Osaka Mathematical Journal
Vol. 5, No. 1, June, 1953

Some Remarks on the Uniform Space

By Kiyo AOTANI

In the theory of uniform space the main problems are: 1) the study of the compactness, 2) the study of the completion and 3) the study of Baire's property etc.. As for 2), Prof. K. Kunugui studied the uniform space replacing Weil's[1] condition by the local condition,[2] and concluded in this case that, if R is a uniform space and R^* is an imbedding of R in a complete space S, then the mapping from R to R^* is not only the bi-continuous but also uniformly bi-continuous mapping. But there remains in this case the problem to determine the complete space S depending only on Λ (Λ is the index set determining the uniform structure in R). In this paper we settle this problem and make some considerations about the local conditions.

§1. R is a neighbourhood space such that we can take, corresponding to each point p in R, a family $\{\mathfrak{D}_\lambda ; \lambda \in \Lambda\}$ of fundamental systems of neighbourhoods. Then this space R is called a *uniform space* when $\{\mathfrak{D}_\lambda ; \lambda \in \Lambda\}$ satisfies the following condition (α), where Λ is a set of indices and it is ordered by the condition that $\alpha \leqq \beta$ is possible if and only if for any element $V_\alpha(p)$ in $\mathfrak{D}_\alpha$ there exists some element $V_\beta(p)$ in $\mathfrak{D}_\beta$ and $V_\alpha(p) \subseteqq V_\beta(p)$ for each point p in R.

(α). Let p be any point in R; and if we select an arbitrary element $V_\lambda(p)$ from each $\mathfrak{D}_\lambda (\lambda \in \Lambda)$, then $\{V_\lambda(p); \lambda \in \Lambda\}$ is also a fundamental system of neighbourhoods of p.

(α) is nothing but a condition to agree with the topology of the neighbourhood space R,[3] and then the uniform space R with condition (α) satisfies Hausdorff's condition (B). Therefore the uniform space R is a T-space.

Then we assume that the uniform space R satisfies the following condition (W):

(W). For any index $\lambda \in \Lambda$ and any point s in R there exists some index $\mu = \mu(\lambda, s)$ in Λ such that if for three elements p, q and r in R we can take some neighbourhoods $V_1(r)$, $V_2(r)$ in $\mathfrak{D}_\mu$ that contain p and q respectively, then there exists always a neighbourhood $V_3(p) \in \mathfrak{D}_\lambda$ that contains q when s coincides with one of p, q and r.

We call this μ *a W-index for* λ *and* s.

Theorem 1. *A uniform space* R *satisfying the condition* (W) *is regular.*

This theorem is proved by Prof. K. Kunugui.[4]

The condition (W) can be expressed by the three conditions.

(W_1). When s coincides with r:

For any index $\lambda \in \Lambda$ and any point $r \in R$ there exists some index $\mu = \mu(\lambda, r) \in \Lambda$ such that, if for points p and q in R we can take some neighbourhoods $V_1(r)$, $V_2(r)$ in $\mathfrak{O}_\mu$ that contain p and q respectively, then there exists always a neighbourhood $V_3(p) \in \mathfrak{O}_\lambda$ that contains q.

(W_2). When s coincides with p:

For any index $\lambda \in \Lambda$ and any point p in R there exists some index $\mu = \mu(\lambda, p) \in \Lambda$ such that if for points q and r in R we can take some neighbourhoods $V_1(r)$, $V_2(r)$ in $\mathfrak{O}_\mu$ that contain p and q respectively, then there exists always a neighbourhood $V_3(p) \in \mathfrak{O}_\lambda$ that contains q.

(W_3). When s coincides with q:

For any index $\lambda \in \Lambda$ and any point q in R there exists some index $\mu = \mu(\lambda, q) \in \Lambda$ such that if for p and r in R we can take some neighbourhoods $V_1(r)$, $V_2(r)$ in $\mathfrak{O}_\mu$ that contain p and q respectively, then there exists always a neighbourhood $V_3(p) \in \mathfrak{O}_\lambda$ that contains q.

Then we have the following

Theorem 2. *The condition* (W) *is equivalent to the condition* (W_1, W_2), *where* (W_1, W_2) *means the combination of* (W_1) *aud* (W_2).

Proof. It is clear that (W) implies (W_1, W_2). We shall prove that (W_1, W_2) implies (W).

(A). On (W_1) we can put $q = r$, so for any index $\mu \in \Lambda$ and any point $r \in R$, there exists the index $\gamma = \gamma(\mu, r) \in \Lambda$, such that if for an element $p \in R$ we can take some neighbourhood $V_1(r)$, in $\mathfrak{O}_\gamma$ that contains p, then there exists a neighbourhood $V_3(p) \in \mathfrak{O}_\lambda$ that contains r. Moreover, if we put $r = p$ and $p = q$, then for any index $\lambda \in \Lambda$ and any point $p \in R$ there exist some index $\mu = \mu(\lambda, p) \in \Lambda$ such that if for an point q in R we can take some neighbourhood $V_1(p)$ in $\mathfrak{O}_\mu$ that contains q, then there exists a neighbourhood $V_3(q) \in \mathfrak{O}_\lambda$ that contains p.

(B). Next, by (W_2), for $\mu \in \Lambda$ which is determined above and for $p \in R$ there exists an index $\gamma = \gamma(\mu, p) \in \Lambda$ such that, if for elements q and r in R we can take some neighbourhood $V_4(r)$, $V_5(r)$ in $\mathfrak{O}_\gamma$ that contain p and q respectively, then there exists a neighbourhood $V_6(p) \in \mathfrak{O}_\mu$ that contains q.

Applying (A) to the fact that $q \in V_6(p) \in \mathfrak{O}_\mu$, we can take a neighbourhood $V_7(q)$ in $\mathfrak{O}_\lambda$ that contains p. Therefore, if we take, for any index $\lambda \in \Lambda$ and for a point $p \in R$, an index $\gamma = \gamma(\mu(\lambda), p) = \gamma(\lambda, p)$ in Λ, then we have our theorem.

Remark. The examples that any one of the (W_1), (W_2) and (W_3) is not equivalent to (W) were shown by Prof. K. Kunugui.[5]

§ 2. An arbitrary subset Λ_1 of a pseudo-ordered set Λ is called *a cofinal subset of* Λ if, for any element $\lambda \in \Lambda$, there exist some indices μ and ν in Λ_1 and $\nu \leqq \lambda \leqq \mu$. Any subset Λ_1 of Λ is called *a rest in* Λ if Λ_1 is a cofinal set in Λ and if for any index $\lambda_1 \in \Lambda_1$ the element $\lambda \in \Lambda$ such that $\lambda \geqq \lambda_1$ is contained in Λ_1.

Let Λ_1 *be a rest in* Λ. *Then* Λ_1 *is a residual*[6] *set of* Λ.

A sequence $\{x_\lambda\}(\lambda \in \Lambda_1 \subseteq \Lambda)$ of the elements of R is called *a conditional point sequence* if the index set Λ_1 is cofinal in Λ. We put $\Lambda' = \{\lambda; x_\lambda \in V(p),\ \lambda \in \Lambda_1 \subseteq \Lambda\}$. Then a point sequence $\{x_\lambda\}(\lambda \in \Lambda_1 \subseteq \Lambda)$ *converges to the point* p *of* R if for any neighbourhood $V(p)$ of $p \in R$ Λ' contains a rest in Λ.

A point sequence $\{x_\lambda\}(\lambda \in \Lambda_1 \subseteq \Lambda)$ of R is called *a Cauchy sequence in* R if it satisfies the following conditions:

C_1). Λ_1 is a cofinal subset of Λ.

C_2). Given any index $\lambda \in \Lambda_1$, there exists some rest $\mathfrak{v} = \mathfrak{v}(\lambda)$ in Λ_1 and for every pair of elements λ_1 and λ_2 of $\mathfrak{v}$, $x_{\lambda_1} \in V(x_{\lambda_2}) \in \mathfrak{O}_\lambda$.

C_3). For any index $\lambda \in \Lambda_1 \subseteq \Lambda$, there exists an index $\mu = \mu(\lambda)$ which is the W-index of $x_{\lambda'}$ for every element λ' of $\mathfrak{v}$.

Then obviously the following proposition holds,

In a uniform space R satisfying the conditon (W), a convergent conditional point sequence is a Cauchy sequence.

If $\{x_\lambda\}(\lambda \in \Lambda_1 \subseteq \Lambda)$ and $\{y_\mu\}(\mu \in \Lambda_2 \subseteq \Lambda)$ are Cauchy sequences and satisfy the following condition (C_4) then we say that $\{x_\lambda\}$ *is equivalent to* $\{y_\mu\}$ *and write* $\{x_\lambda\} \sim \{y_\mu\}$.

C_4). For any $\lambda \in \Lambda_1$ and $\mu \in \Lambda_2$ there exist a pair of rests $\mathfrak{v}_1 = \mathfrak{v}_1(\lambda, \mu)$ and $\mathfrak{v}_2 = \mathfrak{v}_2(\lambda, \mu)$ in Λ_1 and Λ_2 respectively such that for any pair of elements (λ_1, μ_2), $\lambda_1 \in \mathfrak{v}_1$, $\mu_2 \in \mathfrak{v}_2$, there exist some neighbourhoods $V(y_{\mu_2})$ of y_{μ_2} and $V(x_{\lambda_1})$ of x_{λ_1}, and $x_{\lambda_1} \in V(y_{\mu_2}) \in \mathfrak{O}_\mu$ and $y_{\mu_2} \in V'(x_{\lambda_1}) \in \mathfrak{O}_\lambda$ hold.

Theorem 3. *In the family of Cauchy sequences the equivolence law* *with respect to the equivalence relation* $\sim$ *holds. That is, if* $\{x_\lambda\}(\lambda \in \Lambda_1 \subseteq \Lambda)$, $\{y_\mu\}(\mu \in \Lambda_2 \subseteq \Lambda)$ *and* $\{z_\nu\}(\nu \in \Lambda_3 \subseteq \Lambda)$ *are Cauchy sequences, then*

(1) $\{x_\lambda\} \sim \{x_\lambda\}$

(2) if $\{x_\lambda\} \sim \{y_\mu\}$ *then* $\{y_\mu\} \sim \{x_\lambda\}$

(3) if $\{x_\lambda\} \sim \{y_\mu\}$ *and* $\{y_\mu\} \sim \{z_\nu\}$ *then* $\{x_\lambda\} \sim \{z_\nu\}$.

Proof. (1) and (2) are trivial, so we shall prove (3).

(A). For any index $\lambda \in \Lambda$ there exist an index μ_1 in Λ_2 such that $\mu_1 \leqq \lambda$, since Λ_2 is the cofinal set of Λ. But $\{y_\mu\}(\mu \in \Lambda_2 \subseteq \Lambda)$ is a Cauchy sequence, so for the index $\mu = \mu(\mu_1) \in \Lambda_2$ and the rest $\mathfrak{d}_2 = \mathfrak{d}_2(\mu_1)$ in Λ_2, μ is the W-index for every index $\mu' \in \mathfrak{d}_2$, namely, if for a pair of points p and q we can take, for every $\mu' \in \mathfrak{d}_2$, some neighbourhoods $V_1(y_{\mu'})$ and $V_2(y_{\mu'})$ of $y_{\mu'}$ in $\mathfrak{O}_\mu$ that contain p and q respectively then there exists a neighbourhood $V(p)$ of p in $\mathfrak{O}_{\mu'}$ that contains q. Now, according to $\mu_1 \leqq \lambda$ for this $V(p)$, there exists a neighbourhood $V'(p)$ of p such that $V(p) \subseteq V'(p)$ and $V'(p) \in \mathfrak{O}_\lambda$. Therefore, given any index $\lambda \in \Lambda$, there exist $\mu = \mu(\mu_1(\lambda)) = \mu(\lambda)$ and the rest $\mathfrak{d}_2 = \mathfrak{d}_2(\mu_1(\lambda)) = \mathfrak{d}(\lambda)$ in Λ_2 such that, if for some elements p and q in R we can take some neighbourhoods $V_1(y_{\mu'})$ and $V_2(y_{\mu'})$ in $\mathfrak{O}_\mu$ that contain p and q respectively, then there exists a neighbourhood $V'(p) \in \mathfrak{O}_\lambda$ that contains q.

From $\{x_\lambda\} \sim \{y_\mu\}(\lambda \in \Lambda_1 \subseteq \Lambda)$, $(\mu \in \Lambda_2 \subseteq \Lambda)$, we see that, for any $\lambda \in \Lambda_1$ and $\mu \in \Lambda_2$ these exist some rests $\mathfrak{d}_1$ and $\mathfrak{d}_2'$ in Λ_1 and Λ_2 respectively such that for any pair of element (λ_1, μ_2), $\lambda_1 \in \mathfrak{d}_1$, $\mu_2 \in \mathfrak{d}_2'$ we can take certain neighbourhoods of y_{μ_2} and x_{λ_1} which satisfy the conditions: $x_{\lambda_1} \in V(y_{\mu_2}) \in \mathfrak{O}_\mu$ and $y_{\mu_2} \in V(x_{\lambda_1}) \in \mathfrak{O}_\lambda$.

Moreover, $\{y_\mu\} \sim \{z_\nu\}$, $(\mu \in \Lambda_2 \subseteq \Lambda)$, $(\nu \in \Lambda_3 \subseteq \Lambda)$, namely, for any indices $\mu \in \Lambda_2$ and $\nu \in \Lambda_3$ there exist some rests $\mathfrak{d}_2$ and $\mathfrak{d}_3$ in Λ_2 and Λ_3 respectively such that for any pair of element (μ_3, ν_1), $\mu_3 \in \mathfrak{d}_2$, $\nu_1 \in \mathfrak{d}_3$ we can take certain neighbourhoods of y_{μ_3} and z_{ν_1} which satisfy the conditions: $y_{\mu_3} \in V(z_{\nu_1}) \in \mathfrak{O}_\nu$ and $z_{\nu_1} \in V(y_{\mu_3}) \in \mathfrak{O}_\mu$.

Now if we put $\mathfrak{d}_2^* = \mathfrak{d}_2 \wedge \mathfrak{d}_2' \wedge \mathfrak{d}$ then $\mathfrak{d}_2^*$ is also a rest in Λ_2. From the above results, we conclude that, if for any indices $\lambda \in \Lambda_1$, $\mu \in \Lambda_2$ and $\nu \in \Lambda_3$ we take $\mathfrak{d}_1$, $\mathfrak{d}_2^*$ and $\mathfrak{d}_3$ which are respectively the rests in Λ_1, Λ_2 and Λ_3, then for each $(\lambda_1, \mu_1^*, \nu_1)$, $\lambda_1 \in \mathfrak{d}_1$, $\mu_1^* \in \mathfrak{d}_2^*$, and $\nu_1 \in \mathfrak{d}_3$, there exist some neighbourhoods $V_1(y_{\mu_1^*})$ of $V_2(y_{\mu_1^*})$ of $y_{\mu_1^*}$ in $\mathfrak{O}_\mu$ such that $x_{\lambda_1} \in V_1(y_{\mu_1^*}) \in \mathfrak{O}_\mu$ and $z_{\nu_1} \in V_2(y_{\mu_1^*}) \in \mathfrak{O}_\mu$.

Therefore, according to (A), there exists some neighbourhood $V(x_{\lambda_1})$ of x_{λ_1} in $\mathfrak{O}_\lambda$ such that $z_{\nu_1} \in V(x_{\lambda_1}) \in \mathfrak{O}_\lambda$.

From the proof of Theorem 2, if $z_{\nu_1} \in V(x_{\lambda_1}) \in \mathfrak{O}_\lambda$, then there exists the neighbourhood $V'(z_{\nu_1})$ of z_{ν_1} such that $x_{\lambda_1} \in V'(z_{\nu_1}) \in \mathfrak{O}_\nu$.

Thus $\{x_\lambda\} \sim \{z_\nu\}$, and the theorem are proved.

According to Theorem 3 the Cauchy sequence can be classified into equivalent classes, and if we consider one class as a point, we can get a new space, which will be denoted by S.

If we fix an arbitrary point p in R and put $p \equiv x_\lambda$ for all λ in a cofinalset Λ_1 of Λ, then $\{x_\lambda\}$ is obviously a Cauchy sequence, therefore, S contains R^* which is isomorphic to R. We introduce a topology in S, defining for any element $P=[\{x_\lambda\}(\lambda \in \Lambda_1 \subseteq \Lambda)] \in S$ the neighbourhood $V_\alpha(P)$ $(\alpha \in \Lambda)$ of P as follows: $V(P) \ni Q = [\{y_\mu\}(\mu \in \Lambda_2 \subseteq \Lambda)]$ if and only if $\mu_0 = \mu_0(\alpha)$ is contained in Λ_2 and the rests $\mathfrak{d}_1 = \mathfrak{d}_1(\alpha)$ and $\mathfrak{d}_2 = \mathfrak{d}_2(\alpha)$ which are respectively the rests in Λ_1 and Λ_2 satisfy the following condition: there exists an index μ_1 in $\mathfrak{d}_2$ such that $z \in V(y_{\mu_1}) \in \mathfrak{O}_\mu$ implies $z \in V(x_{\lambda_1}) \in \mathfrak{O}_\lambda$ for all $\lambda_1 \in \mathfrak{d}_1$.

Now, we can introduce the uniformity with $\{\mathfrak{O}_\lambda(\lambda \in \Lambda)\}$, where $\mathfrak{O}_\lambda$ is defined as follows:

$$\mathfrak{O}_\lambda = \{V_{\lambda'}(P);\ \text{for all } P \in S,\ V_{\lambda'}(P) \subseteq V_\lambda(P)\}.$$

Then we get the following

Theorem 4. 1) *S is a neighbourhood space with respect to this topology,*

2) *S is a uniform space satisfying (W) with respect to $\{\mathfrak{O}_\lambda(\lambda \in \Lambda)\}$,*

3) *S is a complete space,*

4) *the transformation from R to R^* is the nuiformly bi-continuous transformation,*

5) $\overline{R^*} = S$.

Proof. The proofs of 2), 3), 4) and 5) are similar to that of the theorem for the property of the complete space in 4), so we shall prove 1).

Let $P = [\{x_\lambda\}(\lambda \in \Lambda_1 \subseteq \Lambda)]$ be any point S. Then Λ_1 is a cofinal set of Λ, so for any index $\alpha \in \Lambda$ there exists an index λ such that $\lambda \leqq \alpha$, $\lambda \in \Lambda_1$. Moreover, $\{x_\lambda\}(\lambda \in \Lambda_1 \subseteq \Lambda)$ is a Cauchy sequence in R, namely, for this index $\lambda \in \Lambda_1$ there exists a rest $\mathfrak{d} = \mathfrak{d}(\lambda)$ and an index $\mu = \mu(\lambda)$ in Λ_1 satisfying the following condition: the index μ is the W-index of x_{λ_1} for every element λ_1 of $\mathfrak{d}$ and for every pair of elements λ_1 and λ_2 of $\mathfrak{d}, x_{\lambda_2} \in V(x_{\lambda_1}) \in \mathfrak{O}_\mu$. Therefore, $z \in V(x_{\lambda_1}) \in \mathfrak{O}_\mu$ implies $z \in V(x_{\lambda_2}) \in \mathfrak{O}_\lambda$. But $\alpha \geqq \lambda$, then there exists, for $V(x_{\lambda_2}) \in \mathfrak{O}_\lambda$, some neighbourhood $V'(x_{\lambda_2})$ of x_{λ_2} in $\mathfrak{O}_\alpha$ such that $V(x_{\lambda_2}) \subseteq V'(x_{\lambda_2})$, hence $z \in V(x_{\lambda_1}) \in \mathfrak{O}_\mu$ implies $z \in V'(x_{\lambda_2}) \in \mathfrak{O}_\alpha$. Therefore, for any index $\alpha \in \Lambda$, P is contained in $V_\alpha(P)$. Next, let $P = [\{x_\lambda\}, (\lambda \in \Lambda_1 \subseteq \Lambda)]$ be any point of S and let $V_\alpha(P)$ and $V_\beta(P)(\alpha, \beta \in \Lambda)$ be any neighbourhoods of P. If we

assume that for all index $\gamma \in \Lambda$, $V_\gamma(P) \not\subseteq V_\alpha(P) \cdot V_\beta(P)$, then there exists, for arbitrary fixed index $\gamma \in \Lambda$, at least one point $Q = [\{y_\mu\}, (\mu \in \Lambda_2 \subseteq \Lambda)]$ such that $Q \in V_\gamma(P)$ and $Q \notin V_\alpha(P) \cdot V_\beta(P)$, where we can suppose that $Q \in V_\gamma(P)$ and $Q \notin V_\alpha(P)$. However, if we take $\gamma \in \Lambda$ such that $\gamma \leqq \alpha$, then $Q \in V_\gamma(P)$ implies
$Q \in V_\alpha(P)$ q. e. d.

(Received March 17, 1953)

Reference

1) A. Weil: Sur les espaces à structure uniforme et sur la topologie générale, Actualités Sci. Ind. 551, 1937.

2) L. W. Cohen: On imbedding a space in a complete space, Duke Math. J. 5 (1937), 174-183.

3) K. Morita: On the simple extension of a space with respect to a uniformity, I, Proc. Japan Acad. 27 (1951).

4) The lecture on the uniform space by Prof. K. Kunugui in 1951.

5) Loc. cit., 4).

6) A subset A_1 of a pseudo-ordered set A is called a ***residual subset of*** A if, for any element $\alpha \in A$, there exists some element $\beta = \beta(\alpha)$ in A_1 such that $\kappa \geqq \beta$ and $\kappa \in A$ implies $\kappa \in A_1$.