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WEAK ANALYTIC HYPERBOLICITY OF COMPLEMENTS OF GENERIC SURFACES OF HIGH DEGREE IN PROJECTIVE 3-SPACE

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Abstract

In this article we prove that every entire curve in the complement of a generic hypersurface of degree $d \geq 586$ in $\mathbb{P}^3_{\mathbb{C}}$ is algebraically degenerated, i.e. there exists a proper subvariety which contains the entire curve.

1. Introduction

A complex manifold X is hyperbolic in the sense of S. Kobayashi if the hyperbolic pseudodistance defined on X is a distance (see, for example, [10]). The hyperbolicity problem in complex geometry studies the conditions for a given complex manifold X to be hyperbolic. In the case of hypersurfaces in $\mathbb{P}^n$ we have the Kobayashi conjectures [9]:

Conjecture 1. *A generic hypersurface* $X \subset \mathbb{P}^{n+1}$ $(n \geq 2)$ *of degree* $\deg X \geq 2n+1$ *is hyperbolic.*

Conjecture 2. $\mathbb{P}^n \setminus X$ $(n \geq 2)$ *is hyperbolic for a generic hypersurface* $X \subset \mathbb{P}^n$ *of degree* $\deg X \geq 2n+1$.

A new approach which could lead to a positive result for Conjecture 1 has been described by Y.-T. Siu in [17] for a bound $\delta_n \gg n$ on the degree. If we are interested in the lower bound on the degree, Conjecture 1 is recently proved in [13] for $n = 2$, $d \geq 18$ and in [16] we proved a weak form of Conjecture 1 for $n = 3$:

Theorem ([16]). *For* $X \subset \mathbb{P}^4_{\mathbb{C}}$ *a generic hypersurface such that* $d = \deg(X) \geq 593$, *every entire curve* $f\colon \mathbb{C} \to X$ *is algebraically degenerate, i.e. there exists a proper subvariety* $Y \subset X$ *such that* $f(\mathbb{C}) \subset Y$.

Here we study the logarithmic Conjecture 2 (proved for $n = 2$ and $d \geq 15$ in [7]) and prove the following result, which is a weak form of the conjecture for $n = 3$:

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Theorem 3. *For $X \subset \mathbb{P}^3_{\mathbb{C}}$ a generic hypersurface such that $d = \deg(X) \geq 586$, every entire curve $f\colon \mathbb{C} \to \mathbb{P}^3_{\mathbb{C}} \setminus X$ is algebraically degenerated i.e. there exists a proper subvariety $Y \subset \mathbb{P}^3_{\mathbb{C}}$ such that $f(\mathbb{C}) \subset Y$.*

The proof is based on two techniques.

The first one is a generalization in the logarithmic setting of an approach initiated by Clemens [2], Ein [6], Voisin [18] and used by Y.-T. Siu [17] to construct vector fields on the total space of hypersurfaces in the projective space. Here we construct vector fields on logarithmic spaces.

The second one is based on bundles of logarithmic jet differentials (see [5]). The idea, in hyperbolicity questions, is that global sections of these bundles vanishing on ample divisors provide algebraic differential equations for any entire curve $f\colon \mathbb{C} \to X \setminus D$ where D is a normal crossing divisor on X. Therefore, the main point is to produce enough algebraically independent global holomorphic logarithmic jet differentials. In the case of $\mathbb{P}^3 \setminus X$ for a smooth hypersurface $X \subset \mathbb{P}^3$, we have proved the existence of global logarithmic jet differentials when $\deg(X) \geq 92$ in [14]. Therefore to produce enough logarithmic jet differentials we take the derivative of the logarithmic jet differential in the direction of the vector fields constructed in the first part, just as in the compact case [16].

2. Logarithmic jet bundles

In this section we recall the basic facts and results of J. Noguchi in [11] about logarithmic jet bundles following G. Dethloff and S. Lu [5].

Let X be a complex manifold of dimension n. Let $x \in X$. We consider germs $f\colon (\mathbb{C}, 0) \to (X, x)$ of holomorphic curves. Then the usual k-jet bundle, $J_k X$, is the holomorphic fibre bundle whose fiber $J_k X_x$ is the set of equivalence classes of germs, $j_k(f)$, where two germs are equivalent if they have the same Taylor expansions of order k. Let $\pi\colon J_k X \to X$ be the natural projection.

Let T^*_X be the holomorphic cotangent bundle over X. Take a holomorphic section $\omega \in H^0(O, T^*_X)$ for some open subset O. For $j_k(f) \in J_k X_{|O}$, we have $f^*\omega = Z(t)\,dt$ and a well defined holomorphic mapping

$$\widetilde{\omega}\colon J_k X_{|O} \to \mathbb{C}^k;\ j_k(f) \to \left(\frac{d^j Z}{dt^j}(0)\right)_{0 \leq j \leq k-1}.$$

If, moreover $\omega_1, \ldots, \omega_n$ are holomorphic 1-forms on O such that $\omega_1 \wedge \cdots \wedge \omega_n$ does not vanish anywhere, then we have a biholomorphic map

$$(\widetilde{\omega_1}, \ldots, \widetilde{\omega_n}) \times \pi\colon J_k X_{|O} \to (\mathbb{C}^k)^n \times O$$

which gives the trivialization associated to $\omega_1, \ldots, \omega_n$.

Let $\overline{X}$ be a complex manifold with a normal crossing divisor D. The pair $(\overline{X}, D)$ is called a log manifold. Let $X = \overline{X} \setminus D$.

The logarithmic cotangent sheaf $\overline{T}^*_X = T^*_{\overline{X}}(\log D)$ is defined as the locally free subsheaf of the sheaf of meromorphic 1-forms on $\overline{X}$, whose restriction to X is T^*_X and whose localization at any point $x \in D$ is given by

$$\overline{T}^*_{X,x} = \sum_{i=1}^{l} \mathcal{O}_{\overline{X},x} \frac{dz_i}{z_i} + \sum_{j=1+1}^{n} \mathcal{O}_{\overline{X},x}\, dz_j$$

where the local coordinates $z_1, \ldots, z_n$ around x are chosen such that $D = \{z_1 \cdots z_l = 0\}$.

Its dual, the logarithmic tangent sheaf $\overline{T}_X = T_{\overline{X}}(-\log D)$ is a locally free subsheaf of the holomorphic tangent bundle $T_{\overline{X}}$, whose restriction to X is T_X and whose localization at any point $x \in D$ is given by

$$\overline{T}_{X,x} = \sum_{i=1}^{l} \mathcal{O}_{\overline{X},x} z_i \frac{\partial}{\partial z_i} + \sum_{j=1+1}^{n} \mathcal{O}_{\overline{X},x} \frac{\partial}{\partial z_j}.$$

Given log-manifolds $(\overline{X}, D)$ and $(\overline{X}', D')$, a holomorphic map $F\colon \overline{X}' \to \overline{X}$ such that $F^{-1}(D) \subset D'$ is called a log-morphism from $(\overline{X}', D')$ to $(\overline{X}, D)$. It induces vector bundle morphisms

$$F^*\colon \overline{T}^*_X \to \overline{T}^*_{X'};$$
$$F_*\colon \overline{T}_{X'} \to \overline{T}_X.$$

Let $s \in H^0(O, J_k\overline{X})$ be a holomorphic section over an open subset $O \subset \overline{X}$. We say that s is a logarithmic k-jet field if the map $\widetilde{\omega} \circ s_{|O'}\colon O' \to \mathbb{C}^k$ is holomorphic for all $\omega \in H^0(O', \overline{T}^*_X)$ for all open subsets O' of O. The set of logarithmic k-jet fields over open subsets of $\overline{X}$ defines a subsheaf of the sheaf $J_k\overline{X}$, which we denote by $\overline{J}_kX$. $\overline{J}_kX$ is the sheaf of sections of a holomorphic fibre bundle over $\overline{X}$, denoted again $\overline{J}_kX$ and called the logarithmic k-jet bundle of $(\overline{X}, D)$.

A log-morphism $F\colon (\overline{X}', D') \to (\overline{X}, D)$ induces a canonical map

$$F_k\colon \overline{J}_kX' \to \overline{J}_kX.$$

We can express the local triviality of $\overline{J}_kX$ explicitly in terms of coordinates. Let $z_1, \ldots, z_n$ be coordinates in an open set $U \subset \overline{X}$ in which $D = \{z_1 z_2 \cdots z_l = 0\}$. Let $\omega_1 = dz_1/z_1, \ldots, \omega_l = dz_l/z_l, \omega_{l+1} = dz_{l+1}, \ldots, \omega_n = dz_n$. Then we have a biholomorphic map

$$(\widetilde{\omega_1}, \ldots, \widetilde{\omega_n}) \times \pi\colon \overline{J}_kX_{|U} \to (\mathbb{C}^k)^n \times U.$$

Let $s \in H^0(U, \overline{J}_kX)$ be given by $s(x) = (\xi^{(i)}_j(x), x)$ in this trivialization where the indices i correspond to the orders of derivative. Then the same s considered as an

element of $H^0(U, J_k\overline{X})$ and trivialized by $\omega_1 = dz_1, \ldots, \omega_n = dz_n$ is given by $s(x) = \left(\hat{\xi}_j^{(i)}(x), x\right)$ where

$$\hat{\xi}_j^{(i)} = \begin{cases} z_i\left(\xi_j^{(i)} + g_i\left(\xi_j^{(1)}, \ldots, \xi_j^{(i-1)}\right)\right): \ j \leq l \\ \xi_j^{(i)}: \ j \geq l+1. \end{cases}$$

The g_i are polynomials in the variables $\xi_j^{(1)}, \ldots, \xi_j^{(i-1)}$, obtained by expressing first the different components $\xi_j^{(i)}$ of $\widetilde{(dz_i/z_i)} \circ s(x)$ in terms of the components $\hat{\xi}_j^{(i)}$ of the components $\hat{\xi}_j^{(i)}$ of $\widetilde{dz_i} \circ s(x)$ by using the chain rule, and then by inverting this system.

3. Logarithmic vector fields

Let $\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{P}^{N_d}$ be the universal surface of degree d given by the equation

$$\sum_{|\alpha|=d} a_\alpha Z^\alpha = 0, \quad \text{where} \quad [a] \in \mathbb{P}^{N_d} \quad \text{and} \quad [Z] \in \mathbb{P}^3.$$

In this section we generalize the approach used in [12] (see Proposition 11 of that article) and [16] to logarithmic jet bundles. We use the notations: for $\alpha = (\alpha_0, \ldots, \alpha_3) \in \mathbb{N}^4$, $|\alpha| = \sum_i \alpha_i$ and if $Z = (Z_0, Z_1, Z_2, Z_3)$ are homogeneous coordinates on $\mathbb{P}^3$, then $Z^\alpha = \prod Z_j^{\alpha_j}$. $\mathcal{X}$ is a smooth hypersurface of degree $(d, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^{N_d}$.

We consider the log-manifold $(\mathbb{P}^3 \times \mathbb{P}^{N_d}, \mathcal{X})$. We denote by $\overline{J_3}(\mathbb{P}^3 \times \mathbb{P}^{N_d})$ the manifold of the logarithmic 3-jets, and $\overline{J_3^v}(\mathbb{P}^3 \times \mathbb{P}^{N_d})$ the submanifold of $\overline{J_3}(\mathbb{P}^3 \times \mathbb{P}^{N_d})$ consisting of 3-jets tangent to the fibers of the projection $\pi_2 \colon \mathbb{P}^3 \times \mathbb{P}^{N_d} \to \mathbb{P}^{N_d}$.

We are going to construct meromorphic vector fields on $\overline{J_3^v}(\mathbb{P}^3 \times \mathbb{P}^{N_d})$.

Let us consider

$$\mathcal{Y} = \left(a_d Z_4^d + \sum_{|\alpha|=d} a_\alpha Z^\alpha = 0\right) \subset \mathbb{P}^4 \times U$$

where $U := (a_{0\cdots 0d} \neq 0) \cap \left(\bigcup_{|\alpha|=d, \alpha_{n+2}=0}(a_\alpha \neq 0)\right) \subset \mathbb{P}^{N_d+1}$. We have the projection $\pi : \mathcal{Y} \to \mathbb{P}^3 \times \mathbb{P}^{N_d}$ and $\pi^{-1}(\mathcal{X}) = (Z_4 = 0) := H$ therefore we obtain a log-morphism $\pi : (\mathcal{Y}, H) \to (\mathbb{P}^3 \times \mathbb{P}^{N_d}, \mathcal{X})$ which induces a dominant map

$$\pi_3 \colon \overline{J_3^v}(\mathcal{Y}) \to \overline{J_3^v}(\mathbb{P}^3 \times \mathbb{P}^{N_d}).$$

Let us consider the set $\Omega_0 := (Z_0 \neq 0) \times (a_d \neq 0) \subset \mathbb{P}^4 \times U$. We assume that global coordinates are given on $\mathbb{C}^4$ and $\mathbb{C}^{N_d+1}$. The equation of $\mathcal{Y}$ becomes

$$\mathcal{Y}_0 := \left(z_4^d + \sum_\alpha a_\alpha z^\alpha = 0\right).$$

Following [5] as explained above, we can obtain explicitly a trivialization of $\overline{J_3}(\Omega_0)$. Let $\omega^1 = dz_1$, $\omega^2 = dz_2$, $\omega^3 = dz_3$, $\omega^4 = dz_4/z_4$. Then we have a biholomorphic map

$$\overline{J_3}(\Omega_0) \to \mathbb{C}^4 \times U \times \mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$$

where the coordinates will be noted $\big(z_i, a_\alpha, \xi_j^{(i)}\big)$.

Let's write the equations of $\overline{J_3^v}(\mathcal{Y}_0)$ in this trivialization. We have $\overline{J_3^v}(\mathcal{Y}_0) = J_3^v(\mathcal{Y}_0) \cap \overline{J_3}(\Omega_0)$. The equations of $J_3^v(\mathcal{Y}_0)$ in the trivialization of $J_3(\Omega_0)$ given by $\hat{\omega}^1 = dz_1$, $\hat{\omega}^2 = dz_2$, $\hat{\omega}^3 = dz_3$, $\hat{\omega}^4 = dz_4$ can be written in $\mathbb{C}^4 \times U \times \mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$ with coordinates $\big(z_i, a_\alpha, \hat{\xi}_j^{(i)}\big)$:

$$z_4^d + \sum_{|\alpha|\le d} a_\alpha z^\alpha = 0,$$

$$dz_4^{d-1}\hat{\xi}_4^{(1)} + \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_j}\hat{\xi}_j^{(1)} = 0,$$

$$dz_4^{d-1}\hat{\xi}_4^{(2)} + d(d-1)z_4^{d-2}\big(\hat{\xi}_4^{(1)}\big)^2 + \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_j}\hat{\xi}_j^{(2)} + \sum_{j,k=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k}\hat{\xi}_j^{(1)}\hat{\xi}_k^{(1)} = 0,$$

$$dz_4^{d-1}\hat{\xi}_4^{(3)} + 3d(d-1)z_4^{d-2}\hat{\xi}_4^{(1)}\hat{\xi}_4^{(2)} + d(d-1)(d-2)z_4^{d-3}\big(\hat{\xi}_4^{(1)}\big)^3 + \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_j}\hat{\xi}_j^{(3)}$$
$$+ 3\sum_{j,k=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k}\hat{\xi}_j^{(2)}\hat{\xi}_k^{(1)} + \sum_{j,k,l=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l}\hat{\xi}_j^{(1)}\hat{\xi}_k^{(1)}\hat{\xi}_l^{(1)} = 0.$$

The relations between the two systems of coordinates can be computed as explained above and are given by

$$\begin{aligned}
&\hat{\xi}_j^{(i)} = \xi_j^{(i)} \quad \text{for} \quad j \le 3,\\
&\hat{\xi}_4^{(1)} = z_4\xi_4^{(1)},\\
&\hat{\xi}_4^{(2)} = z_4\big(\xi_4^{(2)} + \big(\xi_4^{(1)}\big)^2\big),\\
&\hat{\xi}_4^{(3)} = z_4\big(\xi_4^{(3)} + 3\xi_4^{(1)}\xi_4^{(2)} + \big(\xi_4^{(1)}\big)^3\big).
\end{aligned}$$

Therefore, to obtain the equations of $\overline{J_3^v}(\mathcal{Y}_0)$ in the first trivialization, we just have to substitute the previous relations

$$z_4^d + \sum_{|\alpha|\le d} a_\alpha z^\alpha = 0, \tag{1}$$

$$dz_4^d\xi_4^{(1)} + \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_j}\xi_j^{(1)} = 0, \tag{2}$$

$$dz_4^d\xi_4^{(2)} + d^2z_4^d\big(\xi_4^{(1)}\big)^2 + \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_j}\xi_j^{(2)} + \sum_{j,k=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j\,\partial z_k}\xi_j^{(1)}\xi_k^{(1)} = 0, \tag{3}$$

$$\begin{aligned} &dz_4^d\xi_4^{(3)} + 3d^2z_4^d\xi_4^{(1)}\xi_4^{(2)} + d^3z_4^d\big(\xi_4^{(1)}\big)^3 + \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_j}\xi_j^{(3)} \\ &+ 3\sum_{j,k=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j\,\partial z_k}\xi_j^{(2)}\xi_k^{(1)} + \sum_{j,k,l=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j\,\partial z_k\,\partial z_l}\xi_j^{(1)}\xi_k^{(1)}\xi_l^{(1)} = 0. \end{aligned} \tag{4}$$

Following the method used in [16] for the compact case, we are going to prove that $T_{\overline{J_3^v}(\mathcal{Y})} \otimes \mathcal{O}_{\mathbb{P}^4}(c) \otimes \mathcal{O}_{\mathbb{P}^{N_d+1}}(*)$ is generated by its global sections on $\overline{J_3^v}(\mathcal{Y}) \setminus (\Sigma \cup p^{-1}(H))$, where $p\colon \overline{J_3^v}(\mathcal{Y}) \to \mathcal{Y}$ is the natural projection, Σ a subvariety that will be defined below, and $c \in \mathbb{N}$ a constant independant of d. Consider a vector field

$$V = \sum_{|\alpha|\le d} v_\alpha \frac{\partial}{\partial a_\alpha} + \sum_j v_j \frac{\partial}{\partial z_j} + \sum_{j,k} w_j^{(k)} \frac{\partial}{\partial \xi_j^{(k)}}$$

on $\mathbb{C}^4 \times U \times \mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$. The conditions to be satisfied by V to be tangent to $\overline{J_3^v}(\mathcal{Y}_0)$ are the following:

$$\sum_{|\alpha|\le d} v_\alpha z^\alpha + \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_j} v_j + dz_4^{d-1}v_4 = 0, \tag{5}$$

$$\begin{aligned} &\sum_{j=1}^{3}\sum_{|\alpha|\le d} v_\alpha \frac{\partial z^\alpha}{\partial z_j}\xi_j^{(1)} + \sum_{j,k=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j\,\partial z_k} v_j\xi_k^{(1)} + \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_j} w_j^{(1)} \\ &+ d^2z_4^{d-1}v_4\xi_4^{(1)} + dz_4^d w_4^{(1)} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &\sum_{|\alpha|\le d}\left(\sum_{j=1}^{3}\frac{\partial z^\alpha}{\partial z_j}\xi_j^{(2)} + \sum_{j,k=1}^{3}\frac{\partial^2 z^\alpha}{\partial z_j\,\partial z_k}\xi_j^{(1)}\xi_k^{(1)}\right)v_\alpha \\ &+ \sum_{j=1}^{3}\sum_{|\alpha|\le d} a_\alpha\left(\sum_{k=1}^{3}\frac{\partial^2 z^\alpha}{\partial z_j\,\partial z_k}\xi_k^{(2)} + \sum_{k,l=1}^{3}\frac{\partial^3 z^\alpha}{\partial z_j\,\partial z_k\,\partial z_l}\xi_k^{(1)}\xi_l^{(1)}\right)v_j \\ &+ \sum_{|\alpha|\le d}\left(\sum_{j,k=1}^{3} a_\alpha\frac{\partial^2 z^\alpha}{\partial z_j\,\partial z_k}\big(w_j^{(1)}\xi_k^{(1)} + w_k^{(1)}\xi_j^{(1)}\big) + \sum_{j=1}^{3} a_\alpha\frac{\partial z^\alpha}{\partial z_j}w_j^{(2)}\right) \\ &+ v_4 d^2 z_4^{d-1}\big(\xi_4^{(2)} + d\big(\xi_4^{(1)}\big)^2\big) + 2d^2z_4^d w_4^{(1)}\xi_4^{(1)} + dz_4^d w_4^{(2)} = 0, \end{aligned} \tag{7}$$

$$
\begin{aligned}
&\sum_{|\alpha|\le d}\left(\sum_{j=1}^{3}\frac{\partial z^\alpha}{\partial z_j}\xi_j^{(3)}+3\sum_{j,k=1}^{3}\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}\xi_j^{(2)}\xi_k^{(1)}+\sum_{j,k,l=1}^{3}\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}\xi_j^{(1)}\xi_k^{(1)}\xi_l^{(1)}\right)v_\alpha\\
&+\sum_{j=1}^{3}\sum_{|\alpha|\le d}a_\alpha\left(\sum_{k=1}^{3}\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}\xi_k^{(3)}+3\sum_{k,l=1}^{3}\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}\xi_k^{(2)}\xi_l^{(1)}\right.\\
&\qquad\left.+\sum_{k,l,m=1}^{3}\frac{\partial^4 z^\alpha}{\partial z_j\partial z_k\partial z_l\partial z_m}\xi_k^{(1)}\xi_l^{(1)}\xi_m^{(1)}\right)v_j\\
&+\sum_{|\alpha|\le d}\left(\sum_{j,k,l=1}^{3}a_\alpha\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}\left(w_j^{(1)}\xi_k^{(1)}\xi_l^{(1)}+\xi_j^{(1)}w_k^{(1)}\xi_l^{(1)}+\xi_j^{(1)}\xi_k^{(1)}w_l^{(1)}\right)\right.\\
&\qquad+3\sum_{j,k=1}^{3}a_\alpha\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}\left(w_j^{(2)}\xi_k^{(1)}+\xi_j^{(2)}w_k^{(1)}\right)+\left.\sum_{j=1}^{3}a_\alpha\frac{\partial z^\alpha}{\partial z_j}w_j^{(3)}\right)\\
&+d^2z_4^{d-1}v_4\left(\xi_4^{(3)}+3d\xi_4^{(1)}\xi_4^{(2)}+d\left(\xi_4^{(1)}\right)^3\right)\\
&+dz_4^d w_4^{(3)}+3d^2z_4^d\left(\xi_4^{(2)}w_4^{(1)}+\xi_4^{(1)}w_4^{(2)}\right)+3d^3z_4^d w_4^{(1)}\left(\xi_4^{(1)}\right)^2=0.
\end{aligned}
\tag{8}
$$

We can introduce the first package of vector fields tangent to $\overline{J_3^v}(\mathcal{Y}_0)$. We denote by $\delta_j\in\mathbb{N}^3$ the multi-index whose j-component is equal to 1 and the other are zero.

For $\alpha_1\ge 4$:

$$
V_\alpha^{400}:=\frac{\partial}{\partial a_\alpha}-4z_1\frac{\partial}{\partial a_{\alpha-\delta_1}}+6z_1^2\frac{\partial}{\partial a_{\alpha-2\delta_1}}-4z_1^3\frac{\partial}{\partial a_{\alpha-3\delta_1}}+z_1^4\frac{\partial}{\partial a_{\alpha-4\delta_1}}.
$$

For $\alpha_1\ge 3,\ \alpha_2\ge 1$:

$$
\begin{aligned}
V_\alpha^{310}:=&\frac{\partial}{\partial a_\alpha}-3z_1\frac{\partial}{\partial a_{\alpha-\delta_1}}-z_2\frac{\partial}{\partial a_{\alpha-\delta_2}}+3z_1z_2\frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}}\\
&+3z_1^2\frac{\partial}{\partial a_{\alpha-2\delta_1}}-3z_1^2z_2\frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2}}-z_1^3\frac{\partial}{\partial a_{\alpha-3\delta_1}}+z_1^3z_2\frac{\partial}{\partial a_{\alpha-3\delta_1-\delta_2}}.
\end{aligned}
$$

For $\alpha_1\ge 2,\ \alpha_2\ge 2$:

$$
\begin{aligned}
V_\alpha^{220}:=&\frac{\partial}{\partial a_\alpha}-z_2\frac{\partial}{\partial a_{\alpha-\delta_2}}-z_1\frac{\partial}{\partial a_{\alpha-\delta_1}}+z_1z_2^2\frac{\partial}{\partial a_{\alpha-\delta_1-2\delta_2}}\\
&+z_1^2z_2\frac{\partial}{\partial a_{\alpha-2\delta_{-\delta_2}}}-z_1^2z_2^2\frac{\partial}{\partial a_{\alpha-2\delta_1-2\delta_2}}.
\end{aligned}
$$

For $\alpha_1 \geq 2,\ \alpha_2 \geq 1,\ \alpha_3 \geq 1$:

$$\begin{aligned} V_\alpha^{211} := {} & \frac{\partial}{\partial a_\alpha} - z_3 \frac{\partial}{\partial a_{\alpha-\delta_3}} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} - 2z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} + z_2 z_3 \frac{\partial}{\partial a_{\alpha-\delta_2-\delta_3}} \\ & + 2z_1 z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_3}} + 2z_1 z_2 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} + z_1^2 \frac{\partial}{\partial a_{\alpha-2\delta_1}} \\ & - 2z_1 z_2 z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2-\delta_3}} - z_1^2 z_3 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_3}} \\ & - z_1^2 z_2 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2}} + z_1^2 z_2 z_3 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2-\delta_3}}. \end{aligned}$$

Similar vector fields are constructed by permuting the z-variables, and changing the index α as indicated by the permutation. The pole order of the previous vector fields is equal to 4.

Lemma 4. *For any $(v_i)_{1\leq i\leq 4} \in \mathbb{C}^4$, there exist $v_\alpha(a)$, with degree at most 1 in the variables (a_γ), such that $V := \sum_\alpha v_\alpha(a)\, \partial/\partial a_\alpha + \sum_{1\leq j\leq 3} v_j\, \partial/\partial z_j + v_4 z_4\, \partial/\partial z_4$ is tangent to $\overline{J_3^v}(\mathcal{Y}_0)$ at each point.*

Proof. First, we substitute equations 1, 2, 3, 4 in equations 5, 6, 7, 8 to get rid of z_4, $\xi_4^{(i)}$ $(1 \leq i \leq 3)$. Then, we impose the additional conditions of vanishing for the coefficients of $\xi_j^{(1)}$ in the second equation (respectively of $\xi_j^{(1)}\xi_k^{(1)}$ in the third equation and $\xi_j^{(1)}\xi_k^{(1)}\xi_l^{(1)}$ in the fourth equation) for any $1 \leq j \leq k \leq l \leq 3$. Then the coefficients of $\xi_j^{(2)}$ (respectively $\xi_j^{(2)}\xi_k^{(1)}$ and $\xi_j^{(3)}$) are automatically zero in the third (respectively fourth) equation. The resulting equations are

$$\sum_{|\alpha|\leq d} v_\alpha z^\alpha + \sum_{j=1}^{3} \sum_{|\alpha|\leq d} a_\alpha \frac{\partial z^\alpha}{\partial z_j} v_j - d\, v_4 \sum_{|\alpha|\leq d} a_\alpha z^\alpha = 0,$$

$$\sum_{|\alpha|\leq d} v_\alpha \frac{\partial z^\alpha}{\partial z_j} + \sum_{k=1}^{3} \sum_{|\alpha|\leq d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} v_k - d\, v_4 \sum_{|\alpha|\leq d} a_\alpha \frac{\partial z^\alpha}{\partial z_j} = 0,$$

$$\sum_{|\alpha|\leq d} \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} v_\alpha + \sum_{l=1}^{3} \sum_{|\alpha|\leq d} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l} v_l - d\, v_4 \sum_{|\alpha|\leq d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} = 0,$$

$$\sum_{|\alpha|\leq d} \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l} v_\alpha + \sum_{m=1}^{3} \sum_{|\alpha|\leq d} a_\alpha \frac{\partial^4 z^\alpha}{\partial z_j \partial z_k \partial z_l \partial z_m} v_m - d\, v_4 \sum_{|\alpha|\leq d} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l} = 0.$$

Now we can observe that if the $v_\alpha(a)$ satisfy the first equation, they automatically satisfy the other ones because the v_α are constants with respect to z. Therefore it is

sufficient to find (v_α) satisfying the first equation. We identify the coefficients of $z^\rho = z_1^{\rho_1} z_2^{\rho_2} z_3^{\rho_3}$:

$$v_\rho + \sum_{j=1}^{3} a_{\rho+\delta_j} v_j(\rho_j + 1) - d\, v_4 a_\rho = 0. \qquad \square$$

Another family of vector fields can be obtained in the following way. Consider a 4×4-matrix $A = \begin{pmatrix} A_1^1 & A_1^2 & A_1^3 & 0 \\ A_2^1 & A_2^2 & A_2^3 & 0 \\ A_3^1 & A_3^2 & A_3^3 & 0 \\ A_4^1 & A_4^2 & A_4^3 & 0 \end{pmatrix} \in \mathcal{M}_4(\mathbb{C})$ and let $\widetilde{V} := \sum_{j,k} w_j^{(k)}\, \partial/\partial \xi_j^{(k)}$, where $w^{(k)} := A\xi^{(k)}$, for $k = 1, 2, 3$.

Lemma 5. *There exist polynomials $v_\alpha(z, a) := \sum_{|\beta|\le 3} v_\beta^\alpha(a) z^\beta$ where each coefficient v_β^α has degree at most 1 in the variables (a_γ) such that*

$$V := \sum_\alpha v_\alpha(z, a) \frac{\partial}{\partial a_\alpha} + \widetilde{V}$$

is tangent to $\overline{J_3^v}(\mathcal{Y}_0)$ at each point.

Proof. First, we substitute equations 1, 2, 3, 4 in equations 5, 6, 7, 8 to get rid of z_4, $\xi_4^{(i)}$ $(1 \le i \le 3)$. We impose the additional conditions of vanishing for the coefficients of $\xi_j^{(1)}$ in the second equation (respectively of $\xi_j^{(1)}\xi_k^{(1)}$ in the third equation and $\xi_j^{(1)}\xi_k^{(1)}\xi_l^{(1)}$ in the fourth equation) for any $1 \le j \le k \le l \le 3$. Then the coefficients of $\xi_j^{(2)}$ (respectively $\xi_j^{(2)}\xi_k^{(1)}$ and $\xi_j^{(3)}$) are automatically zero in the third (respectively fourth) equation. The resulting equations are

$$\sum_{|\alpha|\le d} v_\alpha z^\alpha = 0, \tag{9}$$

$$\sum_{|\alpha|\le d} v_\alpha \frac{\partial z^\alpha}{\partial z_j} + \sum_{k=1}^{3} \sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_k} A_k^j - d A_4^j \sum_{|\alpha|\le d} a_\alpha z^\alpha = 0, \tag{10_j}$$

$$\sum_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} v_\alpha + \sum_{\alpha,p} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_p} A_p^k + \sum_{\alpha,p} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_k \partial z_p} A_p^j - 2d A_4^j \sum_{|\alpha|\le d} a_\alpha \frac{\partial z^\alpha}{\partial z_k} = 0, \tag{11_{jk}}$$

$$\begin{aligned} &\sum_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l} v_\alpha + \sum_{\alpha,p} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_p \partial z_k \partial z_l} A_p^j + \sum_{\alpha,p} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_p \partial z_l} A_p^k \\ &+ \sum_{\alpha,p} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_p} A_p^l - 3d A_4^l \sum_{|\alpha|\le d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} = 0. \end{aligned} \tag{12_{jkl}}$$

The equations for the unknowns v_β^α are obtained by identifying the coefficients of the monomials z^ρ in the above equations.

The monomials z^ρ in (9) are $z_1^{\rho_1}\ z_2^{\rho_2}\ z_3^{\rho_3}$ with $\sum \rho_i \le d$.

If all the components of ρ are greater than 3, then we obtain the following system:

13. The coefficient of z^ρ in (9) impose the condition

$$\sum_{\alpha+\beta=\rho} v_\beta^\alpha = 0.$$

14_j. The coefficient of the monomial $z^{\rho-\delta_j}$ in (10_j) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j v_\beta^\alpha = l_j(a)$$

where l_j is a linear expression in the a-variables.

14_{jj}. For $j = 1, \ldots, 3$ the coefficient of the monomial $z^{\rho-2\delta_j}$ in (11_{jj}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j(\alpha_j - 1)v_\beta^\alpha = l_{jj}(a).$$

14_{jk}. For $1 \le j < k \le 3$ the coefficient of the monomial $z^{\rho-\delta_j-\delta_k}$ in (11_{jk}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j\alpha_k v_\beta^\alpha = l_{jk}(a).$$

15_{jjj}. For $j = 1, \ldots, 3$ the coefficient of the monomial $z^{\rho-3\delta_j}$ in (12_{jjj}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j(\alpha_j - 1)(\alpha_j - 2)v_\beta^\alpha = l_{jjj}(a).$$

15_{jjk}. For $1 \le j < k \le 3$ the coefficient of the monomial $z^{\rho-2\delta_j-\delta_k}$ in (12_{jjk}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j(\alpha_j - 1)\alpha_k v_\beta^\alpha = l_{jjk}(a).$$

15_{jkl}. For $1 \le j < k < l \le 3$ the coefficient of the monomial $z^{\rho-\delta_j-\delta_k-\delta_l}$ in (12_{jjk}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j\alpha_k\alpha_l v_\beta^\alpha = l_{jkl}(a).$$

The determinant of the matrix associated to the system is not zero. Indeed, for each ρ the matrix whose column C_β consists of the partial derivatives of order at most 3 of the monomial $z^{\rho-\beta}$ has the same determinant, at the point $z_0 = (1, 1, 1)$, as our system. Therefore if the determinant is zero, we would have a non-identically zero polynomial

$$Q(z) = \sum_\beta a_\beta z^{\rho-\beta}$$

such that all its partial derivatives of order less or equal to 3 vanish at z_0. Thus the same is true for

$$P(z) = z^\rho Q\left(\frac{1}{z_1}, \ldots, \frac{1}{z_3}\right) = \sum_\beta a_\beta z^\beta.$$

But this implies $P \equiv 0$.

Finally, we conclude by Cramer's rule. The systems we have to solve are never over determined. The lemma is proved. □

REMARK 6. We have chosen the matrix A with this form because we are interested to prove the global generation statement on $\overline{J_3^v}(\mathcal{Y}) \setminus (\Sigma \cup p^{-1}(H))$ where Σ is the closure of $\Sigma_0 = \left\{(z, a, \xi^{(1)}, \xi^{(2)}, \xi^{(3)}) \in \overline{J_3^v}(\mathcal{Y}_0)/\det\left(\xi_i^{(j)}\right)_{1\le i,j} \le 3 = 0\right\}$.

Proposition 7. *The vector space* $T_{\overline{J_3^v}(\mathcal{Y})} \otimes \mathcal{O}_{\mathbb{P}^4}(12) \otimes \mathcal{O}_{\mathbb{P}^{N_d+1}}(*)$ *is generated by its global sections on* $\overline{J_3^v}(\mathcal{Y}) \setminus (\Sigma \cup p^{-1}(H))$.

Proof. From the preceding lemmas, we are reduced to consider $V = \sum_{|\alpha|\le 3} v_\alpha \partial/\partial a_\alpha$. The conditions for V to be tangent to $\overline{J_3^v}(\mathcal{Y}_0)$ are

$$\sum_{|\alpha|\le 3} v_\alpha z^\alpha = 0,$$

$$\sum_{j=1}^{3} \sum_{|\alpha|\le 3} v_\alpha \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(1)} = 0,$$

$$\sum_{|\alpha|\le 3} \left(\sum_{j=1}^{3} \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(2)} + \sum_{j,k=1}^{3} \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} \xi_j^{(1)} \xi_k^{(1)} \right) v_\alpha = 0,$$

$$\sum_{|\alpha|\le 3} \left(\sum_{j=1}^{3} \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(3)} + 3 \sum_{j,k=1}^{3} \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} \xi_j^{(2)} \xi_k^{(1)} + \sum_{j,k,l=1}^{3} \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l} \xi_j^{(1)} \xi_k^{(1)} \xi_l^{(1)} \right) v_\alpha = 0.$$

We denote by W_{jkl} the wronskian operator corresponding to the variables z_j, z_k, z_l. We have $W_{123} := \det\left(\xi_j^{(i)}\right)_{1\le i,j\le 3} \not\equiv 0$. Then we can solve the previous system with

$v_{000}, v_{100}, v_{010}, v_{001}$ as unknowns. By the Cramer rule, each of the previous quantity is a linear combination of the v_α, $|\alpha| \leq 3$, $\alpha \neq (000), (100), (010), (001)$ with coefficients rational functions in $z, \xi^{(1)}, \xi^{(2)}, \xi^{(3)}$. The denominator is W_{123} and the numerator is a polynomial whose monomials verify either:

i) degree in z at most 3 and degree in each $\xi^{(i)}$ at most 1.

ii) degree in z at most 2 and degree in $\xi^{(1)}$ at most 3, degree in $\xi^{(2)}$ at most 0, degree in $\xi^{(3)}$ at most 1.

iii) degree in z at most 2 and degree in $\xi^{(1)}$ at most 2, degree in $\xi^{(2)}$ at most 2, degree in $\xi^{(3)}$ at most 0.

iv) degree in z at most 1 and degree in $\xi^{(1)}$ at most 4, degree in $\xi^{(2)}$ at most 1, degree in $\xi^{(3)}$ at most 0.

$\xi^{(1)}$ has a pole of order 2, $\xi^{(2)}$ has a pole of order 3 and $\xi^{(3)}$ has a pole of order 4, therefore the previous vector field has order at most 12. □

Corollary 8. *The vector space* $T_{\overline{J_3^v}(\mathbb{P}^3 \times \mathbb{P}^{N_d})} \otimes \mathcal{O}_{\mathbb{P}^3}(12) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(*)$ *is generated by its global sections on* $\overline{J_3^v}(\mathbb{P}^3 \times \mathbb{P}^{N_d}) \setminus (\pi_3(\Sigma) \cup \mathcal{X})$.

REMARK 9. If the third derivative of $f\colon (\mathbb{C}, 0) \to \mathbb{P}^3 \times \mathbb{P}^{N_d} \setminus \mathcal{X}$ lies inside $\pi_3(\Sigma)$ then the image of f is contained in a hyperplane.

4. Logarithmic jet differentials

In this section we recall the basic facts about logarithmic jet differentials following G. Dethloff and S. Lu [5]. Let X be a complex manifold with a normal crossing divisor D.

Let (X, D) be the corresponding complex log-manifold. We start with the directed manifold $(X, \overline{T}_X)$ where $\overline{T}_X = T_X(-\log D)$. We define $X_1 := \mathbb{P}(\overline{T}_X)$, $D_1 = \pi^*(D)$ and $V_1 \subset T_{X_1}$:

$$V_{1,(x,[v])} := \{\xi \in \overline{T}_{X_1,(x,[v])}(-\log D_1); \pi_*\xi \in \mathbb{C}v\}$$

where $\pi\colon X_1 \to X$ is the natural projection. If $f\colon (\mathbb{C}, 0) \to (X \setminus D, x)$ is a germ of holomorphic curve then it can be lifted to $X_1 \setminus D_1$ as $f_{[1]}$.

By induction, we obtain a tower of varieties (X_k, D_k, V_k) with $\pi_k\colon X_k \to X$ as the natural projection. We have a tautological line bundle $\mathcal{O}_{X_k}(1)$ and we denote $u_k := c_1(\mathcal{O}_{X_k}(1))$.

Let's consider the direct image $\pi_{k*}(\mathcal{O}_{X_k}(m))$. It's a locally free sheaf denoted $E_{k,m}\overline{T}^*_X$ generated by all polynomial operators in the derivatives of order $1, 2, \ldots, k$ of f, together with the extra function $\log s_j(f)$ along the j-th component of D, which are moreover invariant under arbitrary changes of parametrization: a germ of operator $Q \in E_{k,m}\overline{T}^*_X$ is characterized by the condition that, for every germ in $X \setminus D$ and every germ $\phi \in \mathbb{G}_k$ of k-jet biholomorphisms of $(\mathbb{C}, 0)$,

$$Q(f \circ \phi) = \phi'^m Q(f) \circ \phi.$$

The following theorem makes clear the use of jet differentials in the study of hyperbolicity:

Theorem ([8], [3], [5]). *Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that*

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_k^* L^{-1}) \simeq H^0(X, E_{k,m}\overline{T}_X^* \otimes L^{-1})$$

has non zero sections $\sigma_1, \ldots, \sigma_N$. Let $Z \subset X_k$ be the base locus of these sections. Then every entire curve $f: \mathbb{C} \to X \setminus D$ is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant polynomial differential operator P with values in L^{-1}, every entire curve $f: \mathbb{C} \to X \setminus D$ must satisfy the algebraic differential equation $P(f) = 0$.

If $X \subset \mathbb{P}^3$ is a smooth hypersurface, we have established in [15] the next result:

Theorem ([15]). *Let X be a smooth hypersurface of $\mathbb{P}^3$ such that $d = \deg(X) \geq 92$, and A an ample line bundle, then $E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes A^{-1}$ has global sections for m large enough and every entire curve $f: \mathbb{C} \to \mathbb{P}^3 \setminus X$ must satisfy the corresponding algebraic differential equation.*

The proof relies on the filtration of $E_{3,m}\overline{T}_X^*$ obtained in [14]:

$$Gr^\bullet E_{3,m}\overline{T}_X^* = \bigoplus_{0 \leq \gamma \leq m/5} \left(\bigoplus_{\{\lambda_1+2\lambda_2+3\lambda_3 = m-\gamma;\, \lambda_i - \lambda_j \geq \gamma,\, i<j\}} \Gamma^{(\lambda_1,\lambda_2,\lambda_3)}\overline{T}_X^* \right)$$

where Γ is the Schur functor.

This filtration provides a Riemann-Roch computation of the Euler characteristic [14]:

$$\chi(\mathbb{P}^3, E_{3,m}\overline{T}^*_{\mathbb{P}^3}) = m^9 \left(\frac{389}{81648000000} d^3 - \frac{6913}{34020000000} d^2 + \frac{6299}{4252500000} d - \frac{1513}{63787500} \right) + O(m^8).$$

In dimension 3 there is no Bogomolov vanishing theorem (cf. [1]) as it is used in dimension 2 to control the cohomology group H^2, therefore we need the following proposition obtained in [15]:

Proposition ([15]). *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition such that $\lambda_1 > \lambda_2 > \lambda_3$ and $|\lambda| = \sum \lambda_i > 3d + 2$. Then*:

$$h^2(\mathbb{P}^3, \Gamma^\lambda \overline{T}^*_{\mathbb{P}^3}) \leq g(\lambda)(d + 14) + r(\lambda)$$

where $g(\lambda) = (3|\lambda|^3/2)\prod_{\lambda_i>\lambda_j}(\lambda_i - \lambda_j)$ *and* r *is polynomial in* λ *with homogeneous components of degrees at most* 5.

This proposition provides the estimate [15]

$$h^2(\mathbb{P}^3, Gr^\bullet E_{3,m}\overline{T}^*_{\mathbb{P}^3}) \le C(d+14)m^9 + O(m^8)$$

where C is a constant.

5. Proof of Theorem 3

Let us consider an entire curve $f: \mathbb{C} \to \mathbb{P}^3 \setminus X$ for a generic hypersurface of $\mathbb{P}^3$. By Riemann-Roch and the proposition of the previous section we obtain the following lemma:

Lemma 10. *Let* X *be a smooth hypersurface of* $\mathbb{P}^3$ *of degree* d, $0 < \delta < 1/18$ *then* $h^0\big(\mathbb{P}^3, E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}\big) \ge \alpha(d,\delta)m^9 + O(m^8)$, *with*

$$\begin{aligned}\alpha(d,\delta) = \frac{1}{408240000000}(&67737600\delta^3 + 1945d^3 - 82956d^2 - 968320\\ &+ 1804680d^2\delta + 12700800d^2\delta^3 - 9408960d^2\delta^2 + 37635840d\delta^2\\ &- 8579520d\delta - 50803200d\delta^3 - 1058400d^3\delta^3 - 105030d^3\delta\\ &- 50181120\delta^2 + 12165120\delta + 604704d + 784080d^3\delta^2).\end{aligned}$$

Proof. $E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}$ admits a filtration with graded pieces

$$\Gamma^{(\lambda_1,\lambda_2,\lambda_3)}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3} = \Gamma^{(\lambda_1-\delta m,\lambda_2-\delta m,\lambda_3-\delta m)}\overline{T}^*_{\mathbb{P}^3}$$

for $\lambda_1 + 2\lambda_2 + 3\lambda_3 = m - \gamma$; $\lambda_i - \lambda_j \ge \gamma$, $i < j$, $0 \le \gamma \le m/5$.

We compute by Riemann-Roch

$$\chi\big(\mathbb{P}^3, E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}\big) = \chi\big(X, Gr^\bullet E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}\big).$$

We use the proposition of the previous section to control

$$\begin{aligned}&h^2\big(X, E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}\big):\\ h^2\big(\mathbb{P}^3, \Gamma^{(\lambda_1-\delta m,\lambda_2-\delta m,\lambda_3-\delta m)}\overline{T}^*_{\mathbb{P}^3}\big) &\le g(\lambda_1-\delta m, \lambda_2-\delta m, \lambda_3-\delta m)(d+14)\\ &\quad + r(\lambda_1-\delta m, \lambda_2-\delta m, \lambda_3-\delta m)\end{aligned}$$

under the hypothesis $\sum\lambda_i - 3\delta m > 3d + 2$. The conditions verified by λ imply $\sum\lambda_i \ge$

$m/6$ therefore the hypothesis will be verified if

$$m\left(\frac{1}{6} - 3\delta\right) > 3d + 2.$$

We conclude with the computation

$$\chi\left(\mathbb{P}^3, E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}\right) - h^2\left(\mathbb{P}^3, Gr^\bullet E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}\right) \leq h^0\left(\mathbb{P}^3, E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}\right). \square$$

REMARK 11. If we denote $(\mathbb{P}^3 \times \mathbb{P}^{N_d})^v_3$ the quotient of $\overline{J^{v\,reg}_3}(\mathbb{P}^3 \times \mathbb{P}^{N_d})$ by the reparametrization group $\mathbb{G}_3$, one can easily verify that each vector field given at Section 3 defines a section of the tangent bundle of the manifold $(\mathbb{P}^3 \times \mathbb{P}^{N_d})^v_3$.

We have a section

$$\sigma \in H^0\left(\mathbb{P}^3, E_{3,m}\overline{T}^*_{\mathbb{P}^3} \otimes \overline{K}^{-\delta m}_{\mathbb{P}^3}\right) \simeq H^0\left((\mathbb{P}^3)_3, \mathcal{O}_{(\mathbb{P}^3)_3}(m) \otimes \pi_3^*\overline{K}^{-\delta m}_{\mathbb{P}^3}\right).$$

with zero set Z and vanishing order $\delta m(d-4)$. Consider the family

$$\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{P}^{N_d}$$

of hypersurfaces of degree d in $\mathbb{P}^3$. General semicontinuity arguments concerning the cohomology groups show the existence of a Zariski open set $U_d \subset \mathbb{P}^{N_d}$ such that for any $a \in U_d$, there exists a divisor

$$Z_a = (P_a = 0) \subset (\mathbb{P}^3_a)_3$$

where

$$P_a \in H^0\left((\mathbb{P}^3_a)_3, \mathcal{O}_{(\mathbb{P}^3_a)_3}(m) \otimes \pi_3^*\overline{K}^{-\delta m}_{(\mathbb{P}^3_a)}\right)$$

such that the family $(P_a)_{a\in U_d}$ varies holomorphically. We consider P as a holomorphic function on $\overline{J_3}(\mathbb{P}^3_a)$. The vanishing order of this function is no more than m at a generic point of $\mathbb{P}^3_a$. We have $f_{[3]}(\mathbb{C}) \subset Z_a$.

Then we invoke Corollary 8 which gives the global generation of

$$T_{\overline{J^v_3}(\mathbb{P}^3\times\mathbb{P}^{N_d})} \otimes \mathcal{O}_{\mathbb{P}^3}(12) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(*)$$

on $\overline{J^v_3}(\mathbb{P}^3 \times \mathbb{P}^{N_d}) \setminus (\pi_3(\Sigma) \cup \mathcal{X})$.

If $f_{[3]}(\mathbb{C})$ lies in $\pi_3(\Sigma)$, f is algebraically degenerated. So we can suppose it is not the case.

At any point of $f_{[3]}(\mathbb{C}) \setminus \pi_3(\Sigma)$ where the vanishing of P is no more than m, we can find global meromorphic vector fields $v_1, \ldots, v_p$ $(p \leq m)$ and differentiate P

with these vector fields such that $v_1 \cdots v_p P$ is not zero at this point. From the above remark, we see that $v_1 \cdots v_p P$ corresponds to an invariant differential operator and its restriction to $(\mathbb{P}^3_a)_3$ can be seen as a section of the bundle

$$\mathcal{O}_{(\mathbb{P}^3_a)_3}(m) \otimes \mathcal{O}_{\mathbb{P}^3}(12p - \delta m(d-4)).$$

Assume that the vanishing order of P is larger than the sum of the pole order of the v_i in the fiber direction of $\pi\colon \mathbb{P}^3 \times \mathbb{P}^{N_d} \to \mathbb{P}^{N_d}$. Then the restriction of $v_1 \cdots v_p P$ to $\mathbb{P}^3_a$ defines a jet differential which vanishes on an ample divisor. Therefore $f_{[3]}(\mathbb{C})$ should be in its zero set.

To finish the proof, we just have to see when the vanishing order of P is larger than the sum of the pole order of the v_i. This will be verified if

$$\delta(d-4) > 12.$$

So we want $\delta > 12/(d-4)$ and $\alpha(d, \delta) > 0$. This is the case for $d \geq 586$.

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