Osaka Mathematical Journal,
Vol. 3, No. 1, March, 1951

# *Some Remarks on Unitary Representations of the Free Group*

By Hisaaki YOSHIZAWA

§ 1. INTRODUCTION. This paper contains three (at least superficially) independent remarks concerning "pathological" phenomena appearing in the theory of unitary representations.[1] In order to show the phenomena in their extreme forms and to simplify arguments, we make the examples of the free group of two generators, though some of these phenomena are not peculiar to the free group and may be more significant and interesting in cases of more "regular" groups.

In § 2 we shall consider the following property:

(K) The constant 1 is uniformly approximated on compact by the convolution of function belonging to $L^2(G)$ with its adjoint.[2]

As shown by Godement,[2] for a unimodular (= with a two-sided invariant measure) group, (K) is equivalent with the property that every positive definite function is similarly approximated or that the integral of an absolutely integrable positive definite function is non-negative. It is of important meaning in the theory of unitary representations,[3] but recently it is known to be valid only in a rather special class of locally compact groups among general ones. In particular, the property (K) is very simply disproved for the free group.[4]

In § 3 it will be shown for the free group that, in the set $\boldsymbol{P}_1$ of

1) For the matters connected with those of §§ 2 and 3 of this paper, which were studied during 1947-8, the athour is much indebted to Professor Kakutani, who, however, is in no way related to the present paper itself which is written in 1950.

2) See A. Weil's book "L'intégration dans les groupes topologiques et ses applications" and R. Godement: *Les fonctions de type positif et la théorie des groupes*, Trans. Amer. Math. Soc., **63**. A. Khintchine proved it for the case of the real number group.

3) Moreover from it follow some interesting properties of the group, e.g., the symmetricity of the $L^1$-algebra and the existence of an invariant mean for bounded continuous functions.

4) The author obtained these results in the winter of 1947-8, but afterwards he was informed that essentially the same result as in § 2 had been published (by a lecture) already in 1944 by Professor Kakutani.

positive definite functions with the value 1 at the identity element with the $L^1$-weak topology (or, what is the same, with the topology of the ordinary convergence), extreme ones constitute a dense subset. Moreover there exists an irreducible representation $(U, H)$ such that the set $\boldsymbol{P}_U$ of the positive definite functions $f_\varphi(s)$ of the form $(U_s\varphi, \varphi)$ $(\varphi \in H)$ is dense in $\boldsymbol{P}_1$. It may be a problem for a general group whether such an aggregate $\boldsymbol{P}_U$ ($U$ being an irreducible representation of $G$) is closed in the weak topology or in the uniform topology.[5] We shall also construct an example on the free group which negatively answers this problem in the uniform topology; that is, there exists an irreducible representation $(U, H)$ such that $(U_s\varphi, \varphi)$ converges uniformly to the constant 1 for some sequence of $\varphi \in H$.

In §4 decompositions of the regular representation of the free group into irreducible components will be considered. By making use of a maximal commutative subgroup, the regular representation is decomposed into a concrete type of the direct integral of *irreducible* (in the ordinary sense) unitary representations. Then, for a suitable pair of commutative subgroups, the corresponding decompositions contain entirely different irreducible components; i.e., any of those irreducible representations contained in the regular representation in one manner of decompodition is not unitary equivalent with any of those appearing in the other decomposition. (Moreover, all the irreducible components appearing in such a decomposition are mutually inequivalent.) Thus a uniqueness assertion of the typeof Hellinger-Hahn theorem does not remain valid for our case of the free group, so long as we adhere to the usual definitions of irreducibility, equivalency, etc.

In the following, $G$ always denotes the free group with two generators $a$, $b$.

## § 2. APPROXIMATION OF POSITIVE DEFINITE FUNCTIONS

*Let $f(x)$ be an arbitrary square integrable function on $G$ (= the free group of two generators $a$, $b$). Then it is impossible that for a sufficiently small $\varepsilon$ the following conditions be satisfied simultaneously:*

$$(1)\qquad \begin{cases} ff^*(e) = 1, \\ |ff^*(a) - 1| < \dfrac{\varepsilon^2}{2}, \\ |ff^*(b) - 1| < \dfrac{\varepsilon^2}{2}, \end{cases}$$

5) Godement, ibid. stated a problem of this type, and also showed by an example that the limit function of extreme ones is not always extreme. It was the impulse for the results in § 3.

*where* $f^*(s)=\overline{f(s^{-1})}$ *and the product-type notation means the convolution.*

PROOF. From (1) follows that

$$(2)\qquad \begin{cases} \int |f(s)-f(as)|^2 ds < \varepsilon^2, \\ \int |f(s)-f(bs)|^2 ds < \varepsilon^2. \end{cases}$$

For: $\int |f(s)-f(as)|^2 ds$

$$=\int f(s)\overline{f(s)}ds+\int f(as)\overline{f(as)}ds-\int f(s)\overline{f(as)}ds-\int \overline{f(s)}f(as)ds$$

$$=2\int |f(s)|^2 ds-2\Re(ff^*(a))$$

$$\leqq 2|ff^*(a)-1|<\varepsilon^2, \qquad \text{etc.}$$

Now suppose that (2) holds for some $\varepsilon$. Let

$$P=(s;\ s=a^p b^q\ldots\ldots,\ p\neq 0),\qquad N=G-P;$$

and

$$\mu(E)=\int_E |f(s)|^2 ds$$

for arbitrary subset $E$.

Then $$|\mu(a^{-1}E)-\mu(E)|=\left|\int_E |f(as)|^2 ds-\int_E |f(s)|^2 ds\right|$$

$$\leqq 2\cdot\sqrt{\int |f(as)-f(s)|^2 ds}<2\varepsilon,\text{[6)]}$$

i. e.,

$$\mu(a^{-1}E)>\mu(E)-2\varepsilon.$$

Put

$$E=N,\ a^{-1}N,\ a^{-2}N,$$

successively. Then, since these sets are disjoint each other, we have

$$1\geqq\mu(N)+\mu(a^{-1}N)+\mu(a^{-2}N)>3\mu(N)-6\varepsilon,$$

that is, $$\mu(N)<\frac{1}{3}+2\varepsilon.$$

Analogously we have

$$\mu(P)<\frac{1}{3}+2\varepsilon;$$

and it must hold

$$\mu(N)+\mu(P)=1<\frac{2}{3}+4\varepsilon$$

and consequently

$$\varepsilon>\frac{1}{12}. \qquad \text{Q.E.D.}$$

---

6) For: $||\alpha|^2-|\beta|^2|=||\alpha|+|\beta||\cdot||\alpha|-|\beta||\leqq||\alpha|+|\beta||\cdot|\alpha-\beta|.$

§3. EXTREME POSITIVE DEFINITE FUNCTIONS. (1°) Let $\boldsymbol{P}_1$ be the set of those positive definite functions $\varphi(\cdot)$ on $G$ for which $\varphi(e)=1$, the topology in $\boldsymbol{P}_1$ being defined by the ordinary convergence. We shall construct *an irreducible representation* $(U(s), H)$ *such that the aggregate of*

$$f_\xi(s)=(U(s)\xi,\ \xi),\quad \xi\in H$$

*is dense in* $\boldsymbol{P}_1$.

For this poupose we use the following two lemmas which are easily verified.

LEMMA 1. *For any unitary operator* $V$, *there exists a* $V'$ *which is uniformly close to* $V$ *and has simple pure point spectrum.*

LEMMA 2. *Let* $V$, $W$ *have simple pure point spectra with the proper vectors* $\varphi_m$, $\psi_n$ *respectively. If* $(\varphi_m,\ T\psi_n)\neq 0$ *for any* $m$, $n$, *then* $V$ *and* $TWT^{-1}$ *reduce no common manifold.*[7]

Now, let us choose a countable system of representations $(W_n(s), H_n)$ such that

$$f_n(s)=(W_n(s)\xi_n,\ \xi_n),\quad n=1,\ 2,\ \ldots\ldots$$

are dense in $\boldsymbol{P}_1$. Then construct the direct sum $(W(s), H)$ of these representations, each being of countable multiplicity:

(H) $\qquad H=H_{11}\oplus H_{12}\oplus H_{21}\oplus H_{13}\oplus\ldots\oplus H_{np}\oplus\ldots,\quad (H_{np}=H_n);$

and, applying LEMMA 1, transform each $W_{np}(a)$ and $W_{np}(b)$ in a suitable manner so that we obtain $(V(s), H)$ such that the operators $V(a)$, $V(b)$ in $H$ have simple pure point spectra with the proper vectors

$$\varphi^1_{np},\ \varphi^2_{np},\ \ldots \text{ and } \psi^1_{np},\ \psi^2_{np},\ \ldots \text{ in } H_{np},\ \text{resp.},$$

and

$$f^{(np)}(s)=(V_{np}(s)\xi_{np},\ \xi_{np}),\quad n,p=1,2,\ \ldots\ldots$$

are still dense in $\boldsymbol{P}_1$.

Then, if we can choose a unitary operator $T$ on $H$ which satisfies the following two conditions, the representation $U(s)$ generated by $V(a)$, $TV(b)T^{-1}$ is the required one (by LEMMA 2):

(1) $\qquad (T\psi^k_{np},\ \varphi^j_{nq})\neq 0$ for any $n$, $m$, $p$, $q$, $k$, $j$,

(2) $f^{(np)}(s)-f^{(H)}_{np}(s)\equiv(V_{np}(s)\xi_{np},\ \xi_{np})_H-(U(s)\xi_{np},\ \xi_{np})_H$ convreges to 0 weakly as $p\to\infty$, for each $n$.

For this, rearrange the squences of the proper vectors $(\varphi^k_{np};\ n,\ p,\ k=1,\ 2,\ \ldots)$ in a certain manner, e.g.,

7) The author owed this formulation to Mr. H. Anzai.

$$\varphi_1^{(0)} = \varphi_{11}^1,\ \varphi_2^{(0)} = \varphi_{11}^2,\ \varphi_3^{(0)} = \varphi_{12}^1,\ \varphi_4^{(0)} = \varphi_{11}^3,\ \ldots\ldots,$$

and so with the $\psi$'s. And define $T_l$ $(l=1, 2, \ldots)$ successively on $H$ so that

$$(T_l\psi_k^{(l-1)}, \varphi_j^{(0)}) \neq 0 \quad \text{for} \quad k, j = 1, 2, \ldots, l,$$

where

$$\psi_k^{(l-1)} = T_{l-1}\psi_k^{(l-2)}$$

and that $T=$ the identity, in a tail of the series (H), i.e., on those $H_{np}$ which do not contain any of $\varphi_j^{(0)}$, $1 \leq j \leq l$. If here we make $T_l$ uniformly tend to the identity operator *sufficiently* rapidly, then

$$T_l T_{l-1} \ldots\ldots T_1$$

uniformly converges to a $T$ which satisfies the condition (1).

Ad (2): Fix a $\xi_{np}$. It suffices to choose a $T$ such that, for every $s$ of a given finite subset of $G$, the corresponding operator

$$U(s) = V^{\alpha}(a)\cdot TV^{\beta}(b)T^{-1}\cdot V^{\gamma}(a)\cdot\ldots\cdot TV^{\delta}(b)T^{-1}$$

transforms $\xi_{np}$ sufficiently close to $V(s)\xi_{np}$. Consider the first $T_l$ which non-trivially operates on $H_{np}$. The effect of the preceding $T_j$'s on $U(s)\xi_{np}$ can be made arbitrarily small if we choose $T_l$ sufficiently close to the identity. Therefore, again if $T_l \to I$ rapidly enough, the condition (2) is satisfied.

(2°) We shall sketch the construction of *a non-trivial irreducible representation* $(U_s, H)$ *such that*

$$f_\varphi(s) = (U_s\varphi, \varphi)$$

*uniformly (on G) converges to the constant* 1 *for a suitable sequence* $\varphi$.

Let $V$ be a unitary operator on $H$ having a pure point spectrum, and let $\gamma_0$ $(=1)$ and $\gamma_m$ $(m=1, 2, \ldots, \gamma_m \neq 1)$ be its proper values, 1 being of countable multiplicity and $\gamma_m$ simple, and $\theta_n$ $(n=1, 2, \ldots)$, $\psi_m$ be the corresponding proper vectors.

We shall choose a suitable transformation $T$ such that $U_a = V$ and $U_b = TVT^{-1}$ generate an irreducible representation of $G$ and that

(1) $\qquad (U_s\theta_n, \theta_n)$ uniformly converges to 1 as $n \to \infty$.

For this, we first arrange the proper vectors $\theta_n$, $\psi_m$ in a sequence:

$$\varphi_1, \varphi_2, \ldots\ldots,$$

and set the following condition on $T$:

(2) $\qquad (T\varphi_n, \varphi_m) \neq 0$ for any $m$, $n$.

Such a $T$ is obtained as in (1°), and moreover the request (1) is easily verified. But because of the multiplicity of the spectrum of $V$, in order to assert the irreducibility of $U_s$ we need some extra conditions, e.g.,

(3) $\sum_{n=1}^{\infty} c_n \cdot T\theta_n$ does not belong to the manifold $N$ spanned by $(\theta_k;\ k=1, 2, \ldots)$, for any $c_n$ with $\sum_{1\leqq n<\infty} |c_n|^2 < \infty$,

and

(4) The projection of $T\psi_m$ on $N$ $(m=1, 2, \ldots)$ spans $N$.

These conditions are fulfilled if, in the notations used in (1°), $\| T_l - I \| \to 0$ rapidly enough. As for (3), i.e., in order to satisfy the condition

$$\sum_{1\leqq n<\infty} c_n \cdot (T\theta_n)' \neq 0,$$

where $(\xi)'$ denotes the projection of $\xi$ on the complementary manifold to $N$, choose $T_j$'s $(j=1, \ldots, l;\ l$ fixed) so that

$$(\theta_1^{(l)})',\ (\theta_2^{(l)})',\ \ldots,\ (\theta_k^{(l)})'$$

are linearly independent, and put

$$p_k = \inf\left( \| \sum_{i=1}^{k} c_i (\theta_i^{(l)})' \| ;\ \sum_{i=1}^{k} |c_i|^2 = 1 \right),$$

and let the magnitude $\| T_{l+1} - I \|$ be restricted by means of $p_k$.

The condition (4) can also be verified by the analogous way of step-by-step considerations.

§ 4. DECOMPOSITIONS OF THE REGULAR REPRESENTATION. (1°) Let $H = L^2(G)$, and for $p \in G$, $x \in H$, define

$$V_p x = x_p, \quad \text{where} \quad x_p(s) = x(sp^{-1}).$$

Consider the representation $(V, H)$, using the natural orthonormal system $S = (s;\ s \in G)$ in $H$: Let $H$ be represented by the coordinates $S$, and let $U_p$ be the matrix corresponding to $V_p$. Then $U_p$ generates a permutation in $S$:

$$s \to sp,$$

and $(U_p, H)$ is the right regular representation of $G$. We shall consider the decomposition of $(U_p, H)$.

Let $A=(a^n; -\infty<n<\infty)$. ($G$ is generated, as before, by $a$, $b$.) Let, for every $\theta: |\theta|=1$,

$$\varphi_\theta(s)\begin{cases} =\theta^n, & \text{for } s=a^n \\ =0, & \text{otherwise}; \end{cases}$$

then $\varphi_\theta(s)$ is a positive definite function on $G$. And let $\Omega=\Omega_A$ be the set of $\varphi_\theta$ ($|\theta|=1$), which is identified with the character group of $A$.

For every $\varphi_\theta$ we construct the (anti-)representation $(V^\theta_s, H^\theta)$ of $G$ using the $L^1$-algebra, by the method of Gelfand-Raikov[8]: An element $x$ of $L^1(G)$ is mapped into $H^\theta$, the image being designated as $(x)$ or $x_\theta$; the inner product is defined as $(x_\theta, y_\theta)_\theta=\int_G \overline{\varphi_\theta(s)}xy^*(s)ds$; and $V^\theta_s x_\theta = (V_s x)_\theta = (x_s)_\theta$.

(2°) Realization of $(V^\theta, H^\theta)$. Let us consider the image $(s)$ of $s\in S$ in $H^\theta$. It is easily verified that $(s)\perp(t)$ (= orthogonal in $H^\theta$) if $st^{-1}\notin A$, or $s\neq a^n t$ for any $n$ and that

$$(a^n s)=\overline{\theta^n}\cdot(s).$$

Let

$$\Gamma=(s;\ s=b^m\ldots b^n,\ m\neq 0,\ n\neq 0)$$

and

$$\Sigma=(s;\ s=b^m\ldots\ldots,\ m\neq 0, \text{ or } s=e).$$

Then $(\Sigma)_\theta=(s_\theta; s\in\Sigma)$ is common for all $\theta: (\Sigma)_\theta=(\Sigma)$, and it constitutes a complete orthonormal system in $H^\theta$. Using this system, consider the representation $(U^\theta_s, H^\theta)$ by the matrix $U^\theta_s$. Since $s\to sp$ under $U_p$ $(s\in S, p\in G)$, $(s)(\in H^\theta)$ is transformed to $(sp)$ by $U^\theta_p$; hence for any pair $(s)$, $(t)\in(\Sigma)$,

$$U^\theta_{s^{-1}t}(s)=(t).$$

In particular, $(e)$ or $((sa^n); -\infty<n<\infty)$ for any $s\in\Gamma$ spans a manifold invariant under $U^\theta_a$:

$$U^\theta_a(e)=\theta\cdot(e),$$

$$U^\theta_a(sa^n)=(sa^{n+1});$$

hence $\theta$ is the only (simple) proper value of $U^\theta_a$ and $(e)$ is its only

---

8) I. Gelfand and D. Raikov: *Irreducible unitary representations of locally compact groups*, Mat. Sbornik, **13**. Or see Godement, 2) or the present author's paper in this Journal, Vol. 1, No. 1.

proper vector.

$U_b^\theta$ is generated by a permutation of $(\Sigma)$, without an element of finite order. Therefore $U_b^\theta$ *has no proper value.*

(3°) IRREDUCIBILITY AND INEQUIVALENCY.

$(U_s^\theta, H^\theta)$ *is irreducible.*

PROOF. Suppose that $T$ commutes with $U_p^\theta$ $(p \in G)$. Then in particular

$$U_a^\theta T(e) = TU_a^\theta(e) = \theta T(e).$$

Therefore it must be that

$$T(e) = \gamma(e) \quad \text{for some complex number } \gamma.$$

Let $(s) \in (\Sigma)$; then, since $U_s^\theta(e) = (s)$,

$$U_s^\theta T(e) = TU_s^\theta(e) = T(s)$$

and, on the other hand,

$$U_s^\theta T(e) = U_s^\theta \gamma(e) = \gamma(s).$$

That is,

$$T(s) = \gamma(s) \quad \text{for} \quad (s) \in (\Sigma). \qquad \text{Q.E.D.}$$

*If* $|\theta| = |\eta| = 1$ *and* $\theta \neq \eta$ *then* $(U^\theta, H^\theta)$ *and* $(U^\eta, H^\eta)$ *are not unitary equivalent.*

PROOF. Suppose that for some unitary $T$

$$TU_s^\theta = U_s^\eta T.$$

Then

$$U_a^\eta Te_0 = TU_a^\theta e_0 = \theta Te_0 \;;$$

but this is impossible since $Te_0 \neq 0$ (in $H^\eta$) and $\theta \neq \eta$. Q.E.D.

(4°) DIRECT INTEGRAL. Let

$$\Xi = \Omega \otimes (\Sigma) \,.$$

We shall show that $(U_s, L^2(G))$ is decomposed into the direct integral of $(U_s^\theta, H^\theta)$. For every $x \in L^1(G)$ (or rather, $\in L^1(S)$), consider $x_\theta$ as a sequence over $(\Sigma)_\theta$ i.e., $x \sim x(\theta, \sigma)$, $\sigma$ running over $(\Sigma)$. Now we shall show that

$$\int_G |x(s)|^2 ds = \int_\Omega \int_{(\Sigma)} |x(\theta,\sigma)|^2 d\sigma d\theta, \text{ for } x \in L^1(G)$$

and that, when $x$ runs through $L^1$, $x(\theta, \sigma)$ spans $L^2(\Xi)$.

Let $x \in L^1(G)$, and put, for every $s \in \Sigma$,

$$x^s(t)\begin{cases} = x(t) \text{ for } t = a^n s, \ -\infty < n < \infty, \\ = 0 \quad \text{otherwise.} \end{cases}$$

Then $x^s \perp x^{s'}$ in $L^2(G)$. For each $s$, $x(\theta, (s))$ is identified with the Fourier transform of $x^s$, hence

$$\int_G |x^s(t)|^2 dt = \int_\Omega |x(\theta, (s))|^2 d\theta = \int_{(\Sigma)} \int_\Omega |x(\theta, (s))|^2 d\theta d(s).$$

Therefore, since $x(\theta, (s))$'s are orthogonal in $L^2(\Xi)$ for different $s$, we have

$$\int_G |x(t)|^2 dt = \sum_s \int |x^s(t)|^2 dt = \sum_{(s)} \int_\Omega |x(\theta, (s))|^2 d\theta.$$

It is analogously proved that $x(\theta, \sigma)$'s span $L^2(\Xi)$.

(5°) Different Decompositions. Using the commutative subgroup $B = (b^n\,;\ -\infty < n < \infty)$, decompose $(U_s, L^2)$ analogously : $(U_s, L^2) \sim (W_s^\eta, E^\eta)$, $(|\eta| = 1)$. Then

*$(U_s^\theta, H^\theta)$ and $(W_s^\eta, E^\eta)$ are not equivalent for any $\theta$, $\eta$.*

Proof. Suppose that for some $T$

$$TU_s^\theta = W_s^\eta T.$$

Then $W_a^\eta Te_\theta = TU_a^\theta e_\theta = \theta Te_\theta$. But in $E^\eta$, $W_a^\eta$ has no proper value, as shown above. Q.E.D.

(Received December 16, 1950)