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On Cartesian Product of Compact Spaces

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While the Cartesian product of any number of compact (=bicompact) spaces is again compact by Tychonoff's theorem [1], there is an $\aleph_0$-compact (=compact in the sense of Fréchet) space R whose product $R\times R$ is not $\aleph_0$-compact,[1] as will be shown in the present note. These circumstances will be somewhat clarified by the introduction of a concept of $\aleph_\alpha$-ultracompactness.

1. Let M be a given set of points and let $\boldsymbol{M}=\{M_\lambda\}$ be an ultrafilter [2], i.e., a collection of subsets M_λ of M such that

(i) $\boldsymbol{M}$ has the finite intersection property, i.e., any finite number of M_λ's have a non-void intersection,

(ii) $\boldsymbol{M}$ is maximal with respect to the property (i), i.e., should any subset M' of M distinct from any one of M_λ be added to $\boldsymbol{M}$, then the resulting collection $\boldsymbol{M}+M'$ fails to satisfy the condition (i).

If $\aleph_\alpha$ denotes the lowest of the potencies of M_λ, we say that $\boldsymbol{M}$ is of *potency* $\aleph_\alpha$. A T_1-space will be called $\aleph_\alpha$-*ultracompact*, if every ultrafilter of potency $\aleph_\alpha$ has a cluster point. Then the proof of C. Chevalley and O. Frink [3] for Tychonoff's theorem yields at once the following

Theorem. *The Cartesian product of any nnmber of* $\aleph_\alpha$-*ultracompact spaces is itself* $\aleph_\alpha$-*ultracompact.*

Here arises the question, *whether or not, if* R *is* $\aleph_\alpha$-*compact, i.e., if every subset* $M\subset R$ *of potency* $\aleph_\alpha$ *has a cluster point, but if* R *is not* $\aleph_\alpha$-*ultracompact, then the product* ΠR *is not necessarily* $\aleph_\alpha$-*compact.* As a partly solution of this question we construct in the following an example of an $\aleph_0$-compact but not $\aleph_0$-ultracompact space R, whose product $R\times R$ is not $\aleph_0$-compact.

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1) The question whether or not such an $\aleph_0$-compact space exists was raised by M. Ohnishi of Osaka University and answered by me in Sizyo Sugaku Danwakai (June 10, 1947): An example of an $\aleph_0$-compact space R whose product $R\times R$ is not $\aleph_0$-compact (In Japanese). After I had written the present note I have been informed by Ohnishi that the question is originally that of Čech, for which an answer is announced to have been given by Novák in Časopis propěst. mat. a fys. 74 (1950).

2. Let

$$X = (x^1, x^2, \dots, x^n, \dots)$$

be a sequence of x^n which is either 0 or 1. The family $\boldsymbol{X}$ of all such X becomes a Boolean algebra, if we introduce the following assumptions and definitions:

1) X and $Y=(y^1, y^2, \dots, y^n, \dots)$ are to be regarded as equal if and only if

$$x^n = y^n$$

for almost all n.

2) If $\max(x^n, y^n)=u^n$, $\min(x^n, y^n)=v^n$, $1-x^n=w^n$, then

$$X \cup Y = (u^1, u^2, \dots, u^n, \dots)$$
$$X \cap Y = (v^1, v^2, \dots, v^n, \dots)$$
$$X^c = (w^1, w^2, \dots, w^n, \dots)$$

3)
$$0 = (0, 0, \dots, 0, \dots)$$
$$1 = (1, 1, \dots, 1, \dots)$$

A *filter* is by definition a collection of elements $A \in \boldsymbol{X}$ with the finite intersection property, and an *ultrafilter* $\boldsymbol{A}$ is a filter with maximal property. Clearly

Lemma 1. *If* $\boldsymbol{A}$ *is an ultrafilter and if* X *is any element of* $\boldsymbol{X}$, *then either* X *or* X^c (not both) *belongs to* $\boldsymbol{A}$. *Conversely if for any* X *either* X *or* X^c *belongs to a filter* $\boldsymbol{A}$, *then* $\boldsymbol{A}$ *must be an ultrafilter.*

Now let

$$E = (\varepsilon^1, \varepsilon^2, \dots, \varepsilon^n, \dots)$$

and let

$$A_i = (a_i^1, a_i^2, \dots, a_i^n, \dots) \qquad (i = 1, 2, \dots)$$

be a sequence of $\boldsymbol{X}$. We denote by $\varepsilon^n A_n$ the element A_n itself if $\varepsilon^n=1$ and the null element if $\varepsilon^n=0$ and denote further by

$$\sum \varepsilon^n A_n$$

any one of the elements A of $\boldsymbol{X}$ which are $\subset \varepsilon^n A_n$ for all n, i.e. a superior of the elements $\varepsilon^n A_n$ $(n=1, 2, \dots)$. Then we have the following useful

Lemma 2 [4]. *If* $\boldsymbol{A}_n=\{A_\lambda^n\}(n=1,2,\dots)$ *and* $\boldsymbol{E}=\{E_\lambda=(\varepsilon_\lambda^1, \varepsilon_\lambda^2, \dots, \varepsilon_\lambda^n, \dots)\}$ *are ultrafilters, so is* $\boldsymbol{A}=\{\sum \varepsilon_\lambda^n A_\mu^n\}$.

Proof.

(i) First we prove that $\boldsymbol{A}$ is a filter. In fact, if

$$A_1 = \sum \varepsilon_{\lambda_1}^n A_{\mu_1}^n, \; A_2 = \sum \varepsilon_{\lambda_2}^n A_{\mu_2}^n, \; \dots, \; A_m = \sum \varepsilon_{\lambda_m}^n A_{\mu_m}^n$$

are a finite number of elements of $\boldsymbol{A}$, we have by our definition

$$A_1 \cap A_2 \cap \dots \cap A_m \supset \varepsilon^n_{\lambda_1} \cdot \varepsilon^n_{\lambda_2} \cdot \dots \varepsilon^n_{\lambda_m} \cdot A^n_{\mu_1} \cdot A^n_{\mu_2} \cdot \dots A^n_{\mu_m} .$$

Since $\boldsymbol{E}$ and $\boldsymbol{A}$ have the finite intersection property, $\varepsilon^n_{\lambda_1} \cdot \varepsilon^n_{\lambda_2} \cdot \dots \varepsilon^n_{\lambda_m} = 1$ for some n and $A^n_{\mu_1} \cdot A^n_{\mu_2} \cdot \dots A^n_{\mu_m} \neq 0$ for every n, and consequently we have

$$A_1 \cap A_2 \cap \dots \cap A_m \neq 0 .$$

(ii) To prove that $\boldsymbol{A}$ is an ultrafilter, let B be an element of $\boldsymbol{X}$ not contained in $\boldsymbol{A}$. For each n let $\eta^n = 1$ or $= 0$ according as B belongs to or not to $\boldsymbol{A}_n$. Then B can be written in the form

$$B = \sum \eta^n A^n_{\lambda_n} ,$$

where $A^n_{\lambda_n} = B$ in case $\eta^n = 1$. Since by the assumption on B $H = (\eta^1, \eta^2, \dots, \eta^n, \dots)$ is non $\in \boldsymbol{E}$, we have $H^c = E = (\varepsilon^1, \varepsilon^2, \dots, \varepsilon^n, \dots) \in \boldsymbol{E}$, where $\varepsilon^n = 1 - \eta^n$, and consequently B^c must be of the form $\sum \varepsilon^n A^n_{\lambda_n} \in \boldsymbol{A}$. Thus we have shown that for every element X of $\boldsymbol{X}$ either X or X^c belongs to the filter $\boldsymbol{A}$, whence we conclude by Lemma 1 that $\boldsymbol{A}$ must be an ultrafilter, and our lemma is proved.

Corresponding to

$$A = (a^1, a^2, \dots, a^n, \dots)$$

let

$$A' = (a^0, a^1, \dots, a^{n-1}, \dots) ,$$

where a^0 stands either for 0 or for 1. Evidently

Lemma 3. *If* $\boldsymbol{A} = \{A_\lambda\}$ *is an ultrafilter, so is* $\boldsymbol{A}' = \{A'_\lambda\}$.

We call $\boldsymbol{A}'$ the first *transposed ultrafilter of* $\boldsymbol{A}$. In general we can speak of the n-th transposed ultrafilter of $\boldsymbol{A}$ for any given integer $n \, (-\infty < n < +\infty)$, provided that the 0-th transposed ultrafilter is $\boldsymbol{A}$ itself and the n-th transposed ultrafilter of $\boldsymbol{A}$ is the first transposed ultrafilter of the $(n-1)$-th transposed ultrafilter of $\boldsymbol{A}$.

3. We now consider the following Hausdorff space R^*:

(i) First let

$$q_1, q_2, \dots, q_n, \dots$$

be introduced and defined to be a countable set of *isolated points* of R^* distinct from each other.

(ii) To define the remaining points of R^*, first make correspond to every subset Q of $q_1, q_2, \dots$ the element $A = (a^1, a^2, \dots, a^n, \dots)$ of $\boldsymbol{X}$ in such a way that for each n $a^n = 1$ or $= 0$ according as q_n belongs to or not to Q. Every ultrafilter $\boldsymbol{A} = \{A_\lambda\}$ of $\boldsymbol{X}$ is then defined as a *point* a of R^*, the neighbourhood $U_\lambda(a)$ (for each λ) of a being the subset Q of

$q_1, q_2, \ldots$ corresponding to A_λ together with all the ultrafilters $\boldsymbol{B}=\{B_\lambda\}$, $B_\lambda \in \boldsymbol{X}$, which contain A_λ.

4. Now we proceed to the construction of the desired $\aleph_0$-compact space R on the basis of R^*.

Since every cluster point of $q_1, q_2, \ldots$ is by its definition an ultrafilter $\boldsymbol{A}$, the potency of all noints of R^* different from $q_1, q_2, \ldots$ is by Pospisil's theorem [5] equal to $\mathfrak{f}=2^{2^{\aleph_0}}$. Applying our Lemma 2 on a given sequence of distinct points $a_1, a_2, \ldots$ of R^* other than $q_1, q_2, \ldots$, we see immediately that the potency of all cluster points of the sequence $a_1, a_2, \ldots$ is likewise of potency $\mathfrak{f}$.

Following Kuratowski and Sierpiński [6] let

$$(a) \qquad a_0, a_1, \ldots, a_\lambda, \ldots \quad (\lambda < \omega_{\mathfrak{f}})$$

$$(M) \qquad M_0, M_1, \ldots, M_\lambda, \ldots \quad (\lambda < \omega_{\mathfrak{f}})$$

be transfinite sequences of all points of R^* other than $q_1, q_2, \ldots$ and of all countable subsets M_λ of R^* respectively, where $\omega_{\mathfrak{f}}$ denotes the first ordinal number of potency $\mathfrak{f}$.

Of all cluster points of M_0 let a_ν be the first one which appears in the transfinite sequence (a) and call a_ν as well as the n-th transposed ultrafilters for all even n *points of class* 1. The rest of all transposed ultrafilters of a_ν will be called *points of class* 2.

Suppose that for every ordinal number $\mu(\eta < \lambda < \omega_{\mathfrak{f}})$ points of class 1 and class 2 have been suitably defined and consider M_λ. Of all the cluster points of M_λ which have not been previously defined as points of class 1 or class 2, let a_ρ be the first one which appears in the transfinite sequence (a) and define as above points of class 1 and class 2.

Let R be the subspace of R^* consisting of all points of class 1 together with all isolated points $q_1, q_2, \ldots$ of R^*. We shall show that R possesses the property we are seeking for.

First R is $\aleph_0$-compact, for if M is a countable subset of R, then M is a member of the sequence of (M), say M_λ, and the cluster point a_ρ considered above is just a cluster point of M in R.

To prove that $R \times R$ fails to be $\aleph_0$-compact, let the points of $R \times R$ be represented by (x, y), where $x, y \in R$. Then the sequence of points Q:

$$(q_1, q_2),\ (q_3, q_4),\ \ldots,\ (q_{2n-1}, q_{2n}),\ \ldots$$

has no cluster point in R. In fact, if Q should have a cluster point (a, a'), then a' must be the first transposed ultrafilter of a and consequently a and a' could not be points of R at the same time, which is absurd.

Thus we have proved that R is the required $\aleph_0$-compact space, whose product $R \times R$ is not $\aleph_0$-compact.

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Bibliography

[1] A. Tychonoff: Über die topologische Erweiterung von Räumen, Math. Ann. 102 (1930), 544-561.

[2] H. Cartan: Filtres et ultrafiltres, C. R. Paris 205 (1937).

[3] C. Chevalley and O. Frink: Bicompactness of Cartesian products, Bull. Amer. Math. Soc. 47 (1941), 612-614.

[4] H. Terasaka: Über die Darstellung der Verbände, Proc. Imp. Acad. Japan 14 (1938), 306-311.

[5] B. Pospišil: Remark on bicompact spaces, Ann. of Math. 38 (1937), 845-846.

[6] C. Kuratowski and W. Sierpiński: Sur un problème de M. Fréchet concernant les dimensions des ensembles linéaires, Fund. Math. 8 (1926), 193-200.