A Characterization of Quasi-Frobenius Rings

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In this note we shall consider the problem: in what ring $A$ can every homomorphism between two left ideals be extended to a homomorphism of $A$? ("Homomorphism" means "operator homomorphism"). We shall call this condition as Shoda's condition.\(^1\) When $A$ is a ring with a unit element, Shoda's condition is equivalent to the next one:

\[(a): \text{every homomorphism between two left ideals is given by the right multiplication of an element of } A.\]

The main purpose of this note is to show that if $A$ is a ring with a unit element satisfying the minimum condition for left and right ideals, then $A$ satisfies Shoda's condition if and only if $A$ is a quasi-Frobenius ring.

T. Nakayama characterized quasi-Frobenius rings as the rings in which the duality relations $l(r(I)) = I$ and $r(l(x)) = x$ hold for every left ideal $I$ and right ideal $x$.\(^2\) Our result gives another characterization of quasi-Frobenius rings.

$A$ denotes always a ring with the minimum condition for left and right ideals. Let $N$ be the radical of $A$ and $\bar{A} = A/N = \bar{A}_1 + \cdots + \bar{A}_n$ be the direct decomposition of $\bar{A}$ into simple two-sided ideals. Then, as is well known, we have two direct decompositions of $A$:

\[A = \sum_{\kappa=1}^{n} \sum_{i=1}^{f(\kappa)} Ae_{\kappa, i} + l(E) = \sum_{\kappa=1}^{n} \sum_{i=1}^{f(\kappa)} e_{\kappa, i} A + r(E) \]  

(1)

where $E = \sum_{\kappa=1}^{n} \sum_{i=1}^{f(\kappa)} e_{\kappa, i}$, $e_{\kappa, i} (\kappa = 1, 2, \ldots, n)$, $i = 1, 2, \ldots, f(\kappa)$ are mutually orthogonal primitive idempotents, $Ae_{\kappa, i} = Ae_{\kappa, 1} = Ae_{\kappa}$ for $i = 1, \ldots, f(\kappa)$, $Ae_{\kappa, i} = Ae_{\kappa, j}$ if $\kappa = \lambda$ and the same is true for $e_{\kappa, i} A$, and $l(*)$ ($r(*)$) is the left annihilator (right annihilator) of $\kappa$. Moreover we use matrix units $e_{\kappa, i, j} (\kappa = 1, \ldots, n; i, j = 1, \ldots, f(\kappa))$.

$c_{\kappa, 1, 1} = e_{\kappa, 1} = e_{\kappa}, \ e_{\kappa, 1, i} = e_{\kappa, i}$ and $c_{\kappa, i, j} c_{\lambda, j, i} = \delta_{\kappa, \lambda} \delta_{i, j} c_{\kappa, i, i}$.

We start with the following preliminary lemmas.

\(1\) This problem was suggested by Prof. K. Shoda. Cf. K. Shoda [4].

\(2\) See T. Nakayama [1], [2].
Lemma 1. If $A$ satisfies (a) for simple left ideals, then $A$ has a right unit element.

Proof. To prove this, we show $l(E) = 0$ in (1). If $l(E) \neq 0$, then it contains a simple left subideal $l \neq 0$. The identity automorphism of $I$ is given, from (a), by the right multiplication of an element $a$. $a = Ea + (a - Ea)$, where $a - Ea \in r(E)$. Since $l(E)$ and $r(E)$ are contained in $N$, $l = l(a - Ea) \subset N^2$. Since $l \subset N^2$, $l = l(a - Ea) \subset N^2$. Thus we have finally $l = 0$, which is a contradiction.

Lemma 2. If $A$ has a left unit element and satisfies (a) for simple left ideals, then $A$ has a unit element and there exists a permutation $\pi$ of $(1, 2, \cdots, n)$ such that the largest completely reducible left subideal of $Ae_\kappa i$ is a direct sum of simple left subideals which are isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$.

Proof. From Lemma 1, $A$ has a unit element. Hence $r(N) = \sum E_\kappa r(N) = \sum E_\kappa E_\lambda \vee N$ where $E_\kappa = \sum E_\lambda \kappa$. $E_\kappa r(N)$ is a two-sided ideal for each $\kappa$, since $AE_\kappa r(N) = (\sum E_\lambda AE_\lambda \vee N)E_\kappa r(N) = E_\kappa r(N)$. If $E_\kappa r(N) \neq 0$ and $a \neq 0$ is an arbitrary element of $E_\kappa r(N)$, then there exists an $e_{\kappa i}$ such that $e_{\kappa i} a \neq 0$. $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$ is obvious. Since $E_\kappa r(N)$ is a direct sum of simple left ideals which are isomorphic to $Ae_\kappa i$, each component has the form $Ae_\kappa a_i$, and this shows that $E_\kappa r(N) = AaA$ and $E_\kappa r(N)$ is a simple two-sided ideal. Hence $E_\kappa r(N)$ is a non-zero two-sided ideal for each $\kappa$, since $AE_\kappa r(N) = \sum E_\lambda AE_\lambda \vee N = E_\kappa r(N)$. Since $r(N)E_\kappa$ is the largest completely reducible left subideal of $AE_\kappa r(N)E_\kappa \neq 0$. Since $r(N)E_\kappa A = r(N)E_\kappa (\sum E_\lambda AE_\lambda \vee N) = r(N)E_\kappa$. Hence $r(N)E_\kappa$ is a non-zero two-sided ideal for each $\kappa$. Then, from $r(N) = \sum E_\kappa r(N) = \sum E_\kappa A = \sum E_\kappa (\sum E_\lambda AE_\lambda \vee N)$, it follows that $r(N)E_\kappa = E_\pi(\kappa) r(N)$ is a non-zero simple two-sided ideal for each $\kappa$, where $\pi$ is a permutation of $(1, 2, \cdots, n)$. This shows that the largest completely reducible left subideal of $Ae_\kappa i$ is a direct sum of simple subideals which are isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$.

In the case of algebras, we have by Lemma 2,

Proposition 1. Let $A$ be an algebra with a finite rank over a field $F$. If $A$ has a left unit element and satisfies (a) for simple left ideals, then $A$ is a quasi-Frobenius algebra.

Proof. To prove this, we show that $r(N)e_{\kappa i}$ is simple for each $\kappa$.

If $r(N)e_{\kappa i}$ is not simple, then, by Lemma 2, $r(N)e_{\kappa i} = \sum m_j$,
where \( s > 1 \) and \( m_j \cong A e_{e(\kappa)} / N e_{e(\kappa)} \). Since \( m_1 \cong A e_{e(\kappa)} / N e_{e(\kappa)} \), the endomorphismring of \( m_1 \), is isomorphic to \( e_{e(\kappa)} A e_{e(\kappa)} / e_{e(\kappa)} N e_{e(\kappa)} \). On the other hand, every endomorphism of \( m_1 \) is given by the right multiplication of an element of \( e_{\alpha, i} A e_{\alpha, i} \). Since \( r(N) \subseteq l(N) \), elements of \( e_{\alpha, i} N e_{\alpha, i} \) induce zero-endomorphism and those elements of \( e_{\alpha, i} A e_{\alpha, i} \) which are not in \( e_{\alpha, i} N e_{\alpha, i} \) induce isomorphisms. Hence we have a natural isomorphism of \( e_{e(\kappa)} A e_{e(\kappa)} / e_{e(\kappa)} N e_{e(\kappa)} \) into \( e_{\alpha, i} A e_{\alpha, i} / e_{\alpha, i} N e_{\alpha, i} \).

Since \( s > 1 \), this isomorphism is not an onto isomorphism, and \( \text{End}(N) = e_{\alpha, i} A e_{\alpha, i} / e_{\alpha, i} N e_{\alpha, i} \). Similarly \( (e_{e(\kappa)} A e_{e(\kappa)} / e_{e(\kappa)} N e_{e(\kappa)} : F) \cong (e_{e(\kappa)} A e_{e(\kappa)} / e_{e(\kappa)} N e_{e(\kappa)} : F) \cong (e_{e(\kappa)} A e_{e(\kappa)} / e_{e(\kappa)} N e_{e(\kappa)} : F) \cong (e_{e(\kappa)} A e_{e(\kappa)} / e_{e(\kappa)} N e_{e(\kappa)} : F) \). This is a contradiction. Hence \( r(N) \) is simple. Then, by Nakayama’s theorem,\(^3\) we have our result.

**Proposition 2.** Let \( A \) be a ring with a left unit element. If \( A \) satisfies \( (a) \) for every left ideal, then \( A \) is a quasi-Frobenius ring.

**Proof.** By Lemma 2, \( r(N) = \sum_{j=1}^{s} m_j \) and \( m_j \cong A e_{e(\kappa)} / N e_{e(\kappa)} \). Hence \( m_j = A e_{e(\kappa)} a_j \) for a suitable element \( a_j \) in \( m_j \). Assume \( s > 1 \), then the correspondences \( e_{e(\kappa)} a_1 \rightarrow e_{e(\kappa)} a_2 \) and \( e_{e(\kappa)} a_2 \rightarrow e_{e(\kappa)} a_1 \) define an automorphism of \( m_1 + m_2 \). Then, by \( (a) \), there is an element \( c \) of \( e_{\kappa} A e_{\kappa} \) such that \( e_{e(\kappa)} a_1 c = e_{e(\kappa)} a_2 \) and \( e_{e(\kappa)} a_2 c = e_{e(\kappa)} a_1 \). Hence \( e_{e(\kappa)} a_1 a_2^2 = e_{e(\kappa)} a_1 \) and \( e_{e(\kappa)} a_1 (a_2^2 - e_{\kappa}) = 0 \). \( e_2 - e_{\kappa} \) is in \( e_{e(\kappa)} N e_{e(\kappa)} \). For, otherwise, it is a unit of \( e_{e(\kappa)} A e_{\kappa} \) and consequently \( e_{e(\kappa)} a_1 = 0 \). Hence \( c = \pm e_{\kappa} + n \), where \( n \) belongs to \( e_{e(\kappa)} N e_{e(\kappa)} \). Since \( r(N) \cong l(N) \), \( e_{e(\kappa)} a_1 \pm e_{\kappa} + n = \pm e_{e(\kappa)} a_1 \). This is a contradiction. Hence \( r(N) \) is simple. Now if \( l(N) e_{e(\lambda)} \cong r(N) e_{e(\lambda)} \), than \( l(N) e_{e(\lambda)} \) contains a left subideal \( I \) such that \( l(r(N) e_{e(\lambda)} \) is irreducible. We suppose \( l(r(N) e_{e(\lambda)} \cong A e_{e(\lambda)} / N e_{e(\lambda)} \). Then it follows easily that \( l(N) e_{e(\lambda)} \cong A e_{e(\lambda)} / N e_{e(\lambda)} \). Hence \( l(N) e_{e(\lambda)} \cong A e_{e(\lambda)} / N e_{e(\lambda)} \). Then \( l(N) e_{e(\lambda)} \cong A e_{e(\lambda)} / N e_{e(\lambda)} \). Thus we have contradictions. Hence \( l(N) e_{e(\lambda)} \). Since \( m_1 \cong A e_{e(\kappa)} / N e_{e(\kappa)} \), the largest completriy reducible right subideal of \( e_{e(\kappa)} A \) is a direct sum of simple right subideals which are isomorphic to \( e_{e(\kappa)} A / e_{e(\kappa)} N \). Since \( Me_{e(\kappa)} \) is simple and is

\(^3\) See T. Nakayama [3].
isomorphic to $Ae_{e(x)}/Ne_{e(x)}$, $Me_x = Ae_{e(x)} me_x$ for a suitable element $e_{e(x)} me_x$ in $Me_x$. Let $x$ be an arbitrary element in $e_{e(x)} Ae_{e(x)}$ but not in $e_{e(x)} Ne_{e(x)}$. Then the correspondence $e_{e(x)} me_x \to xe_{e(x)} me_x$ defines an automorphism of $Me_x$. For if $x'e_{e(x)} me_x = 0$, then $x' \in A(1 - e_{e(x)}) \setminus N$ and $x'xe_{e(x)} me_x \in (A(1 - e_{e(x)}) \setminus N)e_{e(x)} M = 0$. By (a), this automorphism is given by the right multiplication of an element of $e_{e(x)} Ae_{e(x)}$. Furthermore $e_{e(x)} Ne_{e(x)} me_x = 0$ is obvious. Hence $e_{e(x)} Ae_{e(x)} me_x \subseteq e_{e(x)} me_x Ae_{e(x)}$. On the other hand, since $e_{e(x)} me_x A$ is a simple right subideal of $e_{e(x)} M$ and $E_{e(x)} M$ is a simple two-sided ideal, $e_{e(x)} M$ is a direct sum of simple right subideals of the form $\xi e_{e(x)} me_x A$, where $\xi$ is a suitable unit of $e_{e(x)} Ae_{e(x)}$. But, as was shown, $e_{e(x)} Ae_{e(x)} me_x \subseteq e_{e(x)} me_x A$. Thus we see that $e_{e(x)} M = e_{e(x)} me_x A$ is a unique simple left subideal of $e_{e(x)} A$. This completes our proof.

Remark. From the assumption (a) for simple left ideals, we can not conclude that $A$ has a left unit element. For example, let $F$ be a field and $A = Fe + Fu$, where $e^2 = e$, $ue = u$, $eu = 0$, $u^2 = 0$. This algebra over $F$ has no left unit element, but it satisfies (a).

If $A$ is a ring and not an algebra, then we can not conclude that $A$ is a quasi-Frobenius ring, from the assumption (a) for simple left ideals and the existence of a left unit element. For example, let $F(x)$ be a rational function field over a field $F$ and $A = F(x) + uF(x)$, where $u^2 = 0$, $ux = xu$. Then this is not a quasi-Frobenius ring, but it has a unit element and (a) is valid for simple left ideals.

Proposition 3. If $A$ is a ring in which (a) is valid for simple left ideals and the same is true for simple right ideals, then $A$ is a quasi-Frobenius ring.

Proof. By Lemma 1, $A$ has a unit element. $r(N) = l(N) = M$, $Me_x = \sum_{j=1}^s m_j$ and $e_{e(x)} M = \sum_{k=1}^t n_k$, by Lemma 2. As was shown in the proof of Theorem 2, $e_{e(x)} Ae_{e(x)} me_x \subseteq e_{e(x)} me_x Ae_{e(x)}$, if we write $m_1 = Ae_{e(x)} me_x$. Similarly $e_{e(x)} Ae_{e(x)} me_x \subseteq e_{e(x)} me_x Ae_{e(x)}$, since $e_{e(x)} me_x A$ is a simple right subideal of $e_{e(x)} M$. Hence $e_{e(x)} Ae_{e(x)} me_x = e_{e(x)} me_x Ae_{e(x)}$. On the other hand, $m_1$ has the form $m_1 \xi = Ae_{e(x)} me_x \xi$, where $\xi$ is an element of $e_{e(x)} Ae_{e(x)}$. Hence $s = 1$ and similarly $r = 1$. Thus $A$ is a quasi-Frobenius ring.

Lemma 3. Let $A$ be a quasi-Frobenius ring and let $I = I_1 \cup I_2$ be a left ideal homorphic to a left ideal $I'$ by a homomorphism $\theta$, where $I_1$ and $I_2$ are two left subideals of $I$. If the homomorphisms from $I_1$ and $I_2$ into $I'$ induced by $\theta$ are given by the right multiplications of elements $a_1$ and $a_2$
respectively, then there is an element \( a \) such that \( \theta \) is given by the right multiplication of \( a \).

Proof. Of course \( \Gamma = IJ \). Then elements \( a_\lambda \) and \( a_2 \) define the same homomorphism for \( Ix f \). Hence \( a_\lambda - a_2 = r_2 - r_1 \) for suitable \( r_1, r_2 \in r (I) \). We write \( a_1 + r_1 = a_2 + r_2 \) as \( a \). Then \( a \) defines \( \theta \) for \( I \). For if \( l_i \) is an element of \( I_i (i = 1, 2) \), then \( l_i a = l_i (a_i + r_i) = l_i a_i = l_i^0 \).

Theorem 1. Let \( A \) be a ring with a unit element. Then \( A \) satisfies Shoda's condition if and only if \( A \) is a quasi-Frobenius ring.

Proof. The "only if" part follows from Proposition 2.

We shall prove the "if" part. If a left ideal \( I' \) is a homomorphic image of a principal left ideal \( l Aa \), then \( I' \) is also a principal ideal. We denote this homomorphism by \( \theta \), and show that \( \theta \) is given by the right multiplication of an element. Since \( \theta \) is a homomorphism, \( l (a) = l (aA) \subseteq l (a^A) = l (a^A) \). Since \( A \) is a quasi-Frobenius ring, \( r (l (aA)) = aA \supseteq r (l (a^A)) = a^A \). Hence there is an element \( c \) such that \( a^c = ac \).

Since every left ideal \( I \) has a finite basis, we can write \( I = \bigvee_{i=1}^n Aa_i \). Then, by Lemma 3, every homomorphism between two left ideals is given by the right multiplication of a suitable element. This completes our proof.

Theorem 2. Let \( A \) be a quasi-Frobenius ring. Then for every isomorphism \( \theta \) between two left ideals we can choose a suitable unit which defines \( \theta \), that is, every isomorphism between two left ideals can be extended to an isomorphism of \( A \).

Proof. Let \( \theta \) be an isomorphism between \( I \) and \( I' \). Then, by Theorem 1, there is an element \( a_\theta \) which defines \( \theta \), that is, \( l a_\theta = l' \). Then \( l a_\theta (l') = l r (l') = 0 \). This shows that \( a_\theta r (l') \subseteq r (l) \).

Case I. \( a_\theta r (l') = r (l) \).

If \( r \) is an arbitrary element of \( r (l) \), then there is an element \( r' \) in \( r (l') \) such that \( a_\theta r' = r \). Let \( \theta^{-1} \) be the inverse isomorphism of \( \theta \) and let \( b_{\theta^{-1}} \) be the element which defines \( \theta^{-1} \). It is easy to see that \( 1 - a_\theta b_{\theta^{-1}} = r_0 \in r (l) \). Then \( a_\theta (b_{\theta^{-1}}+r_0) = a_\theta b_{\theta^{-1}} + r_0 = 1 \). Hence \( a_\theta \) is a unit.\(^4\)

Case II. \( a_\theta r (l') \supseteq r (l) \).

In this case, \( l = l (a_\theta r (l')) \supseteq l (r (l)) = l, \) since \( A \) is a quasi-Frobenius

\(^4\) Since \( A \) satisfies the minimum condition for left and right ideals, if \( ab = 1 \), then \( ba = 1 \).
ring. It follows, from \( l a^\alpha (l') = 0 \), that \( l a^\alpha \subseteq l' \). But \( l a^\alpha \supseteq l a^\alpha = l' \). Hence \( l a^\alpha = l' \). Let \( I \) be an element of \( I \) and \( l a^\alpha = l' \), then \( l' \) is in \( V \) and there is an element \( l \) of \( I \) such that \( l a^\alpha = l' \). Hence \((l - l)a^\alpha = 0 \). Since no element of \( I \) is annihilated by \( a^\alpha \), \( I \) is the direct sum of \( I \) and \( I_0 \) which is annihilated by \( a^\alpha \). Let \( \beta \) be an element of \( I \) and \( l a^\alpha - V \), then \( l' \) is in \( I \) and there is an element \( I \) of \( I \) such that \( l a^\alpha = l' \). Hence \((I - I)a^\alpha = 0 \). Since no element of \( I \) is annihilated by \( a^\alpha \), \( I \) is the direct sum of \( I \) and \( I_0 \) which is annihilated by \( a^\alpha \). Let \( A \) be a simple left subideal of \( I_0 \). We write \( A = a^\alpha / N \). Since \( r (I)/r (I*) \epsilon \ast \epsilon \ast \), it follows evidently that \( r (I) = r (I*) \) for a suitable element \( r \) of \( r (I) \). Since \( r (I) = r (I*) \), the homomorphism defined by \( a^\alpha + r \beta b \), for an arbitrary \( b \) of \( A \), coincides with \( \beta \) in \( I \). \( I^\beta (a^\alpha + r \beta b) \) is homomorphic to \( I^\beta \) and contains \( I (a^\alpha + r \beta b) = l' \). Now if we take a suitable \( b \), then \( I (a^\alpha + r \beta b) \) is actually different from \( l' \). For otherwise, \( A = a^\alpha / N \) is homomorphic to \( I^\beta \) and contains \( I (a^\alpha + r \beta b) = l' \) for every \( b \) of \( A \). Hence \( A = a^\alpha / N \). Since \( A = a^\alpha / N \) is a simple two-sided ideal, \( ME_\epsilon = A = a^\alpha / N \) and \( I^\beta b^{-1} = I \supset ME_\epsilon \). On the other hand \( ME_\epsilon = E_\epsilon (A) = I^\beta b^{-1} \supset ME_\epsilon \). Thus \( I \) contains \( A = a^\alpha \). But this contradicts \( I \supset I_0 = 0 \). Thus we can take an element \( b \) such that \( I^\beta (a^\alpha + r \beta b) \supset I' \). Obviously \( I^\beta (a^\alpha + r \beta b) \supset I^\beta \). We write the isomorphism between \( I \) and \( I^\beta \) defined by the right multiplication of \( a^\alpha + r \beta b \), by \( \beta \). Then \( \beta \) coincides with \( \beta \) in \( I \), as was shown.

Since our assertion is true for \( A \), suppose now that our assertion is true for every left ideal \( L \) for which \( A/L \) has a shorter composition length than that of \( A/I \). Then we can choose a unit \( a^\alpha \) for \( \Theta \). \( a^\alpha \) defines \( \Theta \) for \( I^\beta \), hence \( a^\alpha \) defines \( \beta \) for \( I \). This completes our proof.

The following lemma is trivial.\(^5\)

**Lemma 4.** Let \( A \) be a ring with a unit element. If every residue class ring of \( A \) satisfies Shoda's condition, then \( A \) is a uni-serial ring, and conversely.

**Theorem 3.**\(^7\) Let \( A \) be such a ring with a unit element that if \( I/m \sim I'/m \) for any two left ideals \( I, I' \) with their common left subideal \( m \), then for every homomorphism \( \Theta \) from \( I/m \) onto \( I'/m \) there is such a homomorphism \( \Theta \) from \( I \) onto \( I' \) that is given by the right multiplication of an element of \( A \) and that coincides with \( \Theta \) in \( I/m \). Then \( A \) is a direct sum of a semi-simple ring and completely primary uni-serial rings, and conversely.

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7) Cf. K. Shoda [4].
Proof. It is clear that every residue class ring satisfies Shoda’s condition. Hence $A$ is a uni-serial ring. Since the above assumption holds for primary components of $A$, we prove our assertion for a primary uni-serial ring $A_1$ satisfying the above assumption. If $A_1$ is neither a simple ring nor a completely primary uni-serial ring, then $A_1$ is a total matrix ring of degree $n > 1$ over a completely primary uni-serial ring $D$. The radical $N_D$ of $D$ is a principal ideal: $N_D = D\pi$. Then the principal ideal $A\pi = \pi A$ is the radical $N$ of $A$. Let $N^{p-1} = 0$ and $N^p = 0$. Then $N^{p-1}e_1 = A\pi^{p-1}e_1 = Ae_1\pi^{p-1}$ and $N^{p-1}e_2 = A\pi^{p-1}e_2 = Ae_2\pi^{p-1}$ are the unique simple left subideals of $Ae_1$ and $Ae_2$, respectively. $Ae_1\pi^{p-1} = Ae_2\pi^{p-1}$ by the correspondence $e_1\pi^{p-1} \leftrightarrow c_{12}\pi^{p-1}$. Then $N^{p-1}(e_1 + c_{12}) = A(e_1 + c_{12})\pi^{p-1}$ is a simple left ideal and contained in $A(e_1 + c_{12})$. Since $A(e_1 + c_{12})$ is an indecomposable left ideal, $N^{p-2}(e_1 + c_{12})$ contains $N^{p-1}(e_1 + c_{12})$ as its unique simple left subideal. It is clear that $N^{p-2}(e_1 + c_{12})/N^{p-1}(e_1 + c_{12}) \approx N^{p-1}e_1 + N^{p-1}e_2 / N^{p-1}(e_1 + c_{12})$. But, as was shown, $N^{p-2}(e_1 + c_{12})$ is not isomorphic to $N^{p-1}e_1 + N^{p-1}e_2$. This contradicts our assumption. Thus if $A_1$ is a primary uni-serial ring satisfying our assumption, then $A_1$ is either a simple ring or a completely primary uni-serial ring. The converse is trivial.

Remark. Let $A$ be such a ring with a unit element that if $I/m \sim I'/m$ for any two left ideal $I$, $I'$ with their common left subideal $m$, then for every homomorphism $\theta$ from $I/m$ onto $I'/m$ and every endomorphism $\varphi$ of $m$ there is a homomorphism $\Theta$ from $I$ onto $I'$ which is given by the right multiplication of an element of $A$ and coincides with $\theta$ in $I/m$ and with $\varphi$ in $m$. Then $A$ is a semi-simple ring and conversely.

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References
