rational numbers. In other words if for any positive rational number ε there exists a natural number n such that for any natural number r greater than n the inequality $|a_r - a_n| < \varepsilon$ holds, then the sequence $\{a_r\}$ is called to define a real number α. This condition on $\{a_r\}$ can be written formally as follows:

$$\forall\varepsilon > 0 \ \exists n \ \forall r > n \quad |a_r - a_n| < \varepsilon .$$

According to $\forall\exists\forall$-schema (1. 31.) there are eight non-equivalent forms, but in this case the proposition $|a_r - a_n| < \varepsilon$ is equivalent to its double negation[4], therefore it follows that in the schema certain members of it are equivalent, namely

$$(++++) \rightleftarrows (++-+) \rightleftarrows (++--),$$
$$(-+++) \rightleftarrows (-+-+) \rightleftarrows (-+--),$$
$$(+-++) \rightleftarrows (+---).$$

Hence there remain only three non-equivalent types of fundamental sequences:

(2. 11.) $$\forall\varepsilon > 0 \ \exists n \ \forall r > n \ |a_r - a_n| < \varepsilon ,$$

(2. 12.) $$\neg\neg \forall\varepsilon > 0 \ \exists n \ \forall r > n \ |a_r - a_n| < \varepsilon ,$$

(2. 13.) $$\forall\varepsilon > 0 \ \neg\neg \exists n \ \forall r > n \ |a_r - a_n| < \varepsilon .$$

Clearly (2. 11.) $\rightarrow$ (2. 12.) $\rightarrow$ (2. 13.).

4) The intuitionistic theory of rational numbers is presupposed here.

$$\forall x\,\exists y\,F(xy) \begin{array}{l} \nearrow \neg\neg\,\forall x\,\exists y\,F(xy) \searrow \\ \searrow \forall x\,\exists y\,\neg\neg\,F(xy) \nearrow \end{array} \neg\neg\,\forall x\,\exists y\,\neg\neg\,F(xy) \to$$

$$\to \forall x\,\neg\neg\,\exists y\,F(xy) \begin{array}{l} \nearrow\!\!\!\swarrow \neg\neg\,\forall x\,\neg\neg\,\exists y\,F(xy) \nwarrow\!\!\!\searrow \\ \nwarrow\!\!\!\searrow \forall x\,\neg\neg\,\exists y\,\neg\neg\,F(xy) \nearrow\!\!\!\swarrow \end{array} \neg\neg\,\forall x\,\neg\neg\,\exists y\,\neg\neg\,F(xy).$$

1.22. **$\exists\forall$-schema.** In an analogous way to above we get from the four schemata:

$$(+\,+\,+) \to (+\,-\,+) \to \begin{Bmatrix} (-\,+\,+) \\ (-\,-\,+) \end{Bmatrix},$$

(+ is adjoined *after* $\exists$-schema)

$$(+\,+\,-) \to (+\,-\,-) \to \begin{Bmatrix} (-\,+\,-) \\ (-\,-\,-) \end{Bmatrix},$$

(− is adjoined *after* $\exists$-schema)

$$(+\,+\,+) \to (+\,-\,+) \to \begin{Bmatrix} (+\,+\,-) \\ (+\,-\,-) \end{Bmatrix},$$

(+ is adjoined *before* $\forall$-schema)

$$(-\,+\,+) \to (-\,-\,+) \to \begin{Bmatrix} (-\,+\,-) \\ (-\,-\,-) \end{Bmatrix},$$

(− is adjoined *before* $\forall$-schema)

the following $\exists\forall$-schema:

$$\exists\forall(+\,+\,+) \to (+\,-\,+) \begin{array}{l} \nearrow \begin{Bmatrix} (-\,+\,+) \\ (-\,-\,+) \end{Bmatrix} \searrow \\ \searrow \begin{Bmatrix} (+\,+\,-) \\ (+\,-\,-) \end{Bmatrix} \nearrow \end{array} \begin{Bmatrix} (-\,+\,-) \\ (-\,-\,-) \end{Bmatrix}.$$

That is,

$$\exists x\,\forall y\,F(xy) \to \exists x\,\neg\neg\,\forall y\,F(xy) \begin{array}{l} \nearrow \neg\neg\,\exists x\,\forall y\,F(xy) \rightleftarrows \\ \searrow \exists x\,\forall y\,\neg\neg\,F(xy) \rightleftarrows \end{array}$$

$$\begin{array}{l} \rightleftarrows \neg\neg\,\exists x\,\neg\neg\,\forall y\,F(xy) \searrow \\ \rightleftarrows \exists x\,\neg\neg\,\forall y\,\neg\neg\,F(xy) \nearrow \end{array} \neg\neg\,\exists x\,\forall y\,\neg\neg\,F(xy) \rightleftarrows \neg\neg\,\exists x\,\neg\neg\,\forall y\,\neg\neg\,F(xy).$$

1.3. Formulae with three quantifiers.

1.31. **$\forall\exists\forall$-schema.** First we adjoin + and − after $\forall\exists$-schema, then befor $\exists\forall$-schema, next combining these four we get the following:

Example. In the development of π into decimal fractions we denote by d_ν the ν-th number after the comma, and when the sequence d_ν $d_{\nu+1} \dots d_{\nu+9}$ is equal to 0123456789 for the n-th time, we set $\nu = k_n$. Then we define a sequence $\{a_r\}$ by the following postulates:

if $\quad r < k_1$, then $a_r = 1$,

if $\quad k_1 \leq r < k_2$, then $a_r = \frac{1}{2}$,

....

generally, if $k_n \leq r < k_{n+1}$, then $a_r = \left(\frac{1}{2}\right)^n$.

Thus defined sequence $\{a_r\}$ furnishes an example of (2.13)-number, which is unable to be proved (2 12)-number.[5]

Of course there exists neither a sequence which satisfies (2.12) and does not satisfy (2.11) (i.e. for which negation of (2.11) holds), nor one which satisfies (2.13) and does not satisfy (2.12).

2.2. Equality of real numbers. For two real numbers, or what is the same, two sequences $\{a_r\}$ and $\{b_r\}$ which satisfy (2.13), we define the equality of them by the formula:

$$\forall \varepsilon > 0 \ \exists n \ \forall r > n \quad |a_r - b_r| < \varepsilon .$$

Again $|a_r - b_r| < \varepsilon$ being equivalent to its double negation, there are three kinds of equalities:

(2.21.) $\quad \forall \varepsilon > 0 \ \exists n \ \forall r > n \ |a_r - b_r| < \varepsilon ,$

(2.22.) $\quad \neg\neg A \varepsilon > 0 \ \exists n \ \forall r > n \ |a_r - b_r| < \varepsilon ,$

(2.23.) $\quad \forall \varepsilon > 0 \ \neg\neg \exists n \ \forall r > n \ |a_r - b_r| < \varepsilon .$

Clearly (2.21.) $\rightarrow$ (2.22.) $\rightarrow$ (2.23.).

2.3. Inequality. Let us define inequality of $\{a_r\}$ and $\{b_r\}$ by

$$\exists \varepsilon > 0 \ \forall n \ \exists r > n \quad |a_r - b_r| \geqq \varepsilon .$$

In this case non-equivalent types are[6]:

(2.31.) $\quad \exists \varepsilon > 0 \ \forall n \ \exists r > n \ |a_r - b_r| \geqq \varepsilon ,$

(2.32.) $\quad \exists \varepsilon > 0 \ \neg\neg \forall n \ \exists r > n \ |a_r - b_r| \geqq \varepsilon ,$

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5) The sequence $\{(\frac{1}{2})^r\}$ satisfies (2.11).
If $r < k_1$, then $a_r = (\frac{1}{2})^r$, and if $r \geqq k_1$, then $a_r = 1$. The sequence $\{a_r\}$ defined by these conditions satisfies (2.12) but it cannot be proved (at least now) to satisfy (2.11).

6) $|a_r - b_r| \geqq \varepsilon$ is also equivalent to its double negation.

(2.33.) $\neg\neg \exists \varepsilon > 0 \ \forall n \exists r > n \ |a_r - b_r| \geqq \varepsilon$,

(2.34.) $\exists \varepsilon > 0 \ \forall n \neg\neg \exists r > n \ |a_r - b_r| \geqq \varepsilon$,

(2.35.) $\neg\neg \exists \varepsilon > 0 \ \forall n \neg\neg \exists r > n \ |a_r - b_r| \geqq \varepsilon$.

The implicative relations between them are obviously (cf. $\exists\forall$-schema):

$$(2.31.) \to (2.32.) \begin{matrix} \nearrow (2.33.) \searrow \\ \searrow (2.34.) \nearrow \end{matrix} (2.35.).$$

2.4. Equality and inequality.

2.41. *The negation of* (2.23.)*-equality coincides with* (2.35.)*-inequality.*

Proof. We must show that $\neg \forall \varepsilon \neg\neg \exists n \forall r |a_r - a_n| < \varepsilon$ is equivalent to $\neg\neg \exists \varepsilon \forall n \neg\neg \exists r |a_r - a_n| \geqq \varepsilon$.

We have
$$\neg \forall \varepsilon \neg\neg \exists n \rightleftarrows \neg\neg \exists \varepsilon \neg \exists n,$$
$$\neg \exists n \forall r \rightleftarrows \forall n \neg \forall r,$$
and
$$\neg\neg \exists r |a_r - a_n| \geqq \varepsilon \rightleftarrows \neg \forall r \neg\neg |a_r - a_n| < \varepsilon;$$
on the other hand
$$\neg\neg |a_r - a_n| < \varepsilon \rightleftarrows |a_r - a_n| < \varepsilon.$$

Combining these equivalences we obtain the desired result.

2.42. *The negations of* (2.21.)- *and* (2.22.)*-equality coincide with each other.*

Proof. Clear.

2.43. *Both negations of* (2.34.)- *and* (2.35.)*-inequality coincide with* (2.23.)*-equality.*

Proof. By 2.41. $\neg(2.23.) \rightleftarrows (2.35.)$, and $(2.35.) \rightleftarrows \neg\neg(2.34.)$, therefore $\neg(2.34.) \rightleftarrows \neg(2.35.) \rightleftarrows \neg\neg(2.23.) \rightleftarrows (2.23.)$.

2.44. *The negations of* (2.31.)-, (2.32.)- *and* (2.33.)*-inequality coincide with each other.*

Proof. Since $(2.31.) \to (2.32.) \to (2.33.) \rightleftarrows \neg\neg(2.31.)$,

we have $\neg(2.33.) \to \neg(2.32.) \to \neg(2.31.) \rightleftarrows \neg(2.33.)$.

2.5. **(2.11.)-number.** Let us consider now only (2.11.)-numbers. Then

2.51. *Three equalities* (2.21.), (2.22.) *and* (2.23.) *are equivalent.*

Proof. We have only to show that if two (2.11.)-numbers α and β are (2.23.)-equal then they are also (2.21.)-equal.

Obviously for any positive rational ε it holds that

$$\exists n \forall r > n \ |a_r - a_n| < \varepsilon, \ |b_r - b_n| < \varepsilon. \qquad (*)$$

Suppose now that $|a_n - b_n| \geqq 3\varepsilon$.
Since

$$|a_r - b_r| \geqq |a_n - b_n| - |a_r - a_n| - |b_r - b_n|,$$

it follows that $\forall r > n \ |a_r - b_r| > \varepsilon$ by (*), which contradicts our assumption that α and β are (2.23.)-equal. Therefore $|a_n - b_n| < 3\varepsilon$ is valid because each side of this inequation represents rational number. Again using (*) we infer that

$$\forall r > n \ |a_r - b_r| \leqq |a_n - b_n| + |a_r - b_n| + |b_r - b_n| < 5\varepsilon.$$

ε being arbitrary, we have established our theorem.[7)]

2.52. (2.31.)-, (2.32.)- *and* (2.34.)-*inequalities are equivalent.*

Proof. The equivalence of (2.31.) and (2 34.) has been proved in Kuroda (3), consequently follows our theorem.

2.53. (2.33)- *and* (2.35.)-*inequalities are equivalent.*

Proof. It holds that $\neg\neg$(2.31.)$\rightleftarrows$(2.33.) and $\neg\neg$ (2.34.)$\rightleftarrows$(2.35.); on the other hand (2.31.)$\rightleftarrows$(2.34.) by 2.52., therefore (2.33.)$\rightleftarrows$(2.35.).

2.54. In Brouwer (1)'s terminology "örtliche Verschiedenheit" of two points (numbers) corresponds to the inequality (2.52.), "Abweichung" to (2.53.) and "Zusammenfallung" to (2.51.).

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References

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(3) S. Kuroda: A study on the foundation of mathematical analysis (Japanese), Kiso-Kagaku **12** (1949), pp. 1–12.

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7) In Kuroda (3) the equivalence of (2.11) and (2.12) has been proved.