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On Differentiation of Set-Functions with some of its Applications

By Syunzi KAMETANI and Shizu ENOMOTO

If we want to generalize the known theorems on the relation between differentiation and integration to the case where the measure is defined on a completely additive class of sets in an abstract space without topology, we ought to avoid the use of the covering theorem of Vitali's type, which seems to depend more or less upon the dimensional consideration of the space.

In 1936 RENE DE POSSEL in his elegant paper: *Sur la dérivation abstraite des fonctions d'ensemble* (Journ. d. Math.) obtained some general results which fulfil such demands. The aim of the present paper is to generalize and develop his theory further and give some of its applications.

§ 1. Preliminary

We shall mean by $\mathfrak{F}$ a completely additive class of sets contained in a fixed set E and satisfying the following two conditions: i) $A, B \in \mathfrak{F}$ implies $A-B \in \mathfrak{F}$, ii) $A_j \in \mathfrak{F}$ $(j=1, 2, \dots)$ implies $\bigvee_j A_j \in \mathfrak{F}$.

A function $m(A)$ of sets $A \in \mathfrak{F}$ will be called a *measure* defined in E if the following conditions 1^0-3^0 are satisfied:

1^0) $0 \leqq m(A) \leqq +\infty$,

2^0) m is completely additive,

3^0) E is decomposed into countably infinite sets E_j of $\mathfrak{F}$ of finite $m(E_j)$:

$$E = \bigvee_j E_j, \quad m(E_j) < +\infty, \quad E_j \in \mathfrak{F}\,.$$

By 3^0), the space E itself is a set of $\mathfrak{F}$.

Sets belonging to $\mathfrak{F}$ will be called *measurable*, and the space E, together with its measure, will be called the *measure space* which will be denoted by $\mathfrak{E} = (E, m) = (E, \mathfrak{F}, m)$.

By means of the measure m, we can define an *outer measure* m^* defined for every subset of E in the usual way:

$$m^*(A) = \inf_{\mathfrak{F} \ni B \supseteq A} m(B)$$

Consequently, there exists to each $A \subseteq E$ a measurable $A^* \supseteq A$ such that $m^*(A) = m(A^*)$. Any set of outer measure m^* zero will be called a set of measure zero. Naturally, the meaning of "*almost*" everywhere is now obvious.

We say that a set-function Φ defined for all the measurable sets of the measure space $\mathfrak{E} = (E, m)$ is *m-absolutely continuous*, if it satisfies the following conditions:

1) for all $A \in \mathfrak{F}$, either $-\infty \leqq \Phi(A) < +\infty$ or $-\infty < \Phi(A) \leqq +\infty$,
2) Φ is completely additive,
3) $m(A) = 0$ implies $\Phi(A) = 0$.

Theorem (Radon-Nikodym) [1]

To each m-absolutely continuous set-function Φ defined in a measure space (E, m), there exists a point-function $f(p)$ defined on E such that

$$\Phi(A) = \int_A f(p)\, dm(p)$$

for every measurable A.

The function f will be called the *derivative* of Φ.

§ 2. Systems of Neighborhoods, Differentiation.

Let $\Lambda(p)$ be a directed set associated with $p \in E$.

We mean by the *system of neighbouhoods* of p a system $(\mathfrak{V})_p$, of families $\mathfrak{V}_\lambda(p)$, associated with $\lambda \in \Lambda(p)$, of m-measurable sets $V(p)$, which satisfies the following condition: $\lambda < \lambda'$ implies $\mathfrak{V}_\lambda(p) \supseteq \mathfrak{V}_{\lambda'}(p)$. This definition clearly does not presuppose any topological consideration of the space E.

Given a system of neighborhoods of each $p \in A$, the subset of E, the totality of these systems for p running over A is called the system of neighborhoods of A and is denoted by $(\mathfrak{V})_A$.

Let $\{V(p)\}$ be any set $\subseteq \bigvee_\lambda \mathfrak{V}(p)$. We shall say that the set $\{V(p)\}$ has the *property* (L) with respect to $(\mathfrak{V})_p$ if $\mathfrak{V}_\lambda(p) \bigwedge \{V(p)\} \neq 0$ for every $\lambda \in \Lambda(p)$.

Now let $\{V(p);\ p \in A\}$ be any set $\subseteq \bigvee_{p \in A} \bigvee_\lambda \mathfrak{V}_\lambda(p)$. We shall say that the set $\{V(p);\ p \in A\}$ has the property (L) with respect to $(\mathfrak{V})_p$, if at each point $p \in A$ the set $\{V(p)\} \subseteq \{V(p);\ p \in A\}$ has the property (L) with respect to $(\mathfrak{V})_p$.

1) S. Saks: Theory of the Integral (1937), Chap. I, § 14.

We may introduce the concept of limit concerning these systems of neighborhoods in the following way:

Let σ be a set-function defined on $\mathfrak{V}_\lambda(p)$, the least upper bound of the numbers μ such that $\{V(p); \sigma(V(p)) > \mu\}$ has the property (L) with respect to $(\mathfrak{V})_p$, is called the superior limit of σ at p concerning $(\mathfrak{V})_p$, and is denoted by $\limsup \sigma(V(p))$. The inferior limit, $\liminf \sigma(V(p))$, is now defined by symmetry.

From these definitions follows immediately:

$$\liminf \sigma(V(p)) \leqq \limsup \sigma(V(p)) .$$

Then as usual we may define $\lim \sigma(V(p))$ concerning $(\mathfrak{V})_p$ as the common value of both sides of the above inequality coincide.

Replacing $\sigma(V(p))$ by the quotient $\Phi(V(p))/m(V(p))$, we obtain the definition of the *upper derivate*, $\overline{D}(p) = \limsup (\Phi(V(p))/m(V(p)))$, the *lower derivate*, $\underline{D}(p) = \liminf (\Phi(V(p))/m(V(p)))$ and the *derivate*, $D(p) = \overline{D}(p) = \underline{D}(p)$ at p concerning $(\mathfrak{V})_p$ of the completely additive set-function Φ with respect to m.

Similiarly the *upper* and the *lower density* at p of a set $A \leqq E$ are

$$\limsup (m(A \wedge V(p))/m(V(p)), \quad \liminf (m(A \wedge V(p))/m(V(p)))$$

respectively, and the *density* at p is their common value when they coincide.

Needless to say, these definitions of derivates of completely additive set functions are the generalization of ordinary ones.

§ 3. The relation between derivatives and derivates.

Let us begin with stating without proofs the following two fundamental lemmas obtained by Possel:

Lemma 1. *Given two measurable functions $f(p)$ and $g(p)$ defined in the measure space (E, m), let us suppose that if, at every point p of a subset $A \leqq E$ with $m^*(A) > 0$, holds $g(p) > a$, then there exists a point $p_0 \in A$ such that $f(p_0) \geqq a$. From this assumption, we can conclude that $f(p) \geqq g(p)$ holds almost everywhere in E. Regarding the opposite sign of inequality, we can conclude similarly.*

Lemma 2. *Let Φ be an m-absolutely continuous set-function defined in the measure space $\mathfrak{E} = (E, m)$. Suppose $f(p)$ is the derivative of Φ with respect to m. If, for each $A \leqq B$ with $m^*(A) > 0$, we have $\Phi(A^*) \geqq a \cdot m(A^*)$, then we have also $f(p) \geqq a$ almost everywhere in B. Regarding the opposite sign of inequality, we can conclude similarly.*

Theorem 1. *Let $(\mathfrak{V})_p$ be a system of neighborhoods of the measure space (E, m), and A be any set of positive outer measure m^*. If we* *have* $\Phi(A^*) \geqq a \cdot m(A^*)$ *or* $\Phi(A^*) \leqq a \cdot m(A^*)$ *according as* $\Phi(V) \geqq a \cdot m(V)$ *or* $\Phi(V) \leqq a \cdot m(V)$ *for every neighborhood* $V(p)$ *belonging to* $\{V(p);\ p \in A\}$ *with the property* (L) *with respect to* $(\mathfrak{V})_A$, *then the derivative of* Φ *is equal to the derivate concerning* $(\mathfrak{V})_p$ *at almost every point* p *of* E.

Proof. If $\overline{D} > a$ at a point p, then by the definition of derivates the class $\{V(p)\}$ with $\Phi(V(p)) > a \cdot m(V(p))$ has the property (L). Suppose $A_1 = \{p;\ \overline{D}(p) > a\}$ has positive outer measure. Since $\{V(p);\ p \in A\}$ has the property (L) with respect to $(\mathfrak{V})_A$, we have $\Phi(B^*) \geqq a \cdot m(B^*)$ for every set $B \subseteq A_1$ with $m^*(B) > 0$. By Lemma 2, we have, almost everywhere on A_1,

$$f(p) \geqq \overline{D}(p).$$

Similarly we have

$$f(p) \leqq \underline{D}(p)$$

almost everywhere on A_1, thus the proof is accomplished.

We see easily from Lemma 2 that the boundedness of the derivative of an absolutely continuous, completely additive set-function Φ amounts to the fact that there exists a constant k for which

$$|\Phi(A)| \leqq k \cdot m(A)$$

for every measurable $A \subseteq E$.

We shall now investigate the property of the system of neighborhoods for which the derivative and the derivate of the set-function of the kind mentioned above coincide with each other almost everywhere. To this purpose, the subsequent Lemmas 3 and 4 are the preparatory, the proofs of which are almost identical with the ones given by Possel in the paper already cited and omitted here.

Lemma 3. *Let* $\mathfrak{E} = (E, m)$ *be a measure space, in which a system* $\{V(p);\ p \in E\}$, *not necessarily of neighborhoods, but merely of measurable sets* $V(p)$ *associated with each* $p \in E$ *is given. Then, the following condition* (*a*) *implies the following* (*b*):

(*a*) *For every* A *with* $m^*(A) > 0$, *there exists a* $p \in A$ *and* $V(p) \in \{V(p);\ p \in E\}$ *such that*

$$m(A^* \frown V(p)) > \alpha \cdot m(V(p))$$

where α *is a constant satisfying* $0 < \alpha < 1$.

(*b*) *In a set* A *of positive outer measure* m^*, *there exists a sequence of points* $\{p_n\}$ *and a sequence,* $V_n(p_n) \in \{V(p);\ p \in E\}$ $(n = 1, 2, \ldots)$ *such that*

$$m^*(A) = m^*(A^* \frown \bigcup_n V_n), \quad \sum_m (V_n) \leqq \frac{1}{\alpha}\cdot m(A^*).$$

Lemma 4. *Let* $\{V(p);\ p\in E\}$ *be a system of measurable sets, in* (E, m), *satisfying the condition* (*b*) *for every* α *with* $0<\alpha<1$. *Suppose a completely additive, m-absolutely continuous set-function* Φ *such that* $\Phi(A) \leqq k\cdot m(A)$ *for some constant k and for every measurable* $A\subseteqq E$ *satisfies, at every point p of A of positive outer measure* m^*,

$$\Phi(V(p)) \geqq a\cdot m(V(p)),$$

where $V(p)\in\{V(p);\ p\in E\}$. *Then, we have*

$$\Phi(A^*) \geqq a\cdot m(A^*).$$

The case of opposite sign of inequality holds similarly.

Theorem 2. *Suppose in a measure space* $\mathfrak{E}=(E, m)$ *the system of neighborhoods* $(\mathfrak{V})_E$ *is given. Then the following* 4 *conditions A)-D) are equivalent with each other.*

A) *For every* $\{V(p);\ p\in E\}$ *having the property* (L) *with respect to the system* $(\mathfrak{V})_E$ *and for every* α *with* $0<\alpha<1$, *we may find a point p in every set* $A(\subseteqq E)$ *of positive outer measure* m^*, *and* $V(p)\in\{V(p);\ p\in E\}$ *such that* $m^*(A\frown V(p)) \geqq \alpha\cdot m(V(p))$.

B) *For every* $\{V(p);\ p\in E\}$ *having the property* (L) *with respect to the system* $(\mathfrak{V})_E$ *and for every* α *such that* $0<\alpha<1$, *there exists a sequence of points* p_n *and a sequence of* $V_n(p_n)\in\{V(p);\ p\in E\}$ *satisfying* $m^*(A)=m^*(A\frown\bigcup_n V_n)$, $\sum m(V_n) \leqq \frac{1}{\alpha}\cdot m^*(A)$.

C) *Let* Φ *be a completely additive, m-absolutely continuous set-function with a bounded derivative. Then, the derivative is identical with the derivate with respect to* $(\mathfrak{V})_E$ *almost everywhere.*

D) *The density of a measurable set A with respect to the system* $(\mathfrak{V})_E$ *is equal to a constant* 1 *at almost every point of A.*

Proof. We have only to show A → B → C → D → A. A → B is obvious from Lemma 3. B → C is obvious from Lemma 4 and Theorem 1. C → D is given in the paper of Possel already cited, D → A is obvious.

If the system of neighborhoods in $\mathfrak{E}=(E, m)$ satisfies the following more restricted condition (B^*) instead of (B), then more precise is the result.

B*) *If* $\{V(p);\ p\in E\}$ *has the property* (L) *with respect to* $(\mathfrak{V})_E$, *then, for every set A of positive outer measure* m^* *and for every* $\varepsilon>0$, *there exists a sequence of points* $p_n\in A$ *and* $V_n(p_n)\in\{V(p);\ p\in E\}$ $(n=1, 2, \dots)$ *satisfying* $m(A^*)=m(A^*\frown\bigcup V_n)$, $\sum m(V_n) < m^*(A)+\varepsilon$,

$V_i \frown V_j = 0$ $(i \neq j)$.

Theorem 3. *If the system of neighborhoods* $(\mathfrak{V})_E$ *satisfies the condition* (B^*), *then the following* (C^*) *holds:*

C*) *If* Φ *is a completely additive, m-absolutely continuous set-function satisfying the condition that* $|\Phi(A)| < +\infty$ *for every measurable* A *of finite measure, then the denivative and the derivate, of* Φ, *are equal to each other almost everywhere on* E.

Proof. Without loss of generality we may assume that Φ is always non-negative by the decomposition theorem of Jordan. Let $\{V(p);\ p \in E\}$ be any set having the property (L) with respect to $(\mathfrak{V})_E$, and $A \subseteq E$ be any set of positive outer measure m^*. To each $\varepsilon_i > 0$ such that $\sum \varepsilon_i < +\infty$, we may find $p_{in} \in A$ and $V_{in}(p_{in}) \in \{V(p);\ p \in E\}$ $(n = 1, 2, \ldots)$ satisfying

$$m(A^*) = m(A^* \frown \bigcup_n V_{in}), \qquad \sum m(V_{in}) < m^*(A) + \varepsilon,$$
$$V_{in} \frown V_{im} = 0 \qquad (n \neq m).$$

Writing $V_i = \bigcup_n V_{in}$, $C_i = V_i - A^*$, we have $m(V_i) = m(C_i) + m(V_i \frown A^*) = m(C_i) + m(A^*) < m(A^*) + \varepsilon_i$, so that $m(C_i) < \varepsilon_i$. Putting $B = \bigcup_i C_i$ we find $m(B) \leq \sum m(C_i) < \sum \varepsilon_i < +\infty$, which shows, from the assumption on Φ, that, to each $\eta > 0$, there exists an integer j such that $|\Phi(C_j)| < \eta$ [2]. If for every $V(p)$ belonging to a system $\{V(p);\ p \in A\}$ associated with A holds $\Phi(V) \geq a \cdot m(V)$, then follows $\Phi(V_{ni}) \geq a \cdot m(V_{ni})$ which shows also $\Phi(V_i) \geq a \cdot m(V_i)$ for every i. If $\Phi(A^*) < a \cdot m(A^*)$, where a is obviously > 0, then taking η so that $a \cdot m(A^*) - \Phi(A^*) > \eta > 0$, we would have, as $|\Phi(C_j)| < \eta$,

$$\Phi(A^*) + \Phi(C_j) < a \cdot m(A^*) \leq a \cdot m(A^*) + a \cdot m(C_j)$$

for some j, and consequently

$$\Phi(V_j) < a \cdot m(V_j)$$

which is a contradiction. Hence, $\Phi(A^*) \geq a \cdot m(A^*)$, which shows, from Theorem 1, that (C^*) holds true.

§ 4. Differentiation in a metric space.

Let R be a locally bicompact metric space composed of countably infinite bicompact sets. Consequently, R is perfectly separable.

Let m be a completely additive measure defined on the class of Borel sets in R which takes a finite value for any bicompact Borel set in R, and m^* be the outer measure of m.

2) S. Saks: Loc. ci. Cap. I, § 1,3.

We shall apply our results obtained to this measure space $\mathfrak{R}=(R, m)$.

Theorem 4. (EXISTENCE OF THE SYSTEM OF NEIGHBORHOODS)

There is, in $\mathfrak{R}$, the system of neighborhoods in R and satisfies the condition (B^*).

Proof. 1°) Let the distance between x and y be $\rho(x, y)$. Since the closure of an open sphere

$$\theta_x^{\varepsilon(x)} = \{y;\ \rho(x, y) < \varepsilon(x)\}$$

is, for a suitable choice of $\varepsilon(x) > 0$, bicompact, we may find, at each point x, a complete system of neighborhoods, $\{\theta_x^{\varepsilon_i(x)}\}$, satisfying the following conditions:

$$\varepsilon_i(x) < 2^{-i}\ (i=1, 2, \ldots), \quad \varepsilon_1(x) \geqq \varepsilon_2(x) \geqq \ldots\ldots,$$

the closure $\bar{\theta}_x^{\varepsilon_i(x)}$ of $\theta_x^{\varepsilon_i(x)}$ is bicompact, and $m(\theta_x^{\varepsilon_i(x)}) = m(\bar{\theta}_x^{\varepsilon_i(x)})$. For the suitable choice of $x_j \in R$ $(j=1, 2, \ldots)$, we may write $R = \bigvee_j \theta_{x_j}^{\varepsilon_1(x_j)}$ by our assumption on R.

Putting $\theta_{x_1}^{\varepsilon_1(x_1)} = I_1$, $\theta_{x_2}^{\varepsilon_1(x_2)} - \bar{I}_1 = I_2$, $\theta_{x_3}^{\varepsilon_1(x_3)} - (\bar{I}_1 \bigvee \bar{I}_2) = I_3, \ldots$, we find

$$R = \bigvee_j \bar{I}_j,\ m(\bar{I}_j) = m(I_j) \quad (j=1, 2, \ldots)$$

where $\bar{I}_j$ are bicompact. To each point $a \in \bar{I}_1$, correspond a sphere $\theta_a^{\varepsilon_2(a)}$, and we shall find a finite number of such spheres $\theta_{a_j}^{\varepsilon_2(a_j)}$ $(j=1, \ldots, n_1)$ such that

$$\bigvee_{i=1}^{n_1} \theta_{a_i}^{\varepsilon_2(a_i)} \geqq \bar{I}_1 .$$

Writing

$$I_{11} = \theta_{a_1}^{\varepsilon_2(a_1)} \bigwedge I_1,\ I_{12} = (I_1 - \bar{I}_{11}) \bigwedge \theta_{a_2}^{\varepsilon_2(a_2)},$$
$$I_{13} = (I_1 - (\bar{I}_{11} \bigvee \bar{I}_{12})) \bigwedge \theta_{a_3}^{\varepsilon_2(a_3)}, \ldots,$$

we shall call $\{I_{11}, \ldots, I_{1n_1}\}$ the division of $\{I_1\}$ which will be denoted by D_1^1. This process may be repeated if we replace I_1 by $\bar{I}_{1j}$ $(j=1, \ldots, n_1)$ and $\theta_a^{\varepsilon_2(a)}$ by $\theta_a^{\varepsilon_3(a)}$ $(a \in \bar{I}_{1j})$, whence we have a finite sequence I_1^j, $I_2^j, \ldots$. Now repeat this argument for each I_j, and we shall obtain a sequence of division $\{D_1^j, D_2^j, \ldots\}$. Let D_n be the superposition of the divisions $D_n^1, D_n^2, \ldots$. We call the n-th grating of R and $I\,(\in D_n)$ a mesh of n-th grating.

The sequence $\{D_j\}$ of the gratings has the following properties:

i) $m(\bar{I}) = m(I)$ for any $I \in D_n$ $(n=1, 2, \ldots)$,

ii) if $I_1, I_2 \in D_n$ and $I_1 \neq I_2$, then $I_1 \bigwedge I_2 = 0$,

iii) D_n is the refinement of D_{n-1} $(n=2, 3, \ldots)$,

iv) at each point $a \in R$, there exists a sequence of meshes $\{I_n\}$ such that $I_n \in D_n$, $I_{n+1} \subseteq I_n$ and $\bigwedge \bar{I}_n \ni a$.

But such $\bigwedge \bar{I}_n$ can have only one point a, for, the diameter of $\bar{I}_{n+1}$ is $\leqq 2^{-(n+1)}$ by the construction of $\theta_a^{\varepsilon_i(a)}$.

2^0) We notice here that, for a fixed point a of R, there is, if any, only one sequence $\{I_n\}$ which satisfies

$$\bigwedge I_n = \{a\}, \quad I_n \in D_n \ (n = 1, 2, \ldots), \quad I_n \geqq I_{n+1}.$$

Then $\{I_n\}$ will be the complete system of neighborhoods of the point a. The set of points a where the system of meshes $\{I_n\}$ existe will be denoted by R_0. Then we find easily $m(R - R_0) = 0$. As regards the point $a \in R - R_0$, let be its complete system of neighborhoods. Thus, at every point of R, we have defined the complete system of neighborhoods, which we shall denote by $\{V_t(a);\ a \in R\}$.

We shall now show that this system satisfies the condition (B^*). Since $m^*(A) = \inf m(U)$, where U is an open set such that $U \geqq A$, we may find to each $\varepsilon > 0$ an open set U such that $U \geqq A$ and

$$m(U) < m^*(A) + \varepsilon\ ^{3)}.$$

For the validity of (B^*), we have only to show that for any complete system $\mathfrak{D}$ of neighborhoods such that $\leqq \{V_t(a);\ a \in R\}$ and for any α $(0 < \alpha < 1)$, the condition (b) is satisfied.

The complete systems of neighborhoods of a set A belonging to $\mathfrak{D}$ will be denoted by $\mathfrak{D}^A$. Let every mesh $\bar{I} \subseteq U$ such that $I \in D_1 \bigwedge \mathfrak{D}^A$ be denoted $J_1^1, \ldots, J_{n_1}^1$, and every mesh $\bar{I} \subseteq (R - \bigvee_j \bar{J}_j^1) \bigwedge U$ such that $I \in D_2 \bigwedge \mathfrak{D}^A$ will be denoted by $J_1^2, \ldots, J_{n_2}^2$. Replace $(R - \bigvee_j J_j^1) \bigwedge U$ by $(R - ((\bigvee_j \bar{J}_j^1) \bigvee (\bigvee_j \bar{J}_j^2))) \bigwedge U$ and D_2 by D_1, and we may go on indefinitely. We have thus obtained a sequence of meshes $\{J_j^n\}$ any two of which have no point in common.

To show

$$A \bigwedge R_0 \leqq \bigvee J_j^n \ (\leqq U),$$

take, at each point $a \in A \bigwedge R_0$, a sequence of meshes $\{I_{a_j}\}$ such that $I_{a_j} \in \mathfrak{D}^A$, $I_{a_j} \in D_{a_j}$, $I_{a_j} \geqq I_{a_{j+1}}$, $\bigwedge I_{a_j} = \{a\}$. For a fixed a, we can take n so large that

$$\theta_a^{2^{-(n-1)}} \leqq U,$$

and consequently, $\bar{I}_{a_n} \leqq \theta_a^{2^{-(n-1)}} \leqq U$ and $I_{a_n} \in D_{a_n}$. Let us notice here

3) K. Kodaira: Über die Beziehung zwischen den Massen und den Topologien in einer Gruppe. Proc. Phys.-Math Soc. Japan, Vol. 23 (1941), p. 83.

that D_{a_n} is, for some k, the k-th grating.

If $a \subset J_1^1 \cup \dots \cup J_{(k-1)_n}^{k-1}$, then $a \in \bigcup J_j^n$. On the other hand, we have $I_{a_n} \in D^A$, $a \in I_{a_n} \in D_k$, from which follows $I_{a_n} \in \{J_1^k, \dots, J_{k_n}^k\}$ so that we have $a \in \bigcup J_j^n$. Writing again $\{J_n\}$ for $\{J_j^n\}$. we find, from $J_n \in \mathfrak{D}^A$, $J_n = V(a_n)$, $a_n \in A$. Hence, $V(a_n) \in \mathfrak{D} \subseteq \{V_i(a),\ a \in R\}$ and at last we find

$$m^*(A) = m^*(A \cap R_0) = m^*(A \cap R_0 \cap \bigcup_n V(a_n)),$$
$$= m^*(A \cap \bigcup_n V(a_n)),$$

from which we have

$$\sum m(V(a_n)) \leq m(U) < m^*(A) + \varepsilon$$

where $V(a_n) \cap V(a_m) = 0$ $(m \neq n)$, and our proof is completed.

Comparing these results with those already mentioned, we can state the following theorem:

Theorem 5. *In the space* $\mathfrak{R} = (R, m)$, *we can define a complete system of neighborhoods* $\{V_i(a);\ a \in R,\ i = 1, 2, \dots\}$ *with respect to which any measurble set has the density* 1 *almost everywhere on it, and any completely additive, m-absolutely continuous set-function* Φ *such that* $|\Phi(A)| < +\infty$ *if* $m(A) < +\infty$ *has the derivate which coincides with its derivative almost everywhere.*

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