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Note on Locally Compact Groups

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§1. The purpose of this note is to study the problem proposed by C. Chevalley: Is it true that a locally compact group which has no arbitrarily small[1] subgroup is a Lie group?

Concerning the above problem two theorems will be proved in this note. One of them is:

Theorem 1. *A locally euclidean group G which has a neighbourhood of the identity containing no non-trivial subgroup, has a neighbourhood $\tilde{U}$ of the identity, through any point of which there exists one and only one one-parameter subgroup*[2].

The other is:

Theorem 2. *If $(U_n)^n$ is contained in $\tilde{U}$, then G is a Lie group, where U_n denotes the aggregate of the n-th roots of elements in a neighbourhood U.*

§2. For an element x of a neighbourhood U of the identity e we denote by $\delta_U(x)$ the smallest number n such that $x^{2^n} \in U$. The group G is said to have the property (S) if there exists a neighbourhood U of e such that $\delta_U(x) < \infty$ for every x in $U-\{e\}$. According to Kuranishi[3] a locally euclidean group G which has the property (S), has a neighbourhood of the identity e, through any point of which one can draw one and only one one-parameter subgroup. Therefore we have only to show that a locally compact group which has no arbitrarily small subgroup has the property (S).

In order to prove this we shall need the following Lemma.

Lemma 1. Let W be a neighbourhood of the identity e which contains no non-trivial subgroup in it. For an arbitrarily small neigh-

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1) A small subgroup means a subgroup contained in a sufficiently small neighbourhood of the identity.

2) This theorem was proved with the co-operation of Dr. Gotô. Cf. the forthcoming Nagoya Math. Journal.

3) See Kuranishi: *Differentiability of locally compact groups*, Nagoya Math. Journal Vol. 1, 1950, 71-81.

bourhood U of e there exists a neighbourhood U^* of e such that if $x, x^k \in U^*$ and $x^i \in W$ for all $i \leqq k$, then $x^i \in U$ for all $i \leqq k$.

Proof.

Put

$$X = \{x^k;\ x^k \in U,\ x^i \in W \ \textit{for all}\ i \leqq k,\ x^j \bar{\in} U \ \textit{for some}\ j \leqq k.\}$$

If $U - X$ does not contain any neighbourhood of e, then there exists a sequence $\{a_n\}$ such that

$$a_n^{k_n} \to e$$

and

$$a_n^{j_n} \bar{\in} U,$$

with

$$j_n \leqq k_n.$$

We may suppose that $a_n^{j_n}$ converges to $\bar{a} \in \overline{W}$.

For an integer r we can easily find integers r_n such that

$$r_n \equiv r j_n \quad (\textit{mod}\ k_n)$$
$$0 < r_n \leqq k_n.$$

Then $\{a_n^{r_n}\}$ converges to $\bar{a}^r$ because

$$a_n^{r_n} = a_n^{r j_n} \cdot a_n^{p_n k_n},$$

where p_n are integers whose absolute values are less than r.

Let us denote by A the aggregate of limit points $\bar{a}^r$. A is clearly a non-trivial subgroup of G contained in W, because $A \ni \bar{a} \neq e$. This contradicts the hypothesis and whence we complete the proof of Lemma 1.

Now we shall have the

Theorem 1′. *A locally compact group which has no arbitrarily small subgroup, has the property* (S).

Proof. Let us take a neighbourhood U of e such that $U^2 \subset W$. Let V be a neighbourhood of e contained in U^*. If $x, x^2, x^3, \ldots x^{2^j} \in U$ and $x^{2^{j+1}} \in V$, then clearly $x^i \in W$ for all $i \leqq 2^{j+1}$ and by Lemma 1 $x^i \in U$ for all $i \leqq 2^{j+1}$. Therefore for a large j, $x^{2^j} \bar{\in} V$, which shows that G has the property (S).

§ 3. Concerning Theorem 2 we shall also use the Kuranishi's[4] results. He proved that if $(x^{1/n} y^{1/n})^n$ converge uniformly to a continuous function over $U \times U$, then G is a Lie group.

We shall need some preparatory lemmas.

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4) l. c. 3).

Lemma 2. Under the assumption of Theorem 2 a metric $\rho(x, y)$ can be defined so that $\rho(x^{\lambda}, e)$ may be differentiable at $\lambda = 0$.

Proof. At first we must define the metric. Without loss of generality $\tilde{U}$ may be taken as a symmetric one.

Let y be an element of the boundary $Bd(\tilde{U})$ of $\tilde{U}$ [5], and let U be a neighbourhood of e such that

$$U^2 \subset \tilde{U}.$$

Let us define a metric $\rho(x, y)$ in U such that

$$\rho(x, e) = \inf_{(y)} \sum |\lambda_i|$$

$$\rho(x, z) = \rho(z^{-1}x, e)$$

where inf (y) means the infimum of $\sum |\lambda_i|$ when we take an arbitrary decomposition

$$x = y_1^{\lambda_1} y_2^{\lambda_2} \dots y_p^{\lambda_p}$$

for a suitable y_i's $\in Bd(\tilde{U})$ and real λ_i's.

It is clear that

$$\rho(x, y) = \rho(y, x)$$

$$\rho(x, y) + \rho(y, z) \geqq \rho(x, z).$$

This metric $\rho(x, y)$ satisfies the metric conditions when we prove that $\rho(x, y) = 0$ implies $x = y$. We may suppose that $y = e$ and G is connected.

For a sufficiently large s

$$x^s \bar{\in} \tilde{U}.$$

We denote by $t(n)$ the smallest integer such that

$$(U_n)^{t(n)} \ni x.$$

Then

$$st(n) > n,$$

$$t(n)/n > 1/s.$$

From this inequality we shall have easily

$$\rho(x, e) > 0.$$

Hence we have proved that $\rho(x, y)$ satisfies the metric conditions. From the definition we can easily see that

5) We may assume that $Bd(\tilde{U})$ intersects in only one point with any one-parameter semi-group. See H. Whitney: *On regular family of curves*, Bull. Amer. Math. Soc. 47, 145-147 (1941).

$$\rho(x^{\lambda}, e) \leqq \lambda$$

for $x \in U$.

Then $\rho(x^{\lambda}, e)$ is differentiable because $\rho(x^{\lambda}, e)$ is subadditive and $\rho(x^{\lambda}, e)/\lambda \leqq 1$[6].

§4. Now let us study some properties of this metric $\rho(x, y)$.

Lemma 3. The metric $\rho(x, y)$ has the following properties:

i) $\rho(x, y)$ is left invariant,

ii) $K_2\rho(y, e) \leqq \rho(yx, x) \leqq K_1\rho(y, e)$,[7]

iii) if $\rho(x, e) = O(\mu)$, then

$$\frac{\rho(x^{-1}yx, e)}{\rho(y, e)} = 1 + O(\mu).$$

Proof. i) is evident. From the definition of this metric we have for some y_i's $\in Bd(\tilde{U})$, real λ_i's and an arbitrarily small ε,

$$\left|\rho(y, e) - \sum|\lambda_i|\right| \leqq \varepsilon$$

and

$$y = y_1^{\lambda_1} y_2^{\lambda_2} \dots y_p^{\lambda_p}.$$

Let us consider a real number $s(y)$ with

$$x^{-1}y^{s(y)}x \in Bd(\tilde{U}).$$

Then for every i

$$0 < K_2 \leqq s(y_i) \leqq K_1 < \infty.$$

Hence by simple calculations we obtain

$$K_2\rho(y, e) \leqq \rho(x^{-1}yx, e) \leqq K_1\rho(y, e).$$

Thus the proposition ii) is proved.

In case $\rho(x, e) = O(\mu)$,

$$s(y) = 1 + O(\mu),$$

because

$$\frac{\rho(x^{-1}yx, e)}{\rho(y, e)} - 1 \leqq \frac{\rho(x, e) + \rho(y^{-1}xy, e)}{\rho(y, e)} \leqq \frac{O(\mu)}{\rho(y, e)}.$$

Hence the proposition iii) is proved and we complete the proof of Lemma 3.

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6) See, Einar Hille: *Functional Analysis and Semigroups,* Amer. Math. Soc. Coll. Pub. p. 143.

7) In this note K_i's are all absolute constants. They could be taken near to 1 except K_4.

§ 5. From the following relations

$$x(xaxa^{-1})\,x^{-1}(xaxa^{-1}) = x^2ax^2a^{-1},$$

we have

$$\rho(xaxa^{-1}, e)\,(1+C(x)) \geqq \rho(x^2ax^2a^{-1}, e),$$

where $C(x)$ denotes

$$\sup_y \frac{\rho(x^{-1}yx, e)}{\rho(y, e)}$$

If $\delta_{\tilde{U}}(a)$ and $\delta_{\tilde{U}}(x)$ are $\leqq n$,
then

$$\begin{aligned}&\rho(xaxa^{-1}, e)\prod_{i=1}^{n}(1+C(x^{2^i}))\\ &\geqq \rho(x^{2^n}ax^{2^n}a^{-1}, e)\\ &\geqq \rho(x^{2^{n+1}}, e) + O(1/2^n)\\ &\geqq 2^{n+1}K_3\rho(x, e)\ .\end{aligned}$$

It is possible to take our neighbourhood $\tilde{U}$ so small as to make K_3 sufficiently near to 1.

On the other hand

$$\prod^{n}(1 + C(x^{2^i}))/2^n$$

can be taken sufficiently near to 1 too.

Then we have

$$\rho(xaxa^{-1}, e) \geqq K_4\rho(x, e)$$

with $K_4>1$.

Put

$$\begin{aligned}x_1 &= x & a_1 &= a\\ &\vdots & &\vdots\\ x_i &= x_{i-1}a_{i-1}x_{i-1}a_{i-1}^{-1}, & a_i &= a^{2^i}\\ &\vdots & &\vdots\end{aligned}$$

Then by simple calculations

$$\rho(x_p, e) \geqq K_4^p\rho(x, e)$$

or

$$\rho((xa)^{2^p}a^{-2^p}, e) \geqq K_4^p\rho(x, e).$$

Now take $x^{1/2^q}y^{1/2^p}x^{-1/2^q}y^{-1/2^p}$ for x and $y^{1/2^p}$ for a in the above inequality.

Then

$$\begin{aligned}&\rho(x^{1/2^q}yx^{-1/2^q}y^{-1}, e)\\ &\geqq K_4^p\rho(x^{1/2^q}y^{1/2^p}x^{-1/2^q}y^{-1/2^p}, e).\end{aligned}$$

This means that

$$2^q \rho\left(x^{1/2^q} y^{1/2^p} x^{-1/2^q} y^{-1/2^p}, e\right)$$

converge to zero when p and q increase to ∞.

Consequently

$$\rho\left(x^{\mu} y^{\lambda} x^{-\mu} y^{-\lambda}, e\right) / \mu$$

converge to zero when λ and μ decrease to zero.

§ 6. Proof of Theorem 2.

Put

$$F_n(x, y) = (x^{1/n} y^{1/n})^n .$$

Then

$$\begin{aligned} & \rho\left(F_n(x, y), \quad F_{np}(x, y)\right) \\ = & \rho\left((x^{1/n} y^{1/n})^n, \quad (x^{1/np} y^{1/np})^{np}\right) \\ \leqq & K_1 n \rho\left(x^{1/n} y^{1/n}, \quad (x^{1/np} y^{1/np})^p\right) \\ \leqq & K_1 n \sum_{i=0}^{p-1} \rho\left(y^{1/np}, x^{-i/np} y^{1/np} x^{i/np}\right) \end{aligned}$$

By the inequality obtained in the last chapter

$$\begin{aligned} & \leqq K_1\, np/np \cdot \varepsilon_n \\ & = K_1 \varepsilon_n , \end{aligned}$$

where ε_n converges to zero when $n \to \infty$.

Hence we proved that $F_n(x, y)$ converge uniformly in $U \times U$, which completes the proof of Theorem 2.

To conclude this note, the author wishes to thank Dr. Gotô and Dr. Kuranishi for their valuable suggestions and advices.

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Added in proof. Theorem 1 is true when the group is locally connected. The proof will be given in the next number of this journal.