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On the Classification of Open Riemann Surfaces

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Introduction

We shall denote by O_{HB} and O_{HD} the classes of Riemann surfaces for which any single-valued harmonic functions that are respectively bounded or of finite Dirichlet integrals must be reduced to constants; furthermore we shall denote by O_G the class of Rieman surfaces without Green's functions. Then $O_G \leqq O_{HB}$ was proved by P. J. Myrberg[1] and $O_{HB} \leqq O_{HD}$ by Virtanen.[2] Recently Ahlfors gave an example[3] to prove that the first inclusion ($O_G < O_{HB}$) is strict. But unfortunately we can show his proof fails to prove this fact.

It was M. Inoue who pointed out for the first time that there is some vague point in Ahlfors' reasoning. He remarked: it is not always possible to conclude unconditionally that $\left(r \dfrac{\partial u}{\partial r}\right)^2$ is subharmonic at the end-points of the concentric circular slits.[4]

We shall show in § **1** that there exists a non-constant single-valued bounded harmonic function of Ahlfors' Riemann surface, in § 2 that Ahlfors' anticipation is right, by constructing a Riemann surface without non-constant single-valued bounded harmonic function but with the Green's function, and in § **3** that Virtanen's inclusion is indeed strict ($O_{HB} < O_{HD}$) by means of an example. I owe this investigation to Ahlfors' paper above mentioned.

For convenience we introduce some definitions. Let $\boldsymbol{D}$ be a domain in the z-plane and $\boldsymbol{F}$ a covering surface over the basic surface $\boldsymbol{D}$. Then, in determining the metric on $\boldsymbol{F}$ by that on $\boldsymbol{D}$, two cases can occur according as the sense of argument: for the mapping $\boldsymbol{F} \rightarrow \boldsymbol{D}$ the positive sense of argument on $\boldsymbol{F}$ is defined either

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1) P. J. Myberg, Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche. Acta math. 61 (1933).

2) K. I. Virtanen, Ueber die Existenz von beschrankten harmonischen Funktionen auf offenen Riemannschen. Flächen. Ann. Acad. Scient. Fenn. A I 75 (1950).

3) L. v. Ahlfors, Remarks on the classification of open Riemann surfaces. Ann. Acad. Sic. Fenn. A. I. 87 (1951).

4) Ahlfors himself has recognized the defect of his proof. Cf. Math. Rev. vol. 13, No. 4, p. 338 (1952).

1) so as to coincide with that on $\boldsymbol{D}$,

or

2) so as to be opposite to that on $\boldsymbol{D}$.

We shall call $\boldsymbol{F}$ the *direct covering surface* over $\boldsymbol{D}$ in the first case and the *indirect covering surface* over $\boldsymbol{D}$ in the second case. Let $\boldsymbol{F}'$ and $\boldsymbol{F}''$ be respectively direct and indirect covering surfaces over $\boldsymbol{D}$. We consider both $\boldsymbol{F}'$ and $\boldsymbol{F}''$ provided with slits s_i' and s_i'' respectively, each projection of which is a common analytic arc s_i on $\boldsymbol{D}$ $(i = 1, 2, \cdots)$. Connect two sides of s_i' and s_i'' with folding (i. e. identify the same edges of s_i' and s_i''), and we obtain a new surface $\boldsymbol{F}$ from $\boldsymbol{F}'$ and $\boldsymbol{F}''$. $\boldsymbol{F}$ shall be called *pseudo-covering surface* over the basic surface $\boldsymbol{D}$.

§ 1. Ahlfors' Riemann surface $\boldsymbol{F}$ is the two sheeted and symmetric pseudo-covering surface over the unit circle $|z|<1$ without any relative boundaries but with foldings over concentric circular slits in the circle $|z|<1$ which converges to the circumference $|z|=1$. Two points on $\boldsymbol{F}$ over a point on the circle $|z|<1$ are symmetric points on $\boldsymbol{F}$; we shall denote them by p and $\tilde{p}$.

Let $G(p, p_0)$ be the Green's function with a pole at the point p_0, the existence of which is already proved in Ahlfors' paper.

Let $T(p)$ be the indirectly conformal mapping such that each point p on $\boldsymbol{F}$ corresponds to the symmetric point $\tilde{p}$ on $\boldsymbol{F}$.

Then $G(T(p), T(p_0))$ is the Green's function on $\boldsymbol{F}$ with a pole at $\tilde{p}_0 = T(p_0)$.

Put
$$g(p\,;\, p_0,\, \tilde{p}) = -\Big\{G(p, p_0) + G(T(p),\ T(p_0))\Big\}\,.$$

Let $h(p\,;\, p_0,\, \tilde{p}_0)$ be a conjugate harmonic function of $g(p\,;\, p_0,\, \tilde{p})$ on $\boldsymbol{F}$.

Then by the function $\exp\{g(p\,;\, p_0,\, \tilde{p}_0)+i\ h(p\,;\, p_0, \tilde{p}_0)\}$ $\boldsymbol{F}$ is mapped one-to-one and conformally to the two sheeted and symmetric surface $\hat{\boldsymbol{F}}$ on the circle $|z|<1$ without relative boundaries but with foldings over radial slits on the circle $|z|<1$, which converges to the circumference $|z|=1$. In order that we show the existence of a single-valued bounded harmonic function on the surface $\boldsymbol{F}$, we have only to show that on the surface $\hat{\boldsymbol{F}}$.

Let $\{r_n\}$ $(n = 1, 2, \cdots)$ be a sequence of positive numbers such that $\lim\limits_{n\to\infty} r_n = 1$, and let $\hat{\boldsymbol{F}}_{r_n}$ be the subsurface of $\hat{\boldsymbol{F}}$ over $|z|<r_n$.

Let $u_n(p)$ be the single-valued bounded harmonic function of $\hat{\boldsymbol{F}}_{r_n}$ with the following boundary values:

$$u_n(p) = \begin{cases} 1 & \text{on the two circular arcs on } \hat{F} \text{ over the arc } r_n e^{i\theta} \ (0 \leq \theta < \pi) \text{ on the circle } |z| < 1. \\ 0 & \text{on the two circular arcs on } \hat{F} \text{ over the arc } r_n e^{i\theta} \ (\pi \leq \theta < 2\pi) \text{ on the circle } |z| < 1. \end{cases}$$

We shall denote by $\tau(p)$ the indirectly conformal mapping such that each point p on $\hat{F}$ corresponds to the symmetric point $\tilde{p}$ on $\hat{F}$.

Put

$$U_n(p) = \frac{1}{2}\left[u_n(p) + u_n(\tau(p))\right].$$

Then $U_n(p)$ is also the single-valued bounded harmonic function on $\hat{F}_{r_n}$ with the same boundary values as $u_n(p)$. Let $V_n(p)$ be the conjugate harmonic function of $U_n(p)$ on $\hat{F}_{r_n}$ satisfying the condition $V_n(p_0) = 0$, where p_0 is a fixed point on $\hat{F}_{r_1}$. Since $V_n(p) = V_n(\tilde{p})$, put $W_n(p) = U_n(p) + \boldsymbol{i}\ V_n(p)$, and $W_n(p) = W_n(\tilde{p})$ and by $W_n(p)$ $\hat{F}_{r_n}$ is mapped one-to-one and conformally to the two sheeted and symmetric pseudo-covering surface $\hat{W}_n$ on the domain $\boldsymbol{D}$, $0 < x < 1$ ($z = x + \boldsymbol{i}y$), without relative boundaries but with foldings over slits parallel to the real axis on $\boldsymbol{D}$. Therefore the family $\{W_n(p)\}$ is mormal, so there exists such a subsequence of $\{W_n(p)\}$, $(n = 1, 2, \cdots)$, that it converges uniformly on $\hat{F}_{r_i}$, $(i = 1, 2, \cdots)$. For convenience we shall denote it by $\{W_n(p)\}$, $(n = 1, 2, \cdots)$.

Put

$$W(p) = \lim_{n\to\infty} W_n(p), \quad W(p) = U(p) + \boldsymbol{i}V(p).$$

Then $W(p)$ is a regular function on $\hat{F}$. If $W(p)$ is a constant α, $U(p_0) = \alpha$, $0 \leq \alpha \leq 1$, $V(p) = 0$ for $V_n(p_0) = 0$.

Let $\varphi_n(q)$ be the inverse function of $W_n(p)$, and ρ_0 be an arbitrary positive number, then there exists such a positive number N that the image of $\hat{F}_{r_1}$ by $W_n(p)$ is contained in the subsurface of $\hat{W}_n$ over the disk $|w - \alpha| < \rho_0$ for all n such as $n > N$.

Now we shall denote by $L(\rho)$ the length of the image of curves on $\hat{W}_n$ over $|w - \alpha| = \rho$ by $\varphi_n(q)$ and we shall denote by $A(\rho)$ the area of the image of subsurface of $\hat{W}_n$ over $\rho < |w - \alpha| < \frac{1}{2}$ by $\varphi_n(q)$.

The following inequality is well-known:

$$\int_{\rho_0}^{\frac{1}{2}} \frac{d\rho}{\rho} \leq 4\pi \int_{\rho_0}^{\frac{1}{2}} \frac{dA(\rho)}{L^2(\rho)}$$

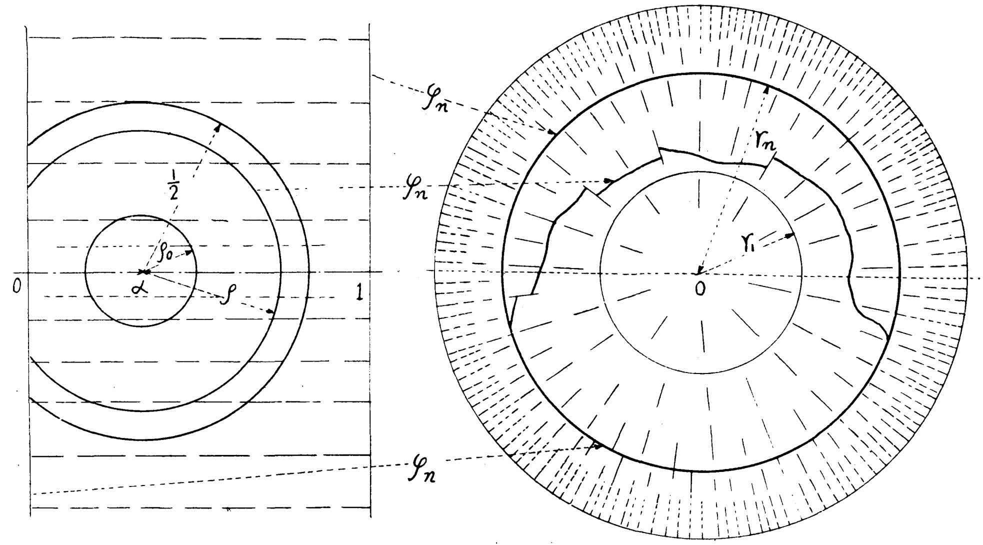


Fig. 1

Since
$$L(\rho) \geq 2\pi r_1,$$
$$\log \frac{1}{2\rho_0} \leq \frac{4\pi}{4\pi^2 r_1^2} A(\rho_0) = \frac{1}{\pi r_1^2} A(\rho_0).$$

But $A(\rho_0) < 2\pi$, and ρ_0 is an arbitrary small positive number. This is a contradiction. So $W(p)$ is not a constant. Then $\hat{\boldsymbol{F}}$ is mapped one-to-one and conformally by the function $W(p)$ to the two sheeted and symmetric pseudo-covering surface $\hat{\boldsymbol{W}}$ over the domain $\boldsymbol{D}$ without any relative boundaries but with foldings over slits parallel to the real axis on $\boldsymbol{D}$. Therefore it is clear that $U(p)$ is a single-valued bounded harmonic function on $\hat{\boldsymbol{F}}$.

§ 2. We shall construct a Riemann surface with a Green's function but without any non-constant single-valued bounded harmonic function. We shall consider the surface $\boldsymbol{F}$ cut along radial slits C_μ^ν ($\mu = 1, 2, 3, \cdots, \nu = 0, 1, \cdots, 2^{\mu+1}-1$) on the unit circle $|z| < 1$ as follows:

$$C_\mu^\nu;\ z = re^{i\theta_\nu},\quad 1-\frac{1}{2\mu} \leq r \leq 1-\frac{1}{2\mu+1},\quad \theta_\nu = \frac{\nu}{2^\mu}\pi.$$

By the relation $\mu = 2^{m-1}(2n-1)$ natural numbers μ correspond one-to-one to the pair of two natural numbers (m, n):

m \ n	1	2	3	4	5	....
1	1	3	5	7	9	....
2	2	6	10	14	18	....
3	4	12	20	28	36	....
4	8	24	40	56	72	....
⋮	....					
⋮	....					

Therefore we shall denote the slits C_μ^ν by $C_{m,n}^\nu$. The slits $C_{m,n}^\nu$ $(m=1\,;\ n=1,2,3,\cdots;\ \nu=0,1,\cdots,2^{\mu+1}-1)$ are symmetric with respect to the real axis. Let $T(z)$ be the indirectly conformal mapping such that each point z corresponds to the point $\bar{z}$.

We shall identify each pair of edges of the same sides[5] of slits belonging to $C_{1,n}^\nu$ $(n=1,2,\cdots,;\ \nu=0,1,\cdots,2^{\mu+1}-1)$ corresponding each other by $T(z)$.

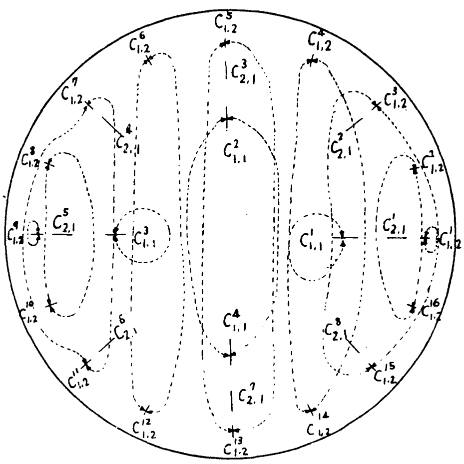


Fig. 2

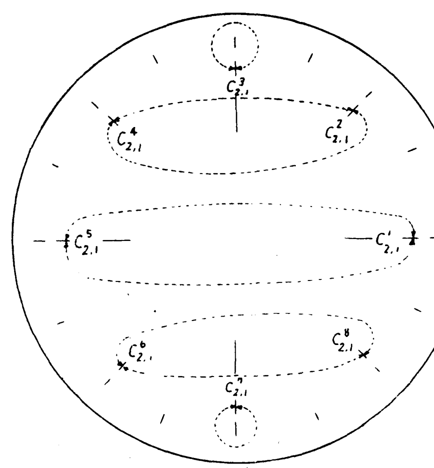


Fig. 3

Next we shall identify each pair of edge of the same sides of slits belonging to $C_{2,n}^\nu$ $(n=1,2,\cdots;\ \nu=0,1,\cdots,2^\mu)$ or to $C_{2,n}^\nu$ $(n=1,2,\cdots;\ \nu=2^\mu,\cdots,2^{\mu+1}-1,0)$ corresponding each other by $T_1(z)$, where $T_1(z) =[T(z^2)]^{\frac{1}{2}}$. And next we shall identify each pair of edges of the same sides of slits belonging to $C_{3,n}^\nu$ $(n=1,2,\cdots;\ \nu=0,1,\cdots,2^{\mu-1})$ or to $C_{?,n}^\nu$ $(n=1,2,\cdots;\ \nu=2^{\mu-1},\cdots,2\cdot 2^{\mu-1})$ or to $C_{?,n}^\nu$ $(n=1,2,\cdots;\ \nu=2\cdot 2^{\mu-1},\cdots,3\cdot 2^{\mu-1})$ or to $C_{?,n}^\nu$ $(n=1,2,\cdots;\ \nu=3\cdot 2^{\mu-1},\cdots,2^{\mu+1}-1,0)$ corresponding each other by $T_2(z)$, where $T_2(z)=[T_1(z^2)]^{\frac{1}{2}}$.

5) If a side of a slit corresponds to another side of the same slit by T(z), we say that these sides are the same sides of the slit.

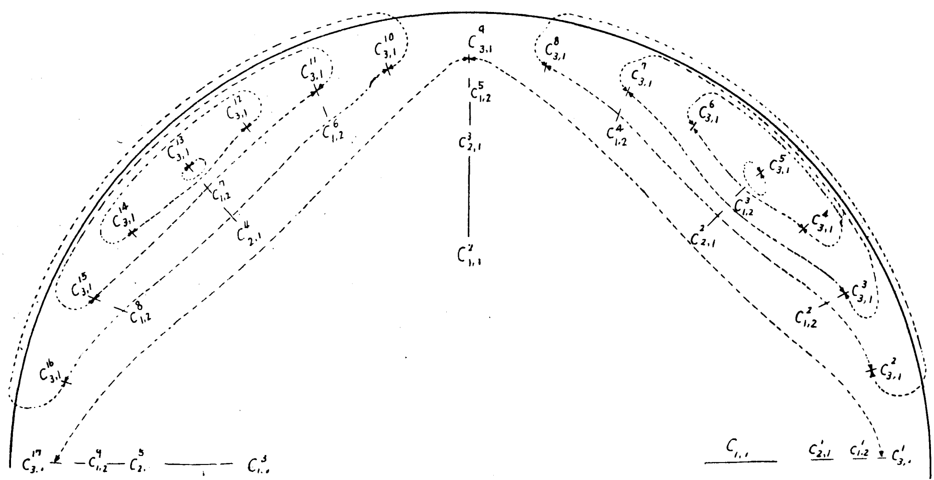


Fig. 4

Generally we shall identify each pair of edges of the same sides of slits belonging to $C^{\nu}_{m,n}(n=1,2,\cdots;\ \nu=0,1,\cdots,2^{\mu-m+2})$ or to $C^{\nu}_{m,n}(n=1,2,\cdots;\ \nu=2^{\mu-m+2},\cdots,2\cdot 2^{\mu-m+2})$ or to $C^{\nu}_{m,n}(n=1,2,\cdots;\ \nu=2\cdot 2^{\mu-m+2},\cdots,3\cdot 2^{\mu-m+2})$ or to $C^{\nu}_{m,n}(n=1,2,\cdots;\ \nu=3\cdot 2^{\mu-m+2},\cdots,4\cdot 2^{\mu-m+2})$ or to $\cdots\cdots$ or to $C^{\nu}_{m,n}(n=1,2,\cdots;\ \nu=(2^{m-1}-2)\cdot 2^{\mu-m+2},\cdots,2^{m-1}-1)\cdot 2^{\mu-m+2})$ or to $C^{\nu}_{m,n}(n=1,2,\cdots;\ \nu=(2^{m-1}-1)\cdot 2^{\mu-m+2},\cdots,2^{\mu+1}-1,0)$ corresponding each other by $T_{m-1}(z)$, where $T_{m-1}(z)=[T_{m-2}(z^2)]^{\frac{1}{2}}=[T(z^{2^{m-1}})]^{\frac{1}{2^{m-1}}}$.

Thus we can construct the Riemann surface $\hat{\boldsymbol{F}}$. Then we shall prove that $\hat{\boldsymbol{F}}$ is just the required Riemann surface.

Let $u(z)$ be a single-valued bounded harmonic function of $\boldsymbol{F}$. We may assume $|u(z)|<1$ without loss of generality. It is clear that $\hat{\boldsymbol{F}}$ is the symmetric surface with respect to the real axis.

Therefore put

$$u_1(z)=\frac{1}{2}\Big[u(z)-u(T(z))\Big],$$

and $u_1(z)$ is a single-valued harmonic function on $\hat{\boldsymbol{F}}$ and vanishes on $C^{\nu}_{1,n}(n=1,2,\cdots;\ \nu=0,1,\cdots,2^{\mu+1}-1)$, and $|u_1(z)|<1$. Let $\omega(z,C^{\nu}_{1,n},R_{1,n})(n=1,2,\cdots)$ be harmonic measures, where $R_{1,n}(n=1,2,\cdots)$ are ring domains $1-\frac{1}{2\mu-1}<|z|<1-\frac{1}{2\mu+2}$ $(\mu=2n-1)$ with radials slits $C^{\nu}_{1,n}(\nu=0,1,\cdots,2^{\mu+1}-1)$.

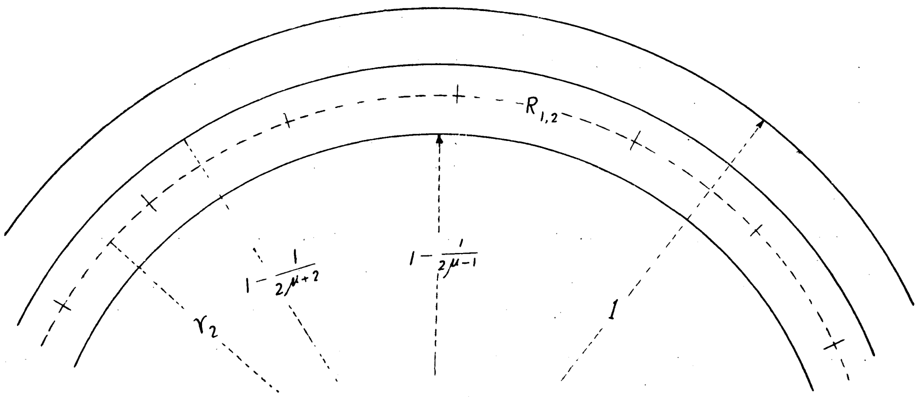


Fig. 5

Then

$$u_1(z) < \omega(z, C_{1,n}^{\nu}, R_{1,n}),$$

$$(1) \qquad (\pi/2^{\mu})/\log\left[\left(1-\frac{1}{2\mu}\right)\Big/\left(1-\frac{1}{2\mu+1}\right)\right] \to 0, \quad \text{for } \mu \to \infty$$

Therefore the sequence $\{\omega(r_n e^{i\theta}, C_{1,n}^{\nu}, R_{1,n})\}$ $(n=1, 2, \cdots)$ converges uniformly to zero with $n \to \infty$, where $r_n = \frac{1}{2}\left[\left(1-\frac{1}{2\mu}\right)+\left(1-\frac{1}{2\mu+1}\right)\right]$ $(\mu = 2n-1)$, and $0 \leq \theta \leq 2\pi$. Hence,

$$u_1(z) = 0 \quad \text{on} \quad \hat{F}.$$

This fact shows that $u(z) = u(T(z)) = u(\bar{z})$ on $\hat{F}$.

Therefore put

$$u_2(z) = \frac{1}{2}\left[u(z) - u(T_1(z))\right],$$

and $u_2(z)$ is a single-valued harmonic function on $\hat{F}$ and vanishes on $C_{2,n}^{\nu}$ $(n=1, 2, \cdots; \nu = 0, 1, \cdots, 2^{\mu+1}-1)$, and $|u_2(z)| < 1$.

Let $\omega(z, C_{2,n}^{\nu}, R_{2,n})$ $(n=1, 2, \cdots)$ be harmonic measures, where $R_{2,n}$ $(n=1, 2, \cdots)$ are ring domains $1-\frac{1}{2\mu-1} < |z| < 1-\frac{1}{2\mu+2}$ $(\mu = 2(2n-1))$ with radial slits $C_{2,n}^{\nu}$ $(\nu = 0, 1, \cdots, 2^{\mu+1}-1)$.

Then

$$u_2(z) < \omega(z, C_{2,n}^{\nu}, R_{2n}),$$

and the sequence $\{\omega(r_n e^{i\theta}, C_{2,n}^{\nu}, R_{2,n})$ $(n=1, 2, \cdots)$ converges uniformly to zero with $n \to \infty$ by (1), where $r_n = \frac{1}{2}\left[\left(1-\frac{1}{2\mu}\right)+\left(1-\frac{1}{2\mu+1}\right)\right]$ $(\mu = 2(2n-1))$, and $0 \leq \theta \leq 2\pi$.

$$u_2(z) = 0 \quad \text{on } \hat{F}.$$

This fact shows that $u(z) = u(T_1(z))$ on $\hat{F}$.
Therefore put

$$u_3(z) = \frac{1}{2}\left[u(z) - u(T_2(z))\right],$$

and $u_3(z)$ is a single-valued harmonic function on $\hat{F}$ and vanishes on $C^{\nu}_{2,n}$ $(n = 1, 2, \cdots;\ \nu = 0, 1, \cdots, 2^{\mu+1}-1)$, and $|u_3(z)| < 1$. From this fact we can prove as above that $u_3(z) = 0$ on $\hat{F}$, so $u(z) = u(T_2(z))$.

In the same way we can prove that $u(z) = u(T_m(z))$ $(m = 3, 4, \cdots)$. Therefore $u(z)$ is a constant on $\hat{F}$. On the other hand $\log\left|\frac{1}{z}\right|$ is a Green's function with pole at the point $z = 0$ on $\hat{F}$.

§ **3.** Now we shall construct a Riemann surface with a single-valued bounded harmonic functions, but with no harmonic function of finite Dirichlet integral.

We shall consider the surface F_0 cut along radial slits C^{ν}_{μ} and C^{ν} $(\mu = 1, 2, \cdots;\ \nu = 0, 1, \cdots, 2^{\mu+1}-1)$ on the ring domain R; $\frac{1}{4} < |z| < 1$ as follows:

$$C^{\nu}_{\mu};\ z = re^{i\theta_\nu} \quad 1 - \frac{1}{4\mu} \leq r \leq 1 - \frac{1}{4\mu+1} \quad \theta_\nu = \frac{\nu}{2^\mu}\pi,$$

$$\tilde{C}^{\nu}_{\mu};\ z = re^{i\theta_\nu} \quad \frac{\mu}{4\mu-1} \geq r \geq \frac{4\mu+1}{16\mu}, \quad \theta_\nu = \frac{\nu}{2^\mu}\pi.$$

Let $F(h)$ $(h = 1, 2, \cdots)$ be one sheeted direct covering surfaces without any relative boundaries over the basic surface F_0, and let $\hat{F}(h)$ $(h = 1, 2, \cdots)$ be one sheeted indirect covering surfaces without any relative boundaries over F_0. We shall denote the slits C^{ν}_{μ} and $\tilde{C}^{\nu}_{\mu}$ by $C^{\nu}_{m,n}$ and $C^{\nu}_{m,n}$ respectively, where m and n are natural numbers with the relation $\mu = 2^{m-1}(2n-1)$.

We shall construct the pseudo-covering surface W over the ring domain R connecting the surfaces $\{F(h)\}$ and $\{\hat{F}(h)\}$. We shall identify the edges of the same sides of each pair of slits $P(k, l, m, n, \nu)$ or $\tilde{P}(k, l, m, n, \nu)$ $(k = 0, 1, \cdots;\ l = 1, 2, \cdots, 2^{m-1},\ m = 1, 2, \cdots;\ n = 1, 2, \cdots;\ \nu = 0, 1, \cdots, 2^{\mu+1}-1)$, where $P(k, l, m, n, \nu)$ and $\tilde{P}(k, l, m, n, \nu)$ are as follows:

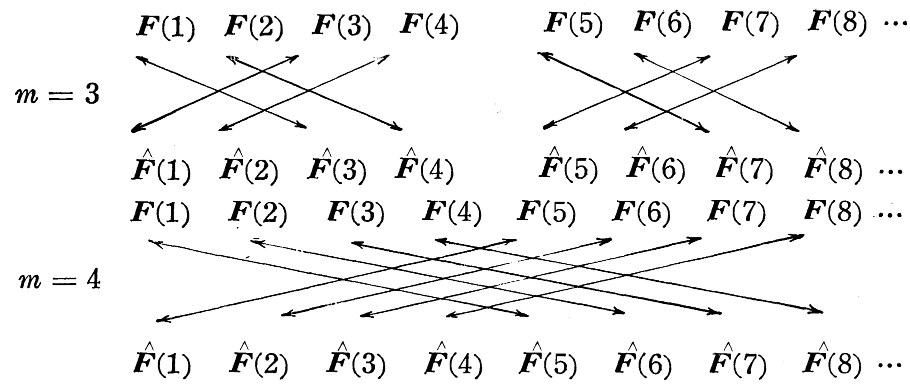
The pair of slits $P(k, 1, 1, n, \nu)$ consists of a slit on $F(k+1)$ and a slit on $\hat{F}(k+1)$ which cover simultaneously a slit $C^{\nu}_{1,n}$.

The pair of slits $P(k, 1, 1, n, \nu)$	consist of a slit on $F(k+1)$	and a slit on $\hat{F}(k+1)$	which cover simultaneously a slit $C^{\nu}_{1,n}$
$P(k, 1, 2, n, \nu)$	$F(2^k+1)$	$\hat{F}(2^k+2)$	$C^{\nu}_{2,n}$
$P(k, 2, 2, n, \nu)$	$F(2^k+2)$	$\hat{F}(2^k+1)$	
$P(k, 1, 3, n, \nu)$	$F(2^{2k}+1)$	$\hat{F}(2^{2k}+3)$	$C^{\nu}_{3,n}$
$P(k, 2, 3, n, \nu)$	$F(2^{2k}+2)$	$\hat{F}(2^{2k}+4)$	
$P(k, 3, 3, n, \nu)$	$F(2^{2k}+3)$	$\hat{F}(2^{2k}+1)$	
$P(k, 4, 3, n, \nu)$	$F(2^{2k}+4)$	$\hat{F}(2^{2k}+2)$	
$\vdots$	$\cdots$	$\vdots$	
$P(k, 1, m, n, \nu)$	$F(2^{(m-1)k}+1)$	$\hat{F}(2^{(m-1)k}+2^{m-2}+1)$	
$P(k, 2, m, n, \nu)$	$F(2^{(m-1)k}+2)$	$\hat{F}(2^{(m-1)k}+2^{m-2}+2)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P(k, 2^{m-2}, m, n, \nu)$	$F(2^{(m-1)k}+2^{m-2})$	$\hat{F}(2^{(m-1)k}+2^{m-1})$	$C^{\nu}_{m,n}$
$P(k, 2^{m-2}+1, m, n, \nu)$	$F(2^{(m-1)k}+2^{m-2}+1)$	$\hat{F}(2^{(m-1)k}+1)$	
$P(k, 2^{m-2}+2, m, n, \nu)$	$F(2^{(m-1)k}+2^{m-2}+2)$	$\hat{F}(2^{(m-1)k}+2)$	
$\vdots$	$\vdots$	$\vdots$	
$P(k, 2^{m-1}, m, n, \nu)$	$F(2^{(m-1)k}+2^{m-1})$	$\hat{F}(2^{(m-1)k}+2^{m-2})$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\tilde{P}(k, 1, 1, n, \nu)$	$F(k+1)$	$\hat{F}(k+1)$	$\tilde{C}^{\nu}_{1,n}$
$\tilde{P}(k, 1, 2, n, \nu)$	$F(2^k+1)$	$\hat{F}(2^k+2)$	$\tilde{C}^{\nu}_{2,n}$
$\tilde{P}(k, 2, 2, n, \nu)$	$F(2^k+2)$	$\hat{F}(2^k+1)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We shall show above correspondence among $\{F(h)\}$ and $\{\hat{F}(h)\}$ by diagrams.

$$m=1 \quad \begin{array}{cccccccccc} F(1) & F(2) & F(3) & F(4) & F(5) & F(6) & F(7) & F(8) & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \hat{F}(1) & \hat{F}(2) & \hat{F}(3) & \hat{F}(4) & \hat{F}(5) & \hat{F}(6) & \hat{F}(7) & \hat{F}(8) & \cdots \end{array}$$

$$m=2 \quad \begin{tikzcd} F(1) \arrow[dr, leftrightarrow] & F(2) \arrow[dl, leftrightarrow] & F(3) \arrow[dr, leftrightarrow] & F(4) \arrow[dl, leftrightarrow] & F(5) \arrow[dr, leftrightarrow] & F(6) \arrow[dl, leftrightarrow] & F(7) \arrow[dr, leftrightarrow] & F(8) \arrow[dl, leftrightarrow] & \cdots \\ \hat{F}(1) & \hat{F}(2) & \hat{F}(3) & \hat{F}(4) & \hat{F}(5) & \hat{F}(6) & \hat{F}(7) & \hat{F}(8) & \cdots \end{tikzcd}$$



Thus we can construct the Riemann surface $\boldsymbol{W}$. Then we shall prove that $\boldsymbol{W}$ is just the required Riemann surface.

Let $u(p)$ be an arbitrary single-valued bounded harmonic function on $\boldsymbol{W}$. We may assume $|w(p)|<1$ without loss of generality. Let $\boldsymbol{W}_m$ $(m=1,2,\cdots)$ be the pseudo-covering subsurfaces, which have a subsurface in common with $\boldsymbol{F}(1)$, over the ring domains $\boldsymbol{R}_m$ respectively, where

$$\boldsymbol{R}_m;\ \frac{1}{2}\left(\frac{\mu_m}{4\mu_m-1}+\frac{4\mu_m+1}{16\mu_m}\right)<|z|<\frac{1}{2}\left[\left(1-\frac{1}{4\mu_m}\right)+\left(1-\frac{1}{4\mu_m+1}\right)\right],\qquad \mu_m=2^{m-1}.$$

Then

$$\boldsymbol{W}_1\subset\boldsymbol{W}_2\subset\cdots\subset\boldsymbol{W}_m\subset\cdots,\ \boldsymbol{W}=\bigcup_{m=1}^{\infty}\boldsymbol{W}_m,$$

and it is clear that $\boldsymbol{W}_m$ is a 2^m sheeted pseudo-covering surface over $\boldsymbol{R}_m$.

Let $T_m(p)$ $(m=1,2,\cdots)$ be the indirectly conformal mappings of $\boldsymbol{W}$ onto itself as follows:

$T_1(p)$ is the mapping	by which a point on $\boldsymbol{F}(k+1)$ over z	corresponds to a point on $\hat{\boldsymbol{F}}(k+1)$ over the same point z.
$T_2(p)$	$\boldsymbol{F}(2^k+1)$	$\hat{\boldsymbol{F}}(2^k+2)$
	$\boldsymbol{F}(2^k+2)$	$\hat{\boldsymbol{F}}(2^k+1)$
$T_3(p)$	$\boldsymbol{F}(2^{2k}+1)$	$\hat{\boldsymbol{F}}(2^{2k}+3)$
	$\boldsymbol{F}(2^{2k}+2)$	$\hat{\boldsymbol{F}}(2^{2k}+4)$
	$\boldsymbol{F}(2^{2k}+3)$	$\hat{\boldsymbol{F}}(2^{2k}+1)$
	$\boldsymbol{F}(2^{2k}+4)$	$\hat{\boldsymbol{F}}(2^{2k}+2)$
$T_4(p)$	$\boldsymbol{F}(2^{3k}+1)$	$\hat{\boldsymbol{F}}(2^{3k}+5)$
	$\boldsymbol{F}(2^{3k}+2)$	$\hat{\boldsymbol{F}}(2^{3k}+6)$
	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$

Put

$$u_m(p) = \frac{1}{2}\Big[u(p) - u(T_m(p))\Big] \quad (m = 1, 2, \cdots),$$

and $u_m(p)$ are single-valued harmonic functions and vanish on the foldings constituted by the edges of pair of slits $P(k, l, m, n, \nu)$ or $\tilde{P}(k, l, m, n, \nu)$, and $|u_m(p)| < 1$.

We can prove as in § 2 that the values $u_m(p)$ on the boundaries of $\boldsymbol{W}_m$ converge uniformly to zero with $m \to \infty$. Therefore all functions $u_m(p)\,(m = 1, 2, \cdots)$ are identically zero on $\boldsymbol{W}_m$. So $u(p)$ assumes the same value on every points on $\boldsymbol{F}(h)\,(h = 1, 2, \cdots)$ over a point z on $\boldsymbol{R}$. This fact means that $u(p)$ has no finite Dirichlet integral on $\boldsymbol{W}$. Therefore by the Virtanen's theorem[6] there is no harmonic function with finite Dirichlet integral on $\boldsymbol{W}$.

On the other hand if we put $U(p) = \log|z|$ for all p over z, then $U(p)$ is a single-valued bounded harmonic function on $\boldsymbol{W}$.

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6) K. I. Virtanen, 1. c.