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Remarks on the Postulates of Metric Groups

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§ 1. Introduction

Let E denote a topological space, which is an abstract group at the same time. But we do not mean by E a topological group in the ordinary sense. In this note we shall discuss about some relations among the following postulates under the condition of metric completeness, or under that of metric local compactness:

(1) *If* $\lim x_n = x$, *then* $\lim x_n y = xy$,

(2) *if* $\lim y_n = y$, *then* $\lim x y_n = xy$,

(3) *if* $\lim x_n = x$ *and* $\lim y_n = y$, *then* $\lim x_n y_n = xy$,

(4) *if* $\lim x_n = u$ *and* $\lim x_n^{-1} = v$, *then* $u^{-1} = v$,

(5) *if* $\lim x_n = x$, *then* $\lim x_n^{-1} = x^{-1}$.

Our results are the following two theorems:

Theorem I: *If E is a metric complete group, then the property* (3) *can be deduced from* (1) *and* (2).

Theorem II: *If E is a metric lacally compact group, then the property* (5) *can be deduced from* (1) *and* (2), *and E is a metric locally compact group in the ordinary sense.*

BANACH gave in his postumas note[1] a theorem that a metric complete group satisfying (1), (2) and (4) has the property (5). From it and Theorem I follows Theorem II (even in the case of metric completeness instead of metric locally compactness) as their logical consequence. But his proof in the non-separable case is not evident for us. In the following, let "e" denote the unit element of the group E, V_e or W_e spherical neighborhood of e, $S_r(x)$ the spherical neighborhood of $x (\in E)$ with radius r, and $d(x, y)$ the distance between x and y. $\bar{S}_r(x)$ means the closure of $S_r(x)$.

§ 2. Proof of Theorem I.

Before the proof of theorem I we shall prove the following:

1) Remarques sur les groupes et les corps métriques, Studia Math. 10 (1948), p. 178.

Lemma 1: *For each element x_0 of E and a natural number k, there exist an open set G and V_e such that*

$$V_e \cdot G \subset \bar{S}_{k^{-1}}(x_0)\text{ [2]}. \tag{6}$$

Proof. It is evident from (1) that for each element y of $\bar{S}_{(2k)^{-1}}(x_0)$ there exists a W_e such that

$$W_e \cdot y \subset \bar{S}_{(2k)^{-1}}(y). \tag{7}$$

Let $r(y)$ denote the radius of W_e and A_i denote the set of all elements y of $\bar{S}_{(2k)^{-1}}(x_0)$ such that there exists a W_e as (7) whose radius $r(y)$ satisfies the inequality $(i+1)^{-1} \leqq r(y) \leqq i^{-1}$ [3]. Then by the definition of A_i and (7) we have

$$\bar{S}_{(2k)^{-1}}(x_0) = \bigcup_{i=1}^{\infty} A_i .$$

It is easily seen that $\bar{S}_{(2k)^{-1}}(x_0)$ is a set of the second category. Then if the closedness of A_i is proved, there must be some i_0 such that A_{i_0} contains an open set G. This G is a desired one in (6) and as V_e there we may choose $V_{e, i_0^{-1}}$ i. e. the neighborhood of e with radius i_0^{-1}.

The closedness of A_i can be proved as follows: Suppose that $y_n \in A_i$ and $\lim y_n = y_0$, then $y_0 \in \bar{S}_{(2k)^{-1}}(x_0)$ and $(i+1)^{-1} \leqq r(y_n) \leqq i^{-1}$. Without loss of generality we can suppose $\lim r(y_n) = r$, for $\{r(y_n)\}$ is bounded. Then we have $W_e \cdot y_0 \subset \bar{S}_{(2k)^{-1}}(y_0)$, if we choose W_e with radius r. In fact, if it was $xy_0 \notin \bar{S}_{(2k)^{-1}}(y_0)$ for some x of W_e, then

$$d(xy_0, y_0) > (2k)^{-1}. \tag{8}$$

But, since $\lim r(y_n) = r$, we have $x \in W_{e, r(y_n)}$ (the neighborhood of e with radius $r(y_n)$) for sufficiently large n. Hence by (7) we get for sufficiently large n

$$d(xy_n, y_n) < (2k)^{-1}. \tag{9}$$

As $d(x, x')$ is a continuous function on $E \times E$, we obtain from (2) and (9)

$$d(xy_0, y_0) \leq (2k)^{-1},$$

which contradicts with (8). Lemma 1 is thus proved.

Now let $G(k, x_0)$ be any but a definit non-empty open set which satisfies the relation (6) for some V_e but for fixed x_0 and k. Let G_0 be

2) For two subsets A and B of the group E we mean by $A \cdot B$ the set of all elements xy such that $x \in A, y \in B$.

3) We can assume that $0 < r(y) \leqq 1$ for all $y \in \bar{S}_{(2k)^{-1}}(x_0)$.

defined by

$$G_0 = \bigcap_{k=1}^{\infty} \bigcup_{x \in E} G(k, x),$$

then we have:

Lemma 2. *G_0 is a set of the second category. Hence, it is not empty.*

Proof. It is evident that $G(k) = \bigcup_{x \in E} G(k, x)$ is an open set. $G(k)$ is also everywhere dense in E for each fixed k, for we have by the definition of $G(k, x)$, $G(k', x) \subset S_{k'^{-1}}(x)$, $0 \neq G(k', x) \subset G(k, x)$ for all $k' > k$. Then $E - G_0 = \bigcup_{k=1}^{\infty} (E - G(k))$ is a set of the first category, since $E - G(k)$ is closed and non-dense. Therefore G_0 is a set of the second category. Lemma 2 is thus proved.

Proof of Theorem I. First suppose that $\lim x_n = x$, $\lim y_n = y$ and $y \in G_0$. For an arbitrary positive number ε there exist a k_0 such that $2/k_0 < \varepsilon/2$ and N_0 such that

(10) $$d(y_n, y) < \varepsilon/2 \quad \text{for sufficiently large } n.$$

Since $y \in G_0$, there exists a $G(k_0, x_0)$ containing y and a V_e

(11) $$V_e \cdot G(k_0, x_0) \subset S_{k_0^{-1}}(x_0).$$

And since $G(k_0, x_0)$ is open, we have

(12) $$y_n \in G(k_0, x_0) \quad \text{for sufficiently large } n.$$

Furthermore $\lim x^{-1}x_n = e$ by (2), then we have

(13) $$x^{-1}x_n \in V_e \quad \text{for sufficiently large } n.$$

Then, since both $x^{-1}x_n y_n$ and y_n are contained in $S_{k_0^{-1}}(x_0)$ by (11), (12), and (13), we have

(14) $$d(x^{-1}x_n y_n, y_n) < 2/k_0 < \varepsilon/2 \quad \text{for sufficiently large } n.$$

Hence from (10) and (14)

$$d(x^{-1}x_n y_n, y) \leq d(x^{-1}x_n y_n, y_n) + d(y_n, y) < \varepsilon$$

for sufficiently large n, which says that $\lim x^{-1}x_n y_n = y$. Then by (2), $\lim x_n y_n = xy$.

Now for the case $y \notin G_0$ we can take z_0 such that $yz_0 \in G_0$ since G_0 is not empty. By (1), $\lim y_n z_0 = yz_0$ $(yz_0 \in G_0)$, then we get by the first part of the proof that $\lim x_n y_n z_0 = xyz_0$. Therefore again by (1), $\lim x_n y_n = xy$. Thus the proof is completed.

This theorem holds also when E is metric locally compact.

§ 3. Proof of Theorem II.

Before the proof of Theorem II we shall prove the following:

Lemma 3: *For each element x_0 of E, every neighborhood $U(x_0)$ of x_0, and a natural number k, there exist a non-empty open set G' contained in $U(x_0)$ and a natural number i_0 such that*

$$(15) \qquad y \cdot V_{k^{-1}} \supset S_{i_0^{-1}}(y) \quad \text{for all } y \in G',$$

where V_r is a spherical neighborhood of e with radius r.

Proof. We may assume that $\bar{V}_{k^{-1}}$ is compact for all natural number k. Let us take $U'(x_0)$ (neighborhood of x_0) and $r(>0)$ such that

$$\bar{U}'(x_0) \subset U(x_0) \quad \text{and.} \quad r < k^{-1},$$

then $\bar{U}'(x_0)$ is a set of the second category and $\bar{V}_r$ is compact.

Now let A_i' be the set of all elements of y of $\bar{U}'(x_0)$ such that the relation $y \cdot \bar{V}_r \supset S_{i^{-1}}(y)$[4] is satisfied, then we have

$$(16) \qquad \bar{U}'(x_0) = \bigcup_{i=1}^{\infty} A_i'$$

A_i' must be closed. In fact, suppose that $\lim y_n = y_0$ and $y_n \in A_i'$, then $y_n \cdot \bar{V}_r \supset S_{i^{-1}}(y_n)$. Let x be an element of $S_{i^{-1}}(y_0)$, then $d(x, y_n) < i^{-1}$ for sufficiently large n, therefore x can be represented in such a way that $x = y_n v_n$, where $v_n \in \bar{V}_r$. By the compactness of $\bar{V}_r$ we may assume that $\lim v_n = v_0 \in \bar{V}_r$. Then by theorem I we have

$$x = y_0 v_0 \quad \text{where} \quad v_0 \in \bar{V}_r \subset V_{k^{-1}}.$$

Then from (16) and the closedness of A_i', there must be some i_0 such that A_{i_0}', contains an open set G', which is a desired one in (15). Lemma 3 is thus proved.

Now let $G'(k, x_0)$ be any but a definit non-empty open set which satisfies the relation (15) for some i_0, but for fixed x_0 and k. Let G_0' be defined by

$$G_0' = \bigcap_{k=1}^{\infty} \bigcup_{x \in E} G(k, x),$$

then we have:

Lemma 4: *G_0' is a set of the second category. Hence, it is not empty.*

Proof. $G'(k) = \bigcup_{x \in E} G'(k, x)$ is open and everywhere dense in E for each fixed k, for there exists a non empty $G'(k, x) \subset U(x)$ for every $U(x)$ by Lemma 3. By the same way as in the proof of Lemma 2, we obtain easily that G_0' is a set of the second category, which was to be proved.

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4) This is possible, since $y \cdot V_r$ is an open set containing y by (2).

Proof of Theorem II. By (2), for every neighborhood $U(x^{-1})$ of x^{-1}, there exists a neighborhood $V_{k_0^{-1}}$ of e such that

$$(17) \qquad x^{-1}\cdot V_{k_0^{-1}} \subset U(x^{-1}).$$

Suppose that $\lim x_n = x$. Now take an element y_0 of the non-empty G_0'. Then by (1) and (2) we have $\lim y_0 x_n x^{-1} = y_0$. By the definition of $G_0' (\ni y_0)$ there exists an open set $G'(k_0, x_0)$ containing y_0 such that

$$(18) \qquad y\cdot V_{k_0^{-1}} \supset S_{i_0^{-1}}(y) \ (i_0 = i_0(k_0)) \quad \text{for all } y \in G'(k_0, x_0).$$

Since $G'(k_0, x_0)$ is open, (18) holds for $y = y_0 x_n x^{-1}$ for sufficiently large n, i.e.

$$(19) \qquad y_0 x_n x^{-1} V_{k_0^{-1}} \supset S_{i_0^{-1}}(y_0 x_n x^{-1}).$$

But since i_0^{-1} is independent of n and $\lim y_0 x_n x^{-1} = y_0$, we have

$$(20) \qquad S_{i_0^{-1}}(y_0 x_n x^{-1}) \ni y_0 \quad \text{for sufficiently large } n.$$

Hence by (19) and (20), $y_0 x_n x^{-1} V_{k_0^{-1}} \ni y_0$, from which it follows that $x_n^{-1} \in x^{-1} V_{k_0^{-1}}$, then by (17) $x_n^{-1} \subset U(x^{-1})$. From this and Theorem I follows that E is a metric locally compact group in the ordinary sense.

§ 4. Remark

In the general case without metric, the elements-convergence in the five postulates (1)-(5) should be replaced by suitable statements in the term of neighborhood: for example, (1) must be replaced by the following:

(1′) If $xy = z$, then for an arbitrary neighborhood $U(z)$ of z, there exists a neighborhood $U(x)$ of x such that $U(x)\cdot y \subset U(z)$.

In such a general case as this we can give an example of the completely regular space for which Theorem I does not hold, as follows:

Let $E = R^2$ denote a plane i.e, the set of all pairs of real numbers. The group-composition is defined by the ordinary vector-addition. We introduce topology into R^2 by the definition of neighborhoods $U(z)$ of z such that $U(z) = S - A_\alpha$, where S is a sphere of centre $z = (x, y)$, and A_α is the set of all $w = (u, v)$ which satisfies the inequality $\left|\tan^{-1}\frac{v-y}{u-x}\right| \leqq \alpha < \alpha_0 \left(< \frac{\pi}{2}\right)$, α_0 being a fixed constant.

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