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A Characterization of Quasi-Frobenius Rings

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In this note we shall consider the problem: in what ring A can every homomorphism between two left ideals be extended to a homomorphism of A? ("Homomorphism" means "operator homomorphism"). We shall call this condition as *Shoda's condition.*[1] When A is a ring with a unit element, Shoda's condition is equivalent to the next one:

(a): *every homomorphism between two left ideals is given by the right multiplication of an element of A.*

The main purpose of this note is to show that if A is a ring with a unit element satisfying the minimum condition for left and right ideals, then A satisfies Shoda's condition if and only if A is a quasi-Frobenius ring.

T. Nakayama characterized quasi-Frobenius rings as the rings in which the duality relations $l(r(\mathfrak{l}))=\mathfrak{l}$ and $r(l(\mathfrak{r}))=\mathfrak{r}$ hold for every left ideal $\mathfrak{l}$ and right ideal $\mathfrak{r}$.[2] Our result gives another characterization of quasi-Frobenius rings.

A denotes always a ring with the minimum condition for left and right ideals. Let N be the radical of A and $\bar{A}=A/N=\bar{A}_1+\cdots+\bar{A}_n$ be the direct decomposition of $\bar{A}$ into simple two-sided ideals. Then, as is well known, we have two direct decompositions of A:

$$A=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}Ae_{\kappa,i}+l(E)=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}A+r(E) \qquad (1)$$

where $E=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}$, $e_{\kappa,i}\,(\kappa=1,2,\cdots,n;\ i=1,2,\cdots f(\kappa))$ are mutually orthogonal primitive idempotents, $Ae_{\kappa,i}\cong Ae_{\kappa,1}=Ae_\kappa$ for $i=1,\cdots,f(\kappa)$, $Ae_{\kappa,i}\not\cong Ae_{\lambda,j}$ if $\kappa\neq\lambda$ and the same is true for $e_{\kappa,i}A$, and $l(*)$ ($r(*)$) is the left annihilator (right annihilator) of $*$. Moreover we use matric units $c_{\kappa,i,j}\,(\kappa=1,\cdots,n;\ i,j=1,\cdots,f(\kappa))$, $c_{\kappa,1,1}=e_{\kappa,1}=e_\kappa$, $c_{\kappa,i,i}=e_{\kappa,i}$ and $c_{\kappa,i,j}c_{\lambda,k,l}=\delta_{\kappa,\lambda}\delta_{j,k}c_{\kappa,i,l}$.

We start with the following preliminary lemmas.

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1) This problem was suggested by Prof. K. Shoda. Cf. K. Shoda [4].

2) See T. Nakayama [1], [2].

Lemma 1. *If A satisfies (a) for simple left ideals, then A has a right unit element.*

Proof. To prove this, we show $l(E)=0$ in (1). If $l(E)\neq 0$, then it contains a simple left subideal $\mathfrak{l}\neq 0$. The identity automorphism of $\mathfrak{l}$ is given, from (a), by the right multiplication of an element a. $a=Ea+(a-Ea)$, where $a-Ea\in r(E)$. Since $l(E)$ and $r(E)$ are contained in N, $\mathfrak{l}=\mathfrak{l}a=\mathfrak{l}(a-Ea)\subset N^2$. Since $\mathfrak{l}\subset N^2$, $\mathfrak{l}=\mathfrak{l}(a-Ea)\subset N^3$. Thus we have finally $\mathfrak{l}=0$, which is a contradiction.

Lemma 2. *If A has a left unit element and satisfies (a) for simple left ideals, then A has a unit element and there exists a permutation π of $(1, 2, \cdots, n)$ such that the largest completely reducible left subideal of $Ae_{\kappa, i}$ is a direct sum of simple left subideals which are isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$.*

Proof. From Lemma 1, A has a unit element. Hence $r(N) =\sum_{\kappa=1}^{n} E_\kappa r(N)=\sum_{\kappa=1}^{n} r(N)E_\kappa$, where $E_\kappa=\sum_{i=1}^{f(\kappa)} e_{\kappa, i}$. $E_\kappa r(N)$ is a two-sided ideal for each κ, since $AE_\kappa r(N)=(\sum E_\lambda AE_\lambda \cup N)E_\kappa r(N)=E_\kappa r(N)$. If $E_\kappa r(N)\neq 0$ and $a\neq 0$ is an arbitrary element of $E_\kappa r(N)$, then there exists an $e_{\kappa, i}$ such that $e_{\kappa, i}a\neq 0$. $Ae_{\kappa, i}a\cong Ae_\kappa/Ne_\kappa$ is obvious. Since $E_\kappa r(N)$ is a direct sum of simple left ideals which are isomorphic to Ae_κ/Ne_κ, each component has the form $Ae_{\kappa, i}ab$, and this shows that $E_\kappa r(N)=AaA$ and $E_\kappa r(N)$ is a simple two-sided ideal. Hence $E_\kappa r(N)N=0$, $E_\kappa r(N)\subseteqq l(N)$ and consequently $r(N)\subseteqq l(N)$. Since $r(N)E_\kappa$ is the largest completely reducible left subideal of AE_κ, $r(N)E_\kappa\neq 0$. Since $r(N)\subseteqq l(N)$, $r(N)E_\kappa A=r(N)E_\kappa(\sum_\lambda E_\lambda AE_\lambda\cup N)=r(N)E_\kappa$. Hence $r(N)E_\kappa$ is a non-zero two-sided ideal for each κ. Then, from $r(N)=\sum_{\kappa=1}^{n} E_\kappa r(N)=\sum_{\kappa=1}^{n} r(N)E_\kappa$, it follows that $r(N)E_\kappa=E_{\pi(\kappa)}r(N)$ is a non-zero simple two-sided ideal for each κ, where π is a permutation of $(1, 2, \cdots, n)$. This shows that the largest completely reducible left subideal of $Ae_{\kappa, i}$ is a direct sum of simple subideals which are isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$.

In the case of algebras, we have by Lemma 2,

Proposition 1. *Let A be an algebra with a finite rank over a field F. If A has a left unit element and satisfies (a) for simple left ideals, then A is a quasi-Frobenius algebra.*

Proof. To prove this, we show that $r(N)e_{\kappa, i}$ is simple for each κ. If $r(N)e_{\kappa, i}$ is not simple, then, by Lemma 2, $r(N)e_{\kappa, i}=\sum_{j=1}^{s}\mathfrak{m}_j$, where $s>1$ and $\mathfrak{m}_j \simeq Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$. Since $\mathfrak{m}_1 \simeq Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, the endomorphismring of $\mathfrak{m}_1$, is isomorphic to $e_{\pi(\kappa)}Ae_{\pi(\kappa)}/e_{\pi(\kappa)}Ne_{\pi(\kappa)}$. On the other hand, every endomorphism of $\mathfrak{m}_1$ is given by the right multiplication of an element of $e_{\kappa,i}Ae_{\kappa,i}$. Since $r(N)\subseteqq l(N)$, elements of $e_{\kappa,i}Ne_{\kappa,i}$ induce zero-endomorphism and those elements of $e_{\kappa,i}Ae_{\kappa,i}$ which are not in $e_{\kappa,i}Ne_{\kappa,i}$ induce isomorphisms. Hence we have a natural isomorphism of $e_{\pi(\kappa)}Ae_{\pi(\kappa)}/e_{\pi(\kappa)}Ne_{\pi(\kappa)}$ into $e_{\kappa,i}Ae_{\kappa,i}/e_{\kappa,i}Ne_{\kappa,i}$.

Since $s>1$, this isomorphism is not an onto isomorphism, and $(e_{\pi(\kappa)}Ae_{\pi(\kappa)}/e_{\pi(\kappa)}Ne_{\pi(\kappa)}: F) \lneqq (e_{\kappa,i}Ae_{\kappa,i}/e_{\kappa,i}Ne_{\kappa,i}: F) = (e_\kappa Ae_\kappa/e_\kappa Ne_\kappa: F)$. Similarly $(e_{\pi^\nu(\kappa)}Ae_{\pi^\nu(\kappa)}/e_{\pi^\nu(\kappa)}Ne_{\pi^\nu(\kappa)}: F) \leqq (e_{\pi^{\nu-1}(\kappa)}Ae_{\pi^{\nu-1}(\kappa)}/e_{\pi^{\nu-1}(\kappa)}Ne_{\pi^{\nu-1}(\kappa)}: F)$, where $\pi^\nu(\kappa) = \underbrace{\pi(\pi(\cdots\pi}_{\nu}(\kappa))))$. Since π is a permutation, it follows that $(e_\kappa Ae_\kappa/e_\kappa Ne_\kappa: F) \lneqq (e_\kappa Ae_\kappa/e_\kappa Ne_\kappa: F)$. This is a contradiction. Hence $r(N)e_{\kappa,i}$ is simple. Then, by Nakayama's theorem,[3] we have our result.

Proposition 2. *Let A be a ring with a left unit element. If A satisfies (a) for every left ideal, then A is a quasi-Frobenius ring.*

Proof. By Lemma 2, $r(N)e_\kappa = \sum_{j=1}^{s}\mathfrak{m}_j$ and $\mathfrak{m}_j \simeq Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$. Hence $\mathfrak{m}_j = Ae_{\pi(\kappa)}a_j$ for a suitable element a_j in $\mathfrak{m}_j$. Assume $s>1$, then the correspondences $e_{\pi(\kappa)}a_1 \to e_{\pi(\kappa)}a_2$ and $e_{\pi(\kappa)}a_2 \to e_{\pi(\kappa)}a_1$ define an automorphism of $\mathfrak{m}_1+\mathfrak{m}_2$. Then, by (a), there is an element c of $e_\kappa Ae_\kappa$ such that $e_{\pi(\kappa)}a_1c = e_{\pi(\kappa)}a_2$ and $e_{\pi(\kappa)}a_2c = e_{\pi(\kappa)}a_1$. Hence $e_{\pi(\kappa)}a_1c^2 = e_{\pi(\kappa)}a_1$ and $e_{\pi(\kappa)}a_1(c^2-e_\kappa)=0$. c^2-e_κ is in $e_\kappa Ne_\kappa$. For, otherwise, it is a unit of $e_\kappa Ae_\kappa$ and consequently $e_{\pi(\kappa)}a_1=0$. Hence $c=\pm e_\kappa+n$, where n belongs to $e_\kappa Ne_\kappa$. Since $r(N)\subseteqq l(N)$, $e_{\pi(\kappa)}ac = e_{\pi(\kappa)}a_1(\pm e_\kappa+n) = \pm e_{\pi(\kappa)}a_1$. This is a contradiction. Hence $r(N)e_\kappa$ is simple. Now if $l(N)e_\kappa \supsetneqq r(N)e_\kappa$, than $l(N)e_\kappa$ contains a left subideal $\mathfrak{l}$ such that $\mathfrak{l}/r(N)e_\kappa$ is irreducible. We suppose $\mathfrak{l}/r(N)e_\kappa \simeq Ae_{\pi(\lambda)}/Ne_{\pi(\lambda)}$ Since $r(N)e_\lambda \simeq Ae_{\pi(\lambda)}/Ne_{\pi(\lambda)}$, there is a homomorphism θ between $\mathfrak{l}$ and $r(N)e_\lambda$. This homomorphism θ is given by the right multiplication of an element of $e_\kappa Ae_\lambda$. If $\kappa\neq\lambda$, then $e_\kappa Ae_\lambda \subset N$ and $\mathfrak{l}\cdot e_\kappa Ae_\lambda \subseteqq l(N)N=0$. If $\kappa=\lambda$, θ is given by the right multiplication of an element of $e_\kappa Ne_\kappa$, since the homomorphisms defined by the elements of $e_\kappa Ae_\kappa$ which are not in $e_\kappa Ne_\kappa$ induce isomorphisms. Then $\mathfrak{l}\cdot e_\kappa Ne_\kappa \subseteqq l(N)\cdot N=0$. Thus we have contradictions. Hence $l(N)e_\kappa = r(N)e_\kappa$. Then it follows easily that $l(N)e_{\kappa,i} = r(N)e_{\kappa,i}$ and $l(N)=r(N)$. We write $l(N)=r(N)=M$. Since $E_{\pi(\kappa)}M = ME_\kappa$, the largest completry reducible right subideal of $e_{\pi(\kappa)}A$ is a direct sum of simple right subideals which are isomorphic to $e_\kappa A/e_\kappa N$. Since Me_κ is simple and is

3) See T. Nakayama [3].

isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, $Me_{\kappa}=Ae_{\pi(\kappa)}me_{\kappa}$ for a suitable element $e_{\pi(\kappa)}me_{\kappa}$ in Me_{κ}. Let x be an arbitrary element in $e_{\pi(\kappa)}Ae_{\pi(\kappa)}$ but not in $e_{\pi(\kappa)}Ne_{\pi(\kappa)}$. Then the correspondence $e_{\pi(\kappa)}me_{\kappa}\rightarrow xe_{\pi(\kappa)}me_{\kappa}$ defines an automorphism of Me_{κ}. For if $x'e_{\pi(\kappa)}me_{\kappa}=0$, then $x'\in A(1-e_{\pi(\kappa)})\cup N$ and $x'xe_{\pi(\kappa)}me_{\kappa}\in(A(1-e_{\pi(\kappa)})\cup N)e_{\pi(\kappa)}M=0$. By (a), this automorphism is given by the right multiplication of an element of $e_{\kappa}Ae_{\kappa}$. Furthermore $e_{\pi(\kappa)}Ne_{\pi(\kappa)}me_{\kappa}=0$ is obvious. Hence $e_{\pi(\kappa)}Ae_{\pi(\kappa)}me_{\kappa}\subseteqq e_{\pi(\kappa)}me_{\kappa}Ae_{\kappa}$. On the other hand, since $e_{\pi(\kappa)}me_{\kappa}A$ is a simple right subideal of $e_{\pi(\kappa)}M$ and $E_{\pi(\kappa)}M$ is a simple two-sided ideal, $e_{\pi(\kappa)}M$ is a direct sum of simple right subideals of the form $\xi e_{\pi(\kappa)}me_{\kappa}A$, where ξ is a suitable unit of $e_{\pi(\kappa)}Ae_{\pi(\kappa)}$. But, as was shown, $e_{\pi(\kappa)}Ae_{\pi(\kappa)}me_{\kappa}\subseteqq e_{\pi(\kappa)}me_{\kappa}A$. Thus we see that $e_{\pi(\kappa)}M=e_{\pi(\kappa)}me_{\kappa}A$ is a unique simple left subideal of $e_{\pi(\kappa)}A$. This completes our proof.

Remark. From the assumption (a) for simple left ideals, we can not conclude that A has a left unit element. For example, let F be a field and $A=Fe+Fu$, where $e^2=e$, $ue=u$, $eu=0$, $u^2=0$. This algebra over F has no left unit element, but it satisfies (a).

If A is a ring and not an algebra, then we can not conclude that A is a quasi-Frobenius ring, from the assumption (a) for simple left ideals and the existence of a left unit element. For example, let $F(x)$ be a rational function field over a field F and $A=F(x)+uF(x)$, where $u^2=0$, $xu=ux^2$. Then this is not a quasi-Frobenius ring, but it has a unit element and (a) is valid for simple left ideals.

Proposition 3. *If A is a ring in which (a) is valid for simple left ideals and the same is true for simple right ideals, then A is a quasi-Frobenius ring.*

Proof. By Lemma 1, A has a unit element. $r(N)=l(N)=M$, $Me_{\kappa}=\sum_{j=1}^{s}\mathfrak{m}_j$ and $e_{\pi(\kappa)}M=\sum_{k=1}^{r}\mathfrak{n}_k$, by Lemma 2. As was shown in the proof of Theorem 2, $e_{\pi(\kappa)}Ae_{\pi(\kappa)}me_{\kappa}\subseteqq e_{\pi(\kappa)}me_{\kappa}Ae_{\kappa}$, if we write $\mathfrak{m}_1=Ae_{\pi(k)}me_{\kappa}$. Similarly $e_{\pi(\kappa)}Ae_{\pi(\kappa)}me_{\kappa}\supseteqq e_{\pi(\kappa)}me_{\kappa}Ae_{\kappa}$, since $e_{\pi(\kappa)}me_{\kappa}A$ is a simple right subideal of $e_{\pi(\kappa)}M$. Hence $e_{\pi(\kappa)}Ae_{\pi(\kappa)}me_{\kappa}=e_{\pi(\kappa)}me_{\kappa}Ae_{\kappa}$. On the other hand, $\mathfrak{m}_j$ has the form $\mathfrak{m}_1\xi=Ae_{\pi(\kappa)}me_{\kappa}\xi$, where ξ is an element of $e_{\kappa}Ae_{\kappa}$. Hence $s=1$ and similarly $r=1$. Thus A is a quasi-Frobenius ring.

Lemma 3. *Let A be a quasi-Frobenius ring and let $\mathfrak{l}=\mathfrak{l}_1\cup\mathfrak{l}_2$ be a left ideal homorphic to a left ideal $\mathfrak{l}'$ by a homomorphism θ, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two left subideals of $\mathfrak{l}$. If the homomorphisms from $\mathfrak{l}_1$ and $\mathfrak{l}_2$ into $\mathfrak{l}'$ induced by θ are given by the right muliplicatios of elements a_1 and a_2*

respectively, then there is an element a such that θ is given by the right multiplication of a.

Proof. Of course $\mathfrak{l}' = \mathfrak{l}_1^\theta \cup \mathfrak{l}_2^\theta$. Then elements a_1 and a_2 define the same homomorphism for $\mathfrak{l}_1 \cap \mathfrak{l}_2$. Hence $a_1 - a_2 \in r(\mathfrak{l}_1 \cap \mathfrak{l}_2) = r(\mathfrak{l}_1) \cup r(\mathfrak{l}_2)$, since A is a quasi-Frobenius ring. Hence $a_1 - a_2 = r_2 - r_1$ for suitable $r_1 \in r(\mathfrak{l}_1)$ and $r_2 \in r(\mathfrak{l}_2)$. We write $a_1 + r_1 = a_2 + r_2$ as a. Then a defines θ for $\mathfrak{l}$. For if l_i is an element of $\mathfrak{l}_i\ (i = 1, 2)$, then $l_i a = l_i(a_i + r_i) = l_i a_i = l_i^\theta$.

Theorem 1. *Let A be a ring with a unit element. Then A satisfies Shoda's condition if and only if A is a quasi-Frobenius ring.*

Proof The "only if" part follows from Proposition 2.

We shall prove the "if" part. If a left ideal $\mathfrak{l}'$ is a homomorphic image of a principal left ideal $\mathfrak{l} = Aa$, then $\mathfrak{l}'$ is also a principal ideal. We denote this homomorphism by θ, and show that θ is given by the right multiplication of an element. Since θ is a homomorphism, $l(a) = l(aA) \subseteqq l(a^\theta) = l(a^\theta A)$. Since A is a quasi-Frobenius ring, $r(l(aA)) = aA \supseteqq r(l(a^\theta A)) = a^\theta A$. Hence there is an element c such that $a^\theta = ac$.

Since every left ideal $\mathfrak{l}$ has a finite basis, we can write $\mathfrak{l} = \bigcup_{i=1}^{s} Aa_i$. Then, by Lemma 3, every homomorphism between two left ideals is given by the right multiplication of a suitable element. This completes our proof.

Therem 2. *Let A be a quasi-Frobenius ring. Then for every isomorphism θ between two left ideals we can choose a suitable unit which defines θ, that is, every isomorphism between two left ideals can be extended to an isomorphism of A.*

Proof. Let θ be an isomorphism between $\mathfrak{l}$ and $\mathfrak{l}'$. Then, by Theorem 1, there is an element a_θ which defines θ, that is, $\mathfrak{l}a_\theta = \mathfrak{l}'$. Then $\mathfrak{l}a_\theta r(\mathfrak{l}') = \mathfrak{l}' r(\mathfrak{l}') = 0$. This shows that $a_\theta r(\mathfrak{l}') \subseteqq r(\mathfrak{l})$.

Case I. $a_\theta r(\mathfrak{l}') = r(\mathfrak{l})$.

If r is an arbitrary element of $r(\mathfrak{l})$, then there is an element r' in $r(\mathfrak{l}')$ such that $a_\theta r' = r$. Let θ^{-1} be the inverse isomorphism of θ and let $b_{\theta^{-1}}$ be the element which defines θ^{-1}. It is easy to see that $1 - a_\theta b_{\theta^{-1}} = r_0 \in r(\mathfrak{l})$. Then $a_\theta(b_{\theta^{-1}} + r_0') = a_\theta b_{\theta_{-1}} + r_0 = 1$. Hence a_θ is a unit.[4)]

Case II. $a_\theta r(\mathfrak{l}') \subsetneqq r(\mathfrak{l})$.

In this case, $\bar{\mathfrak{l}} = l(a_\theta r(\mathfrak{l}')) \supsetneqq l(r(\mathfrak{l})) = \mathfrak{l}$, since A is a quasi-Frobenius

4) Since A satisfies the minimum condition for left and right ideals, it $ab=1$, then $ba=1$.

ring. It follows, from $\bar{\mathfrak{l}}a_\theta r(\mathfrak{l}')=0$, that $\bar{\mathfrak{l}}a_\theta \subseteqq \mathfrak{l}'$. But $\bar{\mathfrak{l}}a_\theta \supseteqq \mathfrak{l}a_\theta = \mathfrak{l}'$. Hence $\bar{\mathfrak{l}}a_\theta = \mathfrak{l}'$. Let $\bar{l}$ be an element of $\bar{\mathfrak{l}}$ and $\bar{l}a_\theta = l'$, then l' is in $\mathfrak{l}'$ and there is an element l of $\mathfrak{l}$ such that $la_\theta = l'$. Hence $(\bar{l}-l)a_\theta = 0$. Since no element of $\mathfrak{l}$ is annihilated by a_θ, $\bar{\mathfrak{l}}$ is the direct sum of $\mathfrak{l}$ and $\mathfrak{l}_0$ which is annihilated by a_0. Let $Ae_{\pi(\kappa)}a\,(\simeq Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)})$ be a simple left subideal of $\mathfrak{l}_0$. We write $Ae_{\pi(\kappa)}a+\mathfrak{l}=\mathfrak{l}^*$. Since $\mathfrak{l}^*/\mathfrak{l} \simeq Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, it follows evidently that $r(\mathfrak{l})/r(\mathfrak{l}^*) \simeq e_\kappa A/e_\kappa N$. Hence $r(\mathfrak{l}) = re_\kappa A \bigcup r(\mathfrak{l}^*)$ for a suitable element r of $r(\mathfrak{l})$. Since $re_\kappa A \subset r(\mathfrak{l})$, the homomorphism defined by $a_\theta + re_\kappa b$, for an arbitrary b of A, coincides with θ in $\mathfrak{l}$. $\mathfrak{l}^*(a_\theta+re_\kappa b)$ is homomorphic to $\mathfrak{l}^*$ and contains $\mathfrak{l}(a_\theta+re_\kappa b)=\mathfrak{l}'$. Now if we take a suitable b, then $\mathfrak{l}\,(a_\theta+re_\kappa b)$ is actually different from $\mathfrak{l}'$. For otherwise, $Ae_{\pi(\kappa)}a\,(a_\theta+re_\kappa b) = Ae_{\pi(\kappa)}are_\kappa b \subset \mathfrak{l}\,(a_\theta+re_\kappa b)=\mathfrak{l}'$ for every b of A. Hence $Ae_{\pi(\kappa)}are_\kappa A \subset \mathfrak{l}'$. Since $Ae_{\pi(\kappa)}are_\kappa \subset ME_\kappa$[5] and ME_κ is a simple two-sided ideal, $ME_\kappa = Ae_{\pi(\kappa)}are_\kappa A \subset \mathfrak{l}'$ and $\mathfrak{l}'b_{\theta^{-1}} = \mathfrak{l} \supset ME_\kappa$. On the other hand $ME_\kappa = E_{\pi(\kappa)}M$ contains every simple left ideal which is isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$. Hence ME_κ contains $Ae_{\pi(\kappa)}a$. Thus $\mathfrak{l}$ contains $Ae_{\pi(\kappa)}a$. But this contradicts $\mathfrak{l} \bigcap \mathfrak{l}_0 = 0$. Thus we can take an element b such that $\mathfrak{l}^*(a_\theta+re_\kappa b) \supsetneqq \mathfrak{l}'$. Obviously $\mathfrak{l}^*(a_\theta+re_\kappa b) \simeq \mathfrak{l}^*$. We write the isomorphism between $\mathfrak{l}^*$ and $\mathfrak{l}^*(a_\theta+re_\kappa b)$ defined by the right multiplication of $a_\theta+re_\kappa b$, by Θ. Then Θ coindides with θ in $\mathfrak{l}$, as was shown.

Since our assertion is true for A, suppose now that our assertion is true for every left ideal L for which A/L has a shorter composition length than that of $A/\mathfrak{l}$. Then we can choose a unit a_Θ for Θ. a_Θ defines Θ for $\mathfrak{l}^*$, hence a_Θ defines θ for $\mathfrak{l}$. This completes our proof.

The following lemma is trivial.[6]

Lemma 4. *Let A be a ring with a unit element. If every residue class ring of A satisfies Shoda's condition, then A is a uni-serial ring, and conversely.*

Theorem 3.[7] *Let A be such a ring with a unit element that if $\mathfrak{l}/\mathfrak{m} \sim \mathfrak{l}'/\mathfrak{m}$ for any two left ideals $\mathfrak{l}$, $\mathfrak{l}'$ with their common left subideal $\mathfrak{m}$, then for every homomorphism θ from $\mathfrak{l}/\mathfrak{m}$ onto $\mathfrak{l}'/\mathfrak{m}$ there is such a homomorphism Θ from $\mathfrak{l}$ onto $\mathfrak{l}'$ that is given by the right multiplication of an element of A and that coincides with θ in $\mathfrak{l}/\mathfrak{m}$. Then A is a direct sum of a semi-simple ring and completely primary uni-serial rings, and conversely.*

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5) See T. Nakayama [2] p. 10.
6) See M. Ikeda [5] p. 239. Cf. K. Shoda [4].
7) Cf. K. Shoda [4].

Proof. It is clear that every residue class ring satisfies Shoda's condition. Hence A is a uni-serial ring. Since the above assumption holds for primary components of A, we prove our assertion for a primary uni-serial ring A_1 satisfying the above assumption. If A_1 is neither a simple ring nor a completely primary uni-serial ring, then A_1 is a total matric ring of degree $n>1$ over a completely primary uni-serial ring D. The radical N_D of D is a principal ideal: $N_D = D\pi = \pi D$. Then the principal ideal $A\pi = \pi A$ is the radical N of A. Let $N^{\rho-1} \neq 0$ and $N^{\rho} = 0$. Then $N^{\rho-1}e_1 = A\pi^{\rho-1}e_1 = Ae_1\pi^{\rho-1}$ and $N^{\rho-1}e_2 = A\pi^{\rho-1}e_2 = Ae_2\pi^{\rho-1}$ are the unique simple left subideals of Ae_1 and Ae_2 respectively. $Ae_1\pi^{\rho-1} \simeq Ae_2\pi^{\rho-1}$ by the correspondence $e_1\pi^{\rho-1} \leftrightarrow c_{12}\pi^{\rho-1}$. Then $N^{\rho-1}(e_1+c_{12}) = A(e_1+c_{12})\pi^{\rho-1}$ is a simple left ideal and contained in $A(e_1+c_{12})$. Since $A(e_1+c_{12})$ is an indecomposable left ideal, $N^{\rho-2}(e_1+c_{12})$ contains $N^{\rho-1}(e_1+c_{12})$ as its unique simple left subideal. It is clear that $N^{\rho-2}(e_1+c_{12})/N^{\rho-1}(e_1+c_{12}) \simeq N^{\rho-1}e_1+N^{\rho-1}e_2/N^{\rho-1}(e_1+c_{12})$. But, as was shown, $N^{\rho-2}(e_1+c_{12})$ is not isomorphic to $N^{\rho-1}e_1+N^{\rho-1}e_2$. This contradicts our assumption. Thus if A_1 is a primary uni-serial ring satisfying our assumption, then A_1 is either a simple ring or a completely primary uni-serial ring. The converse is trivial.

Remark. Let A be such a ring with a unit element that if $\mathfrak{l}/\mathfrak{m} \sim \mathfrak{l}'/\mathfrak{m}$ for any two left ideal $\mathfrak{l}$, $\mathfrak{l}'$ with their common left subideal $\mathfrak{m}$, then for every homomorphism θ from $\mathfrak{l}/\mathfrak{m}$ onto $\mathfrak{l}'/\mathfrak{m}$ and every endomorphism φ of $\mathfrak{m}$ there is a homomorphism Θ from $\mathfrak{l}$ onto $\mathfrak{l}'$ which is given by the right multiplication of an element of A and coincides with θ in $\mathfrak{l}/\mathfrak{m}$ and with φ in $\mathfrak{m}$. Then A is a semi-simple ring and conversely.

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