Osaka Mathematical Journal
Vol. 4, No. 1, May, 1952

On Essential Components of the Set of Fixed Points

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Let X be a compact metric space and let f be a continuous mapping of X into itself. A fixed point p of f was called by M. K. Fort Jr.[1] an essential fixed point of f, if for every neighbourhood U of p there exists $\delta>0$ such that every $g \in X^X$ with $|g-f|<\delta$ has at least one fixed point in U. Then for example, the identity mapping of the interval $[0, 1]$ has no essential fixed point. We shall introduce in this note a notion of essential components (see below) of the set of fixed points: thus if X is an absolute retract[2], then every continuous mapping of X into itself has essential components of the set of fixed points and if X is an absolute neighbourhood retract[3], then every continuous mapping of X into itself which is homotopic to a constant mapping has the same property.

I express my sincere thanks to Prof. H. Terasaka for his valuable advices.

1. Let X be a compact metric space[4] and let f be a mapping[5] of X into itself. Let f have fixed points and let A be the set of all fixed points, C being a component of A. Then C will be called an *essential component* of A, if for every open set U which contains C there exists δ such that every $g \in X^X$ with $|g-f|<\delta$ has at least one fixed point in U. We say that X has *property F'* if every mapping of X into itself has at least one essential component of the set of fixed points.

Theorem 1. *Property F' is invariant under retraction*[6].

Proof. Let Y be a retract of a compact space X having property

1) M. K. Fort Jr.: Essential and nonessential fixed points, Amer. Jour. Math. 72 (1950), pp. 315-322.

2) In the sense of K. Borsuk. See, K. Borsuk: Sur let rétractes, Fund. Math. 17 (1931), pp. 152-170.

3) In the sense of K. Borsuk. See, K. Borsuk: Ueber eine Klasse von lokal zusammenhängenden Räumen, Fund. Math. 19 (1932), pp. 220-242.

4) In this note we assume that the space is separable metric.

5) In this note every mapping means a continuous mapping.

6) Let Y be a closed subset of X. If there exists a mapping r of X onto Y such that $r(x)=x$ for $x \in Y$, then Y is called by K. Borsuk a retract of X and the mapping r, a retraction of X onto Y. Cf. K. Borsuk, Fund. Math. 17. loc. cit.

F' and let r be a retraction of X onto Y. Let f be a mapping of Y into itself. Then fr is a mapping of X into itself. Since X has property F', there exists an essential component C of the set of fixed points of fr. Clearly C is a component of the set of all fixed points of f. If U is an open subset (of Y) which contains C, then there exists an open subset U' (of X) with $U'\cdot Y=U$. It follows that for U' there exists $\delta>0$ such that every $g'\in X^X$ with $|g'-fr|<\delta$ has at least one fixed point in U'. Let g be a mapping of Y into itself with $|g-f|<\delta$. Since $|gr-fr|<\delta$, it follows that gr has at least one fixed point in U'. Clearly this fixed point is contained in Y. Therefore g has at least one fixed point in $U'\cdot Y=U$, and the proof is complete.

Lemma 1. *The Hilbert cube I_ω has property F'.*

Proof. The Hilbert cube has the fixed point property[7]. Let $f\in I_\omega^{I_\omega}$ and let A be the set of all fixed points of f. Let A be decomposed into components C_α. Then it follows that:

(1) $A=\sum_\alpha C_\alpha$,

(2) $C_\alpha\cdot C_\beta=0\ (\alpha\neq\beta)$,

(3) A and all C_α are compact.

If no C_α is essential component, then for every C_α there exists an open set U_α which contains C_α satisfying the following conditions: for every $\delta>0$ there exists $g_\alpha\in I_\omega^{I_\omega}$ with $|g_\alpha-f|<\delta$ having no fixed point in U_α.

It can easily be seen that there exist two finite open coverings $\{V_i\}$ and $\{W_i\}$ $(i=1,2,\dots,n)$ (of A) which satisfy the following conditions:

(4) $\overline{W}_i\subset V_i$,

(5) $V_i\cdot V_j=0$ for $i\neq j$,

(6) V_i contains at least one C_{α_i} with $U_{\alpha_i}\supset V_i$.

Since $I_\omega-\sum_{i=1}^n W_i$ is compact and f has no fixed point on it, there exists an $a>0$ such that $|x-f(x)|>a$ for $x\in I_\omega-\sum_{i=1}^n W_i$. Since V_i contains at least one C_{α_i} with $U_{\alpha_i}\supset V_i$, there exists a mapping g_i with $|g_i-f|<a$ having no fixed point in V_i.

Using vectorial notation, we construct the mapping φ as follows:

$$\varphi(x)=f(x)\quad \text{for}\quad x\in I_\omega-\textstyle\sum_{i=1}^n V_i,$$

$$\varphi(x)=g_i(x)\quad \text{for}\quad x\in W_i,$$

$$\varphi(x)=\frac{d(x,\overline{W}_i)}{d(x,\overline{W}_i)+d(x,I_\omega-\sum_{i=1}^n V_i)}f(x)+\frac{d(x,I_\omega-\sum_{i=1}^n V_i)}{d(x,\overline{W}_i)+d(x,I_\omega-\sum_{i=1}^n V_i)}g_i(x)^{8)}$$

$$\text{for}\quad x\in V_i-W_i.$$

7) See for instance, C. Kuratowski: Topologie II (1950), p. 263

8) $d(x,A)$ means the distance from x to A.

It is easily seen that $|\varphi-f|<a$, and consequently $\varphi\in I^I$ has no fixed point, which is impossible, and the proof is complete.

By Therem 1 and Lemma 1 it follows immediately the

Theorem 2. *Every absolute retract*[9] *has property* F'.

2. **Lemma 2.** *Let* X *be an absolute neighbourhood retract*[10]. *If* $f\in X^{I_\omega}$, *then for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that for every* $g\in X^X$ *with* $|g-f'|<\delta$ *within* $X^X(f'=f|X)$[11] *there exists an extension* φ *of* g *on* I_ω *relative to* X *with* $|\varphi-f|<\varepsilon$.

Proof. Let X be imbedded in I_ω and let f be a mapping of I_ω into X. Since X is an absolute neghbourhood retract, there exist a neighbourhood U of X and a retraction r of U onto X. For $\varepsilon/2$ there exists $\delta'>0$ such that $d(x,X)<\delta'$ yields $|x-r(x)|<\varepsilon/2$.

By a lemma of K. Borsuk[12], for δ' there exist $\delta>0$ such that for every $g\in X^X$ with $|g-f'|<\delta\,(f'=f|X)$ there exists an extension φ' of g on I_ω relative to I_ω with $|\varphi'-f|<\delta'$.

Using this δ, let $g\in X^X$ with $|g-f'|<\delta$. Then there exists an extension φ' which satisfies the above condition. Let $\varphi=r\varphi'$. Then $|\varphi-\varphi'|<\varepsilon/2$. Since $|\varphi'-f|<\delta'\leqq\delta/2$, it follows $|\varphi-f|<\varepsilon$ and φ is an extension of g on I_ω relative to X, and the proof is complete.

Theorem 3. *Let* X *be an absolute neighbourhood retract. If* $f\in X^X$ *is homotopic to a constant mapping, then* f *has at least one essential component of the set of fixed points.*

Proof. Let X be imbedded in I_ω. If $f\in X^X$ is homotopic to a constant mapping, then there exists an extension φ of f on I_ω relative to X[13]. Since I_ω has property F' by Lemma 1, φ has an essential component C of the set of fixed points, and C is at the same time a component of the set of all fixed points of f. Let U be an open subset (of X) which contains C. Then there exists an open subset U' of I_ω with

9) A compact separable metric space is an absolute retract if and only if it is homeomorphic to a retract of I_ω. K. Borsuk, Fund. Math. 17, loc. cit.

10) A closed subset Y of X is a neighbourhood retract of X if there exists an open set U which contains Y and there exists a retraction of U on Y. A compact separable metric space X is an absolute neighbourhood retract if and only if X is homeomorphic to a neighbourhood retract of I_ω. K. Borsuk, Fund. Math. 19, loc. cit.

11) $f|X$ means the partial mapping of f operating only on X.

12) The lemma of K. Borsuk is as follows: let M be a separable metric space, A a closed subset of M and $f\in I_\omega^M$. Then for every $\varepsilon>0$ there exists $\delta>0$ such that for every $g\in I_\omega^A$ with $|g(x)-f(x)|<\delta$ for $x\in A$ there exists an extension φ of g on M relative to I_ω with $|\varphi-f|<\varepsilon$. K. Borsuk, Fund. Math. 19, loc. cit. p. 227.

13) K. Borsuk, Fund. Math. 19, loc. cit. p. 229.

$U' \cdot X = U$. It follows that for U' there exists $\delta' > 0$ such that every φ' with $|\varphi' - \varphi| < \delta'$ has at least one fixed point in U'. For δ' there exists $\delta > 0$ satisfying the condition of Lemma 2. Then for every $g \in X^X$ with $|g - f| < \delta$ there exists an extension φ' of g on I_ω relative to X with $|\varphi - \varphi'| < \delta'$. Therefore φ' has at least one fixed point in U'. Since this fixed point of φ' is contained in X, g has at least one fixed point in $U' \cdot X = U$, and the proof is complete.

PROBLEM. *Does there exist a space which has the fixed point property but which has not property F'?*

(Received December 1, 1951)