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FUNDAMENTAL THEOREMS OF LAGRANGIAN SURFACES IN $S^2 \times S^2$

Dedicated to Professor Masafumi Okumura on the occasion of his 70-th birthday

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Abstract

Existence and $SO(3) \times SO(3)$-congruence of Lagrangian immersion from oriented 2-dimensional Riemannian manifold to the Riemannian product of 2-spheres are studied. In particular, we will show that two minimal Lagrangian immersions are $SO(3) \times SO(3)$-congruent if and only if the corresponding angle functions are coincide.

1. Introduction

Lagrangian submanifolds in symplectic manifolds are one of the most important object in geometry, and Hermitian symmetric spaces are essential examples among symplectic manifolds. To study Lagrangian submanifolds of Hermitian symmetric spaces from differential geometric viewpoint, the following problems are fundamental: (i) Find the condition for which there exists Lagrangian isometric (in particular minimal) immersion from n-dimensional Riemannian manifold to Hermitian symmetric space $\tilde{M}$. (ii) For given two Lagrangian (minimal) isometric immersions x_1, x_2 from a Riemannian manifold M to Hermitian symmetric space $\tilde{M}$, find the condition for which x_1 and x_2 are congruent by a holomorphic isometry of $\tilde{M}$. When $\tilde{M}$ is a complex space form, the results are already known (cf. [2]), but for higher rank cases, it seems that there are no such results. On the other hand, recently it was shown [5] that totally geodesic Lagrangian torus $S^1 \times S^1$ in $S^2 \times S^2$ has Hamiltonian volume minimizing property. In this paper we will obtain existence and $SO(3) \times SO(3)$-congruence theorems for Lagrangian isometric (minimal) immersions from 2-dimensional oriented Riemannian manifolds to $S^2 \times S^2$ with respect to complex structure (J, J), where J denotes the complex structure on S^2 which is determined by an orientation.

With respect to submanifolds $x \colon M \to M_1 \times M_2$ in product manifolds, the almost product structure $\tilde{P}$ plays an important role (cf. [6]). For example if each tangent space of M is invariant under $\tilde{P}$, then M is decomposed as a product manifold and x is a

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product immersion. For a Lagrangian surface M^2 in $S^2 \times S^2$, we introduce *angle function* φ on M^2, by measuring the behavior of each tangent space T_pM^2 under the action of $\tilde{P}$. We note that φ is also described by the Kähler angle of M^2 in $S^2 \times S^2$ with respect to another complex structure $(J, -J)$ whose associated symplectic structure is the twisted product form (cf. [7], §3.4).

We will show that two Lagrangian isometric immersions from 2-dimensional oriented Riemannian manifold M^2 to $S^2 \times S^2$ are $SO(3) \times SO(3)$-congruent if and only if each second fundamental tensor and angle function coincide (Theorem 2). Here we note that the full holomorphic isometry group G of $S^2 \times S^2$ with respect to (J, J) is generated by $SO(3) \times SO(3)$ and the map $S^2 \times S^2 \to S^2 \times S^2$, $(x_1, x_2) \mapsto (x_2, x_1)$. Then the above result does not hold for G (Remark 1).

Next, we will show that when the Lagrangian isometric immersion is minimal, the congruence class is determined by only the angle function (Theorem 4). For Lagrangian submanifolds in Kähler manifolds, Gauss and Codazzi equations are expressed as intrinsic equations, because the second fundamental form is described by a symmetric $(0, 3)$-tensor fields T on the submanifold. But in general these equations do not guarantee the existence of such Lagrangian isometric immersion. Nevertheless we will prove (Theorem 5) that on a simply connected Riemannian 2-manifold M^2, if certain two equations with respect to the metric and a function φ in M^2 hold, which are essentially equivalent to Gauss and Codazzi equations, then there exists a Lagrangian isometric *minimal* immersion from M^2 to $S^2 \times S^2$ such that φ is the corresponding angle function. As a special case, when M^2 is a domain of $\mathbf{R}^2$ and both of the metric and the function φ are rotationally symmetric, Gauss and Codazzi equations are written as two nonlinear ordinary differential equations of second order. By using a solution of the equation, we can obtain non-trivial minimal Lagrangian surfaces in $S^2 \times S^2$. Note that minimal Lagrangian surfaces in $S^2 \times S^2$ are studied in [1] from different viewpoint.

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2. Lagrangian surfaces in $S^2 \times S^2$

Let $\tilde{M}$ be a Kähler manifold of complex dimension m with Kähler form θ and complex structure J. Let M be a real m-dimensional submanifold and let $x: M \to \tilde{M}$ be a Lagrangian immersion, i.e., $x^*\theta = 0$ on M, or equivalently, for any tangent vector X of M, JX is contained in the normal space to M. We denote the Levi-Civita connection of M by ∇ and σ is the second fundamental form of $M \to \tilde{M}$. Then we have the following (cf. [2]).

$$\langle \sigma(X, Y), JZ\rangle = \langle \sigma(Y, Z), JX\rangle = \langle \sigma(Z, X), JY\rangle, \tag{2.1}$$

$$\nabla^{\perp}_X(JY) = J\nabla_X Y, \tag{2.2}$$

for tangent vectors X, Y and Z of M where $\nabla^{\perp}$ is the connection on the normal bundle.

Let S^2 be a unit sphere in $\mathbf{R}^3$. For any $p \in S^2$, we define a linear transformation J of the tangent space T_pS^2 of S^2 at p as

$$Jv = p \times v \tag{2.3}$$

by the vector product $\times$ of $\mathbf{R}^3$, so J is a complex structure on S^2. Then the special orthogonal group $SO(3)$ acts naturally for S^2 and is the isometry group for the Riemannian metric on S^2 which is induced by the standard inner product of $\mathbf{R}^3$. Moreover $SO(3)$ preserves J. Standard symplectic form θ on S^2 is given by $\theta_p(u, v) = (p \times u) \cdot v$, where $u, v \in T_pS^2$ and $\cdot$ is the induced Riemannian metric on S^2 by the inclusion $S^2 \subset \mathbf{R}^3$.

We define a complex structure $\tilde{J}$ on $S^2 \times S^2$ by

$$\tilde{J}(X_1, X_2) = (JX_1, JX_2) \tag{2.4}$$

for all tangent vectors (X_1, X_2) to $S^2 \times S^2$. Let $\langle\ ,\ \rangle$ be the product metric on $S^2 \times S^2$ defined by

$$\langle (X_1, X_2), (Y_1, Y_2)\rangle = X_1 \cdot Y_1 + X_2 \cdot Y_2.$$

Then $\langle\ ,\ \rangle$ is a Hermitian metric and $S^2 \times S^2$ is a Kähler manifold with respect to the complex structure $\tilde{J}$. $S^2 \times S^2$ is considered as a symplectic manifold with symplectic form $\tilde{\theta} = (\mathrm{pr}_1)^*\theta + (\mathrm{pr}_2)^*\theta$, where $\mathrm{pr}_1, \mathrm{pr}_2 : S^2 \times S^2 \to S^2$ are projection maps into first factor and second factor, respectively, and θ is the standard symplectic form on S^2.

Let $\tilde{P}$ be the tensor field of type (1, 1) on $S^2 \times S^2$, defined by

$$\tilde{P}(X_1, X_2) = (X_1, -X_2). \tag{2.5}$$

Then we have (cf. [6])

$$\tilde{P}^2 = 1, \tag{2.6}$$

$$\langle \tilde{P}X, Y\rangle = \langle X, \tilde{P}Y\rangle, \tag{2.7}$$

$$\operatorname{trace} \tilde{P} = 0, \tag{2.8}$$

$$\bar{\nabla}\tilde{P} = 0 \tag{2.9}$$

where X, Y are any tangent vectors of $S^2 \times S^2$ and $\bar{\nabla}$ denotes the Levi-Civita connection of $S^2 \times S^2$. $\tilde{P}$ is called the *almost product structure* of $S^2 \times S^2$. (2.4) and (2.5) imply

$$\tilde{P}\tilde{J} = \tilde{J}\tilde{P}. \tag{2.10}$$

Let M^2 be an oriented Riemannian manifold of dimension 2 and let

$$x : M^2 \to S^2 \times S^2, \quad x(p) = (x_1(p), x_2(p)) \tag{2.11}$$

be a Lagrangian immersion, i.e., $x^*\tilde{\theta} = 0$. If $\{e_1, e_2\}$ is an orthonormal basis for the tangent space T_pM^2 at $p \in M^2$, which is compatible with the orientation of M^2, then $\{\tilde{J}e_1, \tilde{J}e_2\}$ is an orthonormal basis for an orthogonal complement $T_p^{\perp}M^2$ of T_pM^2. Thus $\{e_1, e_2, \tilde{J}e_1, \tilde{J}e_2\}$ is an (oriented) orthonormal basis for $T_{x(p)}(S^2 \times S^2)$. So we put

$$\tilde{P}x_*X = x_*PX + \tilde{J}x_*QX \tag{2.12}$$

for $X \in T_pM^2$ where P and Q are linear endomorphisms in T_pM^2. Then it follows from (2.6)–(2.10) that

$$\operatorname{trace} P = 0, \quad P^2 - Q^2 = 1, \quad PQ + QP = 0, \tag{2.13}$$

$$\langle PX, Y\rangle = \langle X, PY\rangle, \quad \langle QX, Y\rangle + \langle X, QY\rangle = 0, \tag{2.14}$$

$$\begin{aligned} \langle(\nabla_X P)Y, Z\rangle &= \langle\sigma(X, QY), \tilde{J}Z\rangle + \langle\sigma(X, Y), \tilde{J}QZ\rangle, \\ \langle(\nabla_X Q)Y, Z\rangle &= \langle\sigma(X, Y), \tilde{J}PZ\rangle - \langle\sigma(X, PY), \tilde{J}Z\rangle. \end{aligned} \tag{2.15}$$

Then, from (2.13), there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM^2 compatible with the orientation of M^2 and $\varphi \in [-\pi/4, \pi/4]$ such that

$$\begin{cases} Pe_1 = \cos 2\varphi e_1, & Pe_2 = -\cos 2\varphi e_2, \\ Qe_1 = -\sin 2\varphi e_2, & Qe_2 = \sin 2\varphi e_1. \end{cases} \tag{2.16}$$

Here we note that such $\{e_1, e_2\}$ is uniquely determined up to $e_i \mapsto -e_i$ $(i = 1, 2)$. Clearly φ is continuous and when $\varphi \in (-\pi/4, \pi/4)$, φ is differentiable. We call φ the *angle function* for a Lagrangian immersion x from an oriented 2-dimensional Riemannian manifold M^2 to $S^2 \times S^2$.

Next we show that the angle function φ is essentially same as the Kähler angle of M^2 in $S^2 \times S^2$ with respect to the complex structure $(J, -J)$. Let f be an immersion of an oriented 2-dimensional manifold M into a Kähler manifold $(\tilde{M}, J)$. The Kähler angle of f is defined to be the angle between Jf_*e_1 and f_*e_2 for an orthonormal basis $\{e_1, e_2\}$ compatible with the orientation of M. On $S^2 \times S^2$, we consider another complex structure $(J, -J)$ as

$$(J, -J)(X_1, X_2) = (JX_1, -JX_2)$$

for $(X_1, X_2) \in T(S^2 \times S^2)$. Then the corresponding symplectic form is nothing but the twisted product form $(\mathrm{pr}_1)^*\theta - (\mathrm{pr}_2)^*\theta$.

By (2.5), (2.16) and identifications $TM \cong TM \times \{0\}$, $TM \cong \{0\} \times TM$, we have

$$\begin{aligned} x_*e_1 &= \left(\frac{x_*e_1 + \tilde{P}x_*e_1}{2}, \frac{x_*e_1 - \tilde{P}x_*e_1}{2}\right) \\ &= (\cos\varphi(\cos\varphi x_*e_1 - \sin\varphi \tilde{J}x_*e_2), \sin\varphi(\sin\varphi x_*e_1 + \cos\varphi \tilde{J}x_*e_2)), \end{aligned}$$

$$x_*e_2 = \left(\frac{x_*e_2 + \tilde{P}x_*e_2}{2}, \frac{x_*e_2 - \tilde{P}x_*e_2}{2}\right)$$
$$= (\sin\varphi(\sin\varphi x_*e_2 + \cos\varphi \tilde{J}x_*e_1), \cos\varphi(\cos\varphi x_*e_2 - \sin\varphi \tilde{J}x_*e_1)),$$

where we consider x_*e_1 and x_*e_2 as vectors in $\mathbf{R}^3 \times \mathbf{R}^3$. For $p \in M$, if we put

$$\begin{aligned} V_1(p) &:= \cos\varphi(p)(x_*)_p e_1 - \sin\varphi(p)\tilde{J}(x_*)_p e_2, \\ V_2(p) &:= \cos\varphi(p)(x_*)_p e_2 - \sin\varphi(p)\tilde{J}(x_*)_p e_1, \end{aligned} \tag{2.17}$$

then the above equations are written as

$$x_*e_1 = (\cos\varphi V_1, \sin\varphi \tilde{J}V_2), \quad x_*e_2 = (\sin\varphi \tilde{J}V_1, \cos\varphi V_2). \tag{2.18}$$

So we may regard as

$$\begin{aligned} V_1(p),\ JV_1(p) &= \tilde{J}V_1(p) \in T_{x_1(p)}S^2, \\ V_2(p),\ JV_2(p) &= \tilde{J}V_2(p) \in T_{x_2(p)}S^2, \end{aligned} \tag{2.19}$$

where J is the complex structure of S^2 defined by (2.3), and $(x_1(p), x_2(p)) \in S^2 \times S^2$ as (2.11), and V_1, JV_1, V_2, JV_2 are $\mathbf{R}^3$-valued vector fields on M^2.

Proposition 1. *Let $\varphi\colon M^2 \to [-\pi/4, \pi/4]$ be the angle function of a Lagrangian isometric immersion from an oriented surface M^2 to $(S^2 \times S^2, \tilde{J} = (J, J))$. Then the Kähler angle with respect to the complex structure $(J, -J)$ of $S^2 \times S^2$ is equal to $\pi/2 - 2\varphi$. Consequently, when $\varphi \equiv \pm\pi/4$ the immersion x is $\pm$-holomorphic with respect to $(J, -J)$.*

Proof. Let $\{e_1, e_2\}$ be the orthonormal basis of M^2 compatible with the orientation of M^2 given by (2.16). For the complex structure $(J, -J)$, we have

$$(J, -J)x_*e_1 = (\cos\varphi JV_1, \sin\varphi V_2),$$

by (2.18) and (2.19). Hence, using $\|JV_1\| = \|V_2\| = 1$, we get

$$\langle (J, -J)x_*e_1, x_*e_2\rangle = \sin 2\varphi = \cos\left(\frac{\pi}{2} - 2\varphi\right). \qquad \square$$

Now we study some special class of Lagrangian surfaces in $S^2 \times S^2$. We will calculate the second fundamental tensor σ and the mean curvature vector H of the *product immersion*. Let $x_i\colon I_i \to S^2$ $(i = 1, 2)$ be curves in a 2-sphere with arclength parameter s_i, and let $x\colon I_1 \times I_2 \to S^2$ be the product immersion defined by $x(s_1, s_2) =$

$(x_1(s_1), x_2(s_2))$. If κ_i $(i = 1, 2)$ are curvatures of spherical curves x_i, then we get $x_i''(s_i) = \kappa_i(s_i)Jx_i'(s_i) - x_i(s_i)$. So we have

$$
\begin{aligned}
\sigma\left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_1}\right) &= (\kappa_1(s_1)Jx_1'(s_1), 0),\\
\sigma\left(\frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_2}\right) &= (0, \kappa_2(s_2)Jx_2'(s_2)),
\end{aligned}
\tag{2.20}
$$

and

$$
\begin{aligned}
2H &= \sigma\left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_1}\right) + \sigma\left(\frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_2}\right)\\
&= (\kappa_1(s_1)Jx_1'(s_1), \kappa_2(s_2)Jx_2'(s_2)).
\end{aligned}
$$

Consequently the product immersion x is minimal if and only if $\kappa_1 \equiv \kappa_2 \equiv 0$, that is, each x_i is a great circle of S^2. Hence, we have

Proposition 2. *Let x be a product immersion: $M_1 \times M_2 \to S^2 \times S^2$. If x is a minimal immersion, then x is totally geodesic and each M_i $(i = 1, 2)$ is a great circle of S^2.*

For the Lagrangian immersion x, if $\varphi \equiv 0$, then we have $P^2 = 1$ by (2.13). Hence, we can decompose $TM^2 = T_1M \oplus T_{-1}M$ where T_1M is an eigenspace of eigenvalue 1 of P and $T_{-1}M$ is an eigenspace of eigenvalue -1 of P. Since T_1M and $T_{-1}M$ are totally geodesic distributions on M^2, we can see that M^2 is a product manifold $M_1 \times M_2$ and x is a product immersion.

Now we back to the general case and we will deduce fundamental equations for Lagrangian surfaces in $S^2 \times S^2$. It follows from (2.15) that

$$
\langle(\nabla_X P)e_1, e_1\rangle = 2\sin 2\varphi\langle\tilde{J}\sigma(e_1, e_2), X\rangle. \tag{2.21}
$$

On the other hand,

$$
\begin{aligned}
\langle(\nabla_X P)e_1, e_1\rangle &= \langle\nabla_X(Pe_1) - P\nabla_X e_1, e_1\rangle\\
&= \langle\nabla_X(\cos 2\varphi e_1), e_1\rangle - \langle\nabla_X e_1, Pe_1\rangle\\
&= -2\sin 2\varphi(X\varphi).
\end{aligned}
\tag{2.22}
$$

Hence, we obtain from (2.21) and (2.22)

$$
\sin 2\varphi\{X\varphi + \langle\tilde{J}\sigma(e_1, e_2), X\rangle\} = 0.
$$

We get also from calculation of $\langle(\nabla_X Q)e_1, e_2\rangle$

$$
\cos 2\varphi\{X\varphi + \langle\tilde{J}\sigma(e_1, e_2), X\rangle\} = 0.
$$

Therefore $X\varphi + \langle \tilde{J}\sigma(e_1, e_2), X\rangle = 0$ for all $X \in T_pM^2$, i.e.,

$$\operatorname{grad}\varphi = -\tilde{J}\sigma(e_1, e_2). \tag{2.23}$$

By calculating $\langle(\nabla_X P)e_1, e_2\rangle$, we have

$$\sin 2\varphi\{\langle\sigma(e_1, e_1), \tilde{J}X\rangle - \langle\sigma(e_2, e_2), \tilde{J}X\rangle\} = 2\cos 2\varphi\langle\nabla_X e_1, e_2\rangle. \tag{2.24}$$

Let ω be a connection form with respect to the orthonormal frame field $\{e_1, e_2\}$ on M^2, which is given by

$$\nabla_X e_1 = \omega(X)e_2, \quad \nabla_X e_2 = -\omega(X)e_1.$$

By (2.1), $\langle\sigma(e_i, e_j), \tilde{J}e_k\rangle$ are symmetric for $i, j, k = 1, 2$. We put

$$\begin{aligned} T_0 &= \langle\sigma(e_1, e_1), \tilde{J}e_1\rangle, \quad T_1 = \langle\sigma(e_1, e_1), \tilde{J}e_2\rangle, \\ T_2 &= \langle\sigma(e_1, e_2), \tilde{J}e_2\rangle, \quad T_3 = \langle\sigma(e_2, e_2), \tilde{J}e_2\rangle. \end{aligned} \tag{2.25}$$

It follows from (2.23) that

$$\operatorname{grad}\varphi = T_1e_1 + T_2e_2 \tag{2.26}$$

and from (2.24) that

$$\begin{cases} 2\omega(e_1)\cos 2\varphi = (T_0 - T_2)\sin 2\varphi, \\ 2\omega(e_2)\cos 2\varphi = (T_1 - T_3)\sin 2\varphi. \end{cases} \tag{2.27}$$

Next, we consider the equations of Gauss and Codazzi for a Lagrangian surface M^2 in $S^2 \times S^2$. The curvature tensor $\bar{R}$ of $S^2 \times S^2$ satisfies

$$\bar{R}(X, Y)Z = \frac{\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle\tilde{P}Y, Z\rangle\tilde{P}X - \langle\tilde{P}X, Z\rangle\tilde{P}Y}{2}$$

for any $X, Y, Z \in T(S^2 \times S^2)$ (cf. [6]). So we have

$$\begin{aligned} \langle\bar{R}(e_1, e_2)e_2, e_1\rangle &= \frac{1 + \langle Pe_1, e_1\rangle\langle Pe_2, e_2\rangle - \langle Pe_2, e_1\rangle 2}{2} \\ &= \frac{\sin^2 2\varphi}{2} \end{aligned}$$

for an orthonormal basis $\{e_1, e_2\}$ of M^2 satisfying (2.16). Hence, the Gauss equation is

$$K = \frac{\sin^2 2\varphi}{2} + T_0T_2 + T_1T_3 - (T_1)^2 - (T_2)^2 \tag{2.28}$$

for the Gauss curvature $K = \langle R(e_1, e_2)e_2, e_1\rangle$ of M^2. Normal components $(\bar{R}(e_1, e_2)e_i)^\perp$ of $\bar{R}(e_1, e_2)e_i$ (for $i = 1, 2$) to M^2 are

$$
\begin{aligned}
(\bar{R}(e_1, e_2)e_i)^\perp &= \frac{\langle Pe_2, e_i\rangle \tilde{J}Qe_1 - \langle Pe_1, e_i\rangle \tilde{J}Qe_2}{2} \\
&= \frac{\sin 4\varphi\{\langle e_2, e_i\rangle \tilde{J}e_2 - \langle e_1, e_i\rangle \tilde{J}e_1\}}{4}.
\end{aligned}
$$

We define the covariant derivative of σ as

$$(\nabla_X\sigma)(Y, Z) = \nabla^\perp_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

Then the Codazzi equations are given by

$$(\nabla_{e_1}\sigma)(e_2, e_i) - (\nabla_{e_2}\sigma)(e_1, e_i) = \frac{\sin 4\varphi\{\langle e_2, e_i\rangle \tilde{J}e_2 - \langle e_1, e_i\rangle \tilde{J}e_1\}}{4}.$$

Hence,

$$
\begin{cases}
\langle(\nabla_{e_1}\sigma)(e_1, e_2) - (\nabla_{e_2}\sigma)(e_1, e_1), \tilde{J}e_1\rangle = -\dfrac{\sin 4\varphi}{4}, \\
\langle(\nabla_{e_1}\sigma)(e_2, e_2) - (\nabla_{e_2}\sigma)(e_1, e_2), \tilde{J}e_2\rangle = \dfrac{\sin 4\varphi}{4}, \\
\langle(\nabla_{e_2}\sigma)(e_1, e_1) - (\nabla_{e_1}\sigma)(e_1, e_2), \tilde{J}e_2\rangle = 0, \\
\langle(\nabla_{e_1}\sigma)(e_2, e_2) - (\nabla_{e_2}\sigma)(e_1, e_2), \tilde{J}e_1\rangle = 0.
\end{cases}
\tag{2.29}
$$

By (2.2), we have

$$
\begin{aligned}
\langle(\nabla_{e_i}\sigma)(e_j, e_k), \tilde{J}e_l\rangle = {}& e_i\langle\sigma(e_j, e_k), \tilde{J}e_l\rangle - \langle\sigma(e_j, e_k), \tilde{J}\nabla_{e_i}e_l\rangle \\
&- \langle\sigma(\nabla_{e_i}e_j, e_k), \tilde{J}e_l\rangle - \langle\sigma(e_j, \nabla_{e_i}e_k), \tilde{J}e_l\rangle.
\end{aligned}
$$

Therefore from (2.29), the Codazzi equations are written as

$$e_1T_1 - e_2T_0 + \omega(e_1)(T_0 - 2T_2) + 3\omega(e_2)T_1 = -\frac{\sin 4\varphi}{4}, \tag{2.30}$$

$$e_1T_3 - e_2T_2 + 3\omega(e_1)T_2 + \omega(e_2)(T_3 - 2T_1) = \frac{\sin 4\varphi}{4}, \tag{2.31}$$

$$e_1T_2 - e_2T_1 + \omega(e_1)(2T_1 - T_3) + \omega(e_2)(2T_2 - T_0) = 0.$$

Note that the last equation is also derived from (2.26).

When $\varphi \equiv \pm\pi/4$, it follows from (2.1) and (2.15) that $\sigma \equiv 0$ and we get $K \equiv 1/2$ from (2.28). Hence, we have

Proposition 3. *Let M^2 be a 2-dimensional Riemannian manifold and let $x\colon M^2 \to S^2 \times S^2$ be a Lagrangian isometric immersion. If the angle function φ defined by (2.16) is identically equal to $\pm\pi/4$, then x is totally geodesic and the Gauss curvature of M^2 is $K \equiv 1/2$.*

It is well-known that $S^2 \times S^2$ is holomorphically isometric to complex 2-dimensional complex quadric Q^2. And totally geodesic submanifolds in complex quadrics Q^m are classified by Chen and Nagano [3].

EXAMPLE. Let $\phi_\pm$: $S^2 \to S^2 \times S^2$ be a Lagrangian immersion given by $(x, y, z) \mapsto ((x, y, \mp z), (x, y, \pm z))$, where (x, y, z) is an orthogonal coordinate system on $\mathbf{R}^3$. Then we can see that the angle function φ of $\phi_\pm$ is identically equal to $\pm\pi/4$.

3. Existence of $SO(3) \times SO(3)$-valued frame fields

In this section, we study integrability conditions for existence of Lagrangian isometric immersion $x\colon M^2 \to S^2 \times S^2$ by using some frame field $M \to SO(3) \times SO(3)$.

Now we consider Lagrangian immersion $x : M^2 \to S^2 \times S^2$ with which the angle function satisfies $\varphi \in (-\pi/4, \pi/4)$. Let $V_1(p), V_2(p)$ be vectors in $\mathbf{R}^3$ defined by (2.17). By (2.19), at each $p \in M^2$,

$$\begin{cases} \xi_1(p) = (x_1(p),\ V_1(p),\ JV_1(p)), \\ \xi_2(p) = (x_2(p),\ V_2(p),\ JV_2(p)) \end{cases} \tag{3.1}$$

are orthonormal frames in $\mathbf{R}^3$ respectively. By the definition (2.3) of the complex structure J on S^2, we can see that $(\xi_1(p), \xi_2(p)) \in SO(3) \times SO(3)$.

Now, we calculate Ricci identity (i.e., integrability conditions) for two frame fields $\xi_1(p)$ and $\xi_2(p)$, namely,

$$(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})\xi_1(p) = 0,$$
$$(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})\xi_2(p) = 0,$$

where D is the Euclidean connection of $\mathbf{R}^6 = \mathbf{R}^3 \times \mathbf{R}^3$. We denote the frames as (x_1, V_1, JV_1), (x_2, V_2, JV_2), and also denote e_1, e_2 instead of x_*e_1, x_*e_2 for simplicity. Note that for $e_1 = (e_1 + \tilde{P}e_1)/2 + (e_1 - \tilde{P}e_1)/2$, $(e_1 + \tilde{P}e_1)/2$ (resp. $(e_1 - \tilde{P}e_1)/2$) is an eigenvector of $\tilde{P}$ with eigenvalue 1 (resp. -1) and is contained in $T_{x_1(p)}S^2$ (resp. $T_{x_2(p)}S^2$). Then from (2.12), (2.16), (2.17) and (2.19), we obtain

$$\begin{aligned} D_{e_1}x_1 = D_{(e_1+\tilde{P}e_1)/2}x_1 &= \frac{e_1 + \tilde{P}e_1}{2} \\ &= \cos\varphi V_1, \\ D_{e_1}x_2 = D_{(e_1-\tilde{P}e_1)/2}x_2 &= \frac{e_1 - \tilde{P}e_1}{2} \\ &= \sin\varphi JV_2. \end{aligned}$$

By calculating $D_{e_2}x_1$, $D_{e_2}x_2$ similarly, we have

$$\begin{cases} D_{e_1}x_1 = \cos\varphi V_1, \quad D_{e_2}x_1 = \sin\varphi JV_1, \\ D_{e_1}x_2 = \sin\varphi JV_2, \quad D_{e_2}x_2 = \cos\varphi V_2. \end{cases} \tag{3.2}$$

To get $D_{e_i}V_j$ and $D_{e_i}(JV_j)$, we first calculate $D_{e_i}e_j$ and $D_{e_i}(\tilde{J}e_j)$. By (2.25), (2.17) and (2.19), we have

$$\begin{aligned}
D_{e_1}e_1 &= \langle D_{e_1}e_1, x_1\rangle x_1 + \langle D_{e_1}e_1, x_2\rangle x_2 + \bar{\nabla}_{e_1}e_1\\
&= -\langle e_1, D_{e_1}x_1\rangle x_1 - \langle e_1, D_{e_1}x_2\rangle x_2 + \nabla_{e_1}e_1 + \sigma(e_1, e_1)\\
&= -\cos^2\varphi x_1 - \sin^2\varphi x_2 + \omega(e_1)e_2 + T_0\tilde{J}e_1 + T_1\tilde{J}e_2,
\end{aligned}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on $S^2 \times S^2$. Note that x_1 and x_2 are considered as unit normal vector fields of the inclusion $S^2 \times S^2 \hookrightarrow \mathbf{R}^3 \times \mathbf{R}^3$. Similar computations yield

$$\begin{aligned}
D_{e_2}e_1 &= \omega(e_2)e_2 + T_1\tilde{J}e_1 + T_2\tilde{J}e_2,\\
D_{e_1}e_2 &= -\omega(e_1)e_1 + T_1\tilde{J}e_1 + T_2\tilde{J}e_2,\\
D_{e_2}e_2 &= -\sin^2\varphi x_1 - \cos^2\varphi x_2 - \omega(e_2)e_1 + T_2\tilde{J}e_1 + T_3\tilde{J}e_2,\\
D_{e_1}(\tilde{J}e_1) &= -T_0e_1 - T_1e_2 + \omega(e_1)\tilde{J}e_2,\\
D_{e_2}(\tilde{J}e_1) &= -\cos\varphi\sin\varphi x_1 + \cos\varphi\sin\varphi x_2 - T_1e_1 - T_2e_2 + \omega(e_2)\tilde{J}e_2,\\
D_{e_1}(\tilde{J}e_2) &= \cos\varphi\sin\varphi x_1 - \cos\varphi\sin\varphi x_2 - T_1e_1 - T_2e_2 - \omega(e_1)\tilde{J}e_1,\\
D_{e_2}(\tilde{J}e_2) &= -T_2e_1 - T_3e_2 - \omega(e_2)\tilde{J}e_1.
\end{aligned}$$

(2.17), (2.26), (2.27) and these equations imply

$$\begin{aligned}
D_{e_1}V_1 &= D_{e_1}\{\cos\varphi e_1 - \sin\varphi\tilde{J}e_2\}\\
&= (e_1\varphi)\{-\sin\varphi e_1 - \cos\varphi\tilde{J}e_2\} + \cos\varphi D_{e_1}e_1 - \sin\varphi D_{e_1}(\tilde{J}e_2)\\
&= -\cos\varphi x_1 + \{\omega(e_1)\cos\varphi + \sin\varphi T_2\}e_2 + \{\omega(e_1)\sin\varphi + \cos\varphi T_0\}\tilde{J}e_1\\
&= -\cos\varphi x_1 + \{\omega(e_1)\sin 2\varphi + T_0\cos^2\varphi + T_2\sin^2\varphi\}JV_1.
\end{aligned}$$

Similar computations yield

$$\begin{cases}
D_{e_1}V_1 = -\cos\varphi x_1 + \alpha JV_1, \quad D_{e_2}V_1 = \beta JV_1,\\
D_{e_1}(JV_1) = -\alpha V_1, \quad D_{e_2}(JV_1) = -\sin\varphi x_1 - \beta V_1,\\
D_{e_1}V_2 = \gamma JV_2, \quad D_{e_2}V_2 = -\cos\varphi x_2 + \delta JV_2,\\
D_{e_1}(JV_2) = -\sin\varphi x_2 - \gamma V_2, \quad D_{e_2}(JV_2) = -\delta V_2
\end{cases} \tag{3.3}$$

where

$$\begin{cases}
\alpha = \omega(e_1)\sin 2\varphi + T_0\cos^2\varphi + T_2\sin^2\varphi,\\
\beta = \omega(e_2)\sin 2\varphi + T_1\cos^2\varphi + T_3\sin^2\varphi,\\
\gamma = -\omega(e_1)\sin 2\varphi + T_0\sin^2\varphi + T_2\cos^2\varphi,\\
\delta = -\omega(e_2)\sin 2\varphi + T_1\sin^2\varphi + T_3\cos^2\varphi.
\end{cases} \tag{3.4}$$

Using (2.27), we get

$$\begin{cases} \alpha \sin\varphi = \omega(e_1)\cos\varphi + T_2\sin\varphi, \\ \beta\cos\varphi = \omega(e_2)\sin\varphi + T_1\cos\varphi, \\ \gamma\cos\varphi = -\omega(e_1)\sin\varphi + T_2\cos\varphi, \\ \delta\sin\varphi = -\omega(e_2)\cos\varphi + T_1\sin\varphi. \end{cases} \tag{3.5}$$

By differentiating (3.4) and using (2.27), we obtain

$$\begin{cases} e_i\alpha = e_i(\omega(e_1))\sin 2\varphi + (e_iT_0)\cos^2\varphi + (e_iT_2)\sin^2\varphi, \\ e_i\beta = e_i(\omega(e_2))\sin 2\varphi + (e_iT_1)\cos^2\varphi + (e_iT_3)\sin^2\varphi, \\ e_i\gamma = -e_i(\omega(e_1))\sin 2\varphi + (e_iT_0)\sin^2\varphi + (e_iT_2)\cos^2\varphi, \\ e_i\delta = -e_i(\omega(e_2))\sin 2\varphi + (e_iT_1)\sin^2\varphi + (e_iT_3)\cos^2\varphi. \end{cases} \tag{3.6}$$

(3.2), (3.3) and (3.5) imply

$$\begin{aligned} D_{e_1}D_{e_2}x_1 - D_{e_2}D_{e_1}x_1 - D_{[e_1,e_2]}x_1 &= 0, \\ D_{e_1}D_{e_2}x_2 - D_{e_2}D_{e_1}x_2 - D_{[e_1,e_2]}x_2 &= 0. \end{aligned} \tag{3.7}$$

Next, we compute $(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})V_1$. From (3.2), (3.3), (3.5) and (3.6), we get

$$\begin{aligned} &D_{e_1}D_{e_2}V_1 - D_{e_2}D_{e_1}V_1 - D_{[e_1,e_2]}V_1 \\ &= \bigg[\bigg\{\frac{1}{2} + e_1(\omega(e_2)) - e_2(\omega(e_1)) + \omega(e_1)^2 + \omega(e_2)^2\bigg\}\sin 2\varphi \\ &\quad + \{e_1T_1 - e_2T_0 + \omega(e_1)T_0 + \omega(e_2)T_1\}\cos^2\varphi \\ &\quad + \{e_1T_3 - e_2T_2 + \omega(e_1)T_2 + \omega(e_2)T_3\}\sin^2\varphi\bigg]JV_1. \end{aligned}$$

By the definition of the Gauss curvature,

$$K = \langle R(e_2, e_1)e_1, e_2\rangle = e_2(\omega(e_1)) - e_1(\omega(e_2)) - \omega(e_1)^2 - \omega(e_2)^2, \tag{3.8}$$

the equation $(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})V_1 = 0$ is equivalent to

$$\begin{aligned} &\left(\frac{1}{2} - K\right)\sin 2\varphi + \{e_1T_1 - e_2T_0 + \omega(e_1)T_0 + \omega(e_2)T_1\}\cos^2\varphi \\ &+ \{e_1T_3 - e_2T_2 + \omega(e_1)T_2 + \omega(e_2)T_3\}\sin^2\varphi = 0. \end{aligned}$$

By similar computations for JV_1, V_2 and JV_2, we obtain

Proposition 4. *Let $x\colon M^2 \to S^2 \times S^2$ be a Lagrangian isometric immersion such that x is not $\pm$-holomorphic with respect to $(J, -J)$. Let V_1, JV_1, V_2, JV_2 be $\mathbf{R}^3$-valued vector fields on M^2 defined by* (2.17) *and* (2.19). *Then* (i) $(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})V_1 = 0$ *and* $(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})(JV_1) = 0$ *are equivalent to*

$$\begin{aligned}&\left(\frac{1}{2} - K\right)\sin 2\varphi + \{e_1T_1 - e_2T_0 + \omega(e_1)T_0 + \omega(e_2)T_1\}\cos^2\varphi \\ &+ \{e_1T_3 - e_2T_2 + \omega(e_1)T_2 + \omega(e_2)T_3\}\sin^2\varphi = 0,\end{aligned} \tag{3.9}$$

(ii) $(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})V_2 = 0$ *and* $(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})JV_2 = 0$ *are equivalent to*

$$\begin{aligned}&\left(-\frac{1}{2} + K\right)\sin 2\varphi + \{e_1T_1 - e_2T_0 + \omega(e_1)T_0 + \omega(e_2)T_1\}\sin^2\varphi \\ &+ \{e_1T_3 - e_2T_2 + \omega(e_1)T_2 + \omega(e_2)T_3\}\cos^2\varphi = 0.\end{aligned} \tag{3.10}$$

4. The Maurer-Cartan equation for Lagrangian immersions

Let G be a Lie group and $\mathfrak{g}$ be Lie algebra of G. we denote a basis for $\mathfrak{g}$ by $e_1, \dots, e_n$ and the dual basis for $e_1, \dots, e_n$ by $\psi_1, \dots, \psi_n$. Then $\mathfrak{g}$-valued 1-form Ω on G is defined as $\Omega = \sum_{i=1}^n e_i \otimes \psi_i$. If we put $d(e_i \otimes \psi_i) = e_i \otimes d\psi_i$ and $[e_i \otimes \psi_i \wedge e_j \otimes \psi_j] = [e_i, e_j] \otimes \psi_i \wedge \psi_j$, then we have

$$\begin{aligned}
d\Omega(e_k, e_l) &= \sum_i e_i \otimes d\psi_i(e_k, e_l) \\
&= \sum_i e_i \otimes \{e_k(\psi_i(e_l)) - e_l(\psi_i(e_k)) - \psi_i([e_k, e_l])\} \\
&= -\sum_i e_i \otimes \psi_i([e_k, e_l]) \\
&= -[e_k, e_l], \\
\Omega t(e_k, e_l) &= \left(\sum_{i,j}[e_i, e_j] \otimes \psi_i \wedge \psi_j\right)(e_k, e_l) \\
&= \sum_{i,j}[e_i, e_j](\psi_i(e_k)\psi_j(e_l) - \psi_i(e_l)\psi_j(e_k)) \\
&= [e_k, e_l] - [e_l, e_k] \\
&= 2[e_k, e_l].
\end{aligned}$$

Hence Ω satisfies

$$d\Omega = -\frac{1}{2}[\Omega \wedge \Omega].$$

This equation is called the *Maurer-Cartan equation* and the $\mathfrak{g}$ valued 1-form Ω on G which satisfies the equation is called the *Maurer-Cartan form*. The following theorem, due to Cartan (cf. [4]), is the key result of this paper.

Theorem 1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form Ω.* (i) *Let M be a manifold on which there exists a $\mathfrak{g}$-valued 1-form Φ satisfying*

$$d\Phi = -\frac{1}{2}[\Phi \wedge \Phi]. \tag{4.1}$$

Then for any point $p \in M$ there exists a neighborhood U of p and a map $f : U \to G$ such that $f^*\Omega = \Phi$. (ii) *Given maps f_1, $f_2 : M \to G$, then $f_1^*\Omega = f_2^*\Omega$ if and only if $f_1 = L_a \circ f_2$ for some fixed $a \in G$, where L is the left translation on G.*

When G is the special orthogonal group $SO(n)$, the Lie algebra $\mathfrak{g}$ for $SO(n)$ is $\mathfrak{o}(n)$ which is the set of all skew-symmetric matrices of degree n. It is known that the Maurer-Cartan form Ω on $SO(n)$ is given by $\Omega = g^{-1}\,dg$ with the condition $\Omega + {}^t\Omega = 0$ for $g \in SO(n)$ (cf. [4]).

Now, we want to find the conditions for existence and congruence for Lagrangian isometric immersions $M^2 \to S^2 \times S^2$ by using Theorem 1. It is known that S^2 is a homogeneous space of $SO(3)$ and we may identify S^2 with the quotient space $SO(3)/SO(2)$. Thus $S^2 \times S^2$ is identified with the homogeneous space $SO(3) \times SO(3)/SO(2) \times SO(2)$.

Let $\{e_1, e_2\}$ be an oriented orthonormal frame field on M^2 satisfying (2.16), and let $\{\psi_1, \psi_2\}$ be the dual 1-forms for $\{e_1, e_2\}$. Then $(\xi_1(p), \xi_2(p))$ is a $SO(3) \times SO(3)$-valued frame field over M^2, given by (3.1). So we consider the Maurer-Cartan equation for this frame field.

For $\xi_1 = (x_1, V_1, JV_1)$ and $\xi_2 = (x_2, V_2, JV_2)$, it follows from (3.2) and (3.3) that

$$D\xi_1 = \xi_1\Phi_1, \quad D\xi_2 = \xi_2\Phi_2,$$

where Φ_1 and Φ_2 are $\mathfrak{o}(3)$-valued 1-forms on M^2, given by

$$\Phi_1 = \begin{pmatrix} 0 & -\cos\varphi\psi_1 & -\sin\varphi\psi_2 \\ \cos\varphi\psi_1 & 0 & -\alpha\psi_1 - \beta\psi_2 \\ \sin\varphi\psi_2 & \alpha\psi_1 + \beta\psi_2 & 0 \end{pmatrix}, \tag{4.2}$$

$$\Phi_2 = \begin{pmatrix} 0 & -\cos\varphi\psi_2 & -\sin\varphi\psi_1 \\ \cos\varphi\psi_2 & 0 & -\gamma\psi_1 - \delta\psi_2 \\ \sin\varphi\psi_1 & \gamma\psi_1 + \delta\psi_2 & 0 \end{pmatrix}. \tag{4.3}$$

Now $SO(3) \times SO(3)$-congruence Theorem for Lagrangian isometric immersion $M^2 \to S^2 \times S^2$ is obtained as:

Theorem 2. *Let M^2 be a connected and oriented 2-dimensional Riemannian manifold and let $x^1, x^2 \colon M^2 \to S^2 \times S^2$ be Lagrangian isometric immersions with which the angle functions φ^1, φ^2 take the values in $(-\pi/4, \pi/4)$. We denote σ^i $(i = 1, 2)$ the second fundamental forms of x^i, respectively, and $T^i(X, Y, Z) = \langle \sigma^i(X, Y), \tilde{J}Z \rangle$, the corresponding symmetric tensor fields on M^2. Then there is an isometry $g \in SO(3) \times SO(3)$ such that $x^2 = g \circ x^1$ if and only if $\varphi^1 = \varphi^2$, and $T^1 = T^2$ hold.*

Proof. If $\varphi^1 = \varphi^2$, $T^1 = T^2$, then $\mathfrak{o}(3)$ valued 1-forms Φ_1, Φ_2 on M^2 given by (4.2), (4.3) are the same values respectively. Thus the result follows from Theorem 1 (ii). $\square$

REMARK 1. The full holomorphic isometry group of $S^2 \times S^2$ with respect to the standard product metric and the complex structure (J, J) is generated (cf. [1]) by $SO(3) \times SO(3)$ and

$$\tau : S^2 \times S^2 \to S^2 \times S^2, \quad \tau(x_1, x_2) = (x_2, x_1).$$

Then the congruence theorem for the full holomorphic isometry group of $S^2 \times S^2$ is not true as Theorem 2. Let $C(\kappa_i)$ $(i = 1, 2)$ be oriented circles in S^2 with constant curvature κ_i and suppose $\kappa_1 < \kappa_2$. If we put $M_1 = C(\kappa_1) \times C(\kappa_2)$ and $M_2 = C(\kappa_2) \times C(\kappa_1)$, then both M_1 and M_2 are Lagrangian surfaces in $S^2 \times S^2$ by product immersions. We can see that M_1 and M_2 are *not* congruent under $SO(3) \times SO(3)$ but congruent under *full holomorphic isometry group* of $S^2 \times S^2$. The angle functions φ of M_i are both identically equal to 0, but the quantities T_0, T_3 defined by (2.25) are different, because of (2.20). As we saw in Proposition 2, the product immersion of $C(\kappa_1) \times C(\kappa_2)$ into $S^2 \times S^2$ is minimal if and only if $\kappa_1 = \kappa_2 = 0$.

We prove the equivalence of the Ricci identity for the frame field (3.1) and the Maurer-Cartan equation for $\mathfrak{o}(3)$-valued 1-forms (4.2) and (4.3).

Proposition 5. *Let M^2 be an oriented Riemannian manifold of dimension 2, let $\{e_1, e_2\}$ be an orthonormal frame field on M^2 compatible with the orientation of M^2, and let $\{\psi_1, \psi_2\}$ be the dual 1-forms for $\{e_1, e_2\}$. Suppose $\varphi : M^2 \to (-\pi/4, \pi/4)$ and $T_0, T_1, T_2, T_3 : M^2 \to \mathbf{R}$ are functions on M^2 such that the equations* (2.26) *and* (2.27) *hold. Let Φ_1, Φ_2 be $\mathfrak{o}(3)$-valued 1-forms on M^2 defined by* (4.2) *and* (4.3), *respectively. Then the Maurer-Cartan equations* (4.1) *for Φ_1, Φ_2*

$$d\Phi_i = -\frac{1}{2}[\Phi_i \wedge \Phi_i]$$

are equivalent to the Ricci identity, (3.9) *and* (3.10).

Proof. $D\xi_i = \xi_i \Phi_i$ implies

$$D_{e_i} D_{e_2} \xi_i = D_{e_1}(\xi_i \Phi_i(e_2)) = \xi_i(\Phi_i(e_1)\Phi_i(e_2) + e_1 \Phi_1(e_2)),$$

$$D_{e_2} D_{e_1} \xi_i = \xi_i(\Phi_i(e_2)\Phi_i(e_1) + e_2\Phi_1(e_1)), \quad D_{[e_1,e_2]}\xi_i = \xi_i\Phi_i([e_1, e_2]).$$

Hence we obtain

$$\begin{aligned}(D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]})\xi_i &= \xi_i(d\Phi_1(e_1, e_2) + [\Phi_i(e_1), \Phi_i(e_2)]) \\ &= \xi_i\left(d\Phi_i + \frac{1}{2}[\Phi_i \wedge \Phi_i]\right)(e_1, e_2).\end{aligned}$$

This equation means that the Ricci identity ((3.9) and (3.10)) is equivalent to $d\Phi_i + (1/2)[\Phi_i \wedge \Phi_i] = 0$. □

According to Theorem 1 (i), we get the existence theorem for Lagrangian isometric immersion $M^2 \to S^2 \times S^2$.

Theorem 3. *Let (M^2, g) be a simply connected oriented Riemannian manifold of dimension 2. Suppose that there exists an orthonormal frame field $\{e_1, e_2\}$ on M^2 compatible with the orientation of M^2, functions $T_i\colon M^2 \to \mathbf{R}$ $(i = 0, 1, 2, 3)$ and $\varphi\colon M^2 \to (-\pi/4, \pi/4)$ such that they satisfy*

$$(4.4) \qquad \begin{cases} \operatorname{grad}\varphi = T_1e_1 + T_2e_2, \\ 2\omega(e_1)\cos 2\varphi = (T_0 - T_2)\sin 2\varphi, \\ 2\omega(e_2)\cos 2\varphi = (T_1 - T_3)\sin 2\varphi \end{cases}$$

for the connection form ω on M^2 with respect to $\{e_1, e_2\}$. If the Gauss equation

$$(4.5) \qquad K = \frac{\sin^2 2\varphi}{2} + T_0T_2 + T_1T_3 - (T_1)^2 - (T_2)^2$$

and two equations of Codazzi

$$(4.6) \qquad e_1T_1 - e_2T_0 + \omega(e_1)(T_0 - 2T_2) + 3\omega(e_2)T_1 = -\frac{\sin 4\varphi}{4},$$

$$(4.7) \qquad e_1T_3 - e_2T_2 + 3\omega(e_1)T_2 + \omega(e_2)(T_3 - 2T_1) = \frac{\sin 4\varphi}{4}$$

hold, then there is the Lagrangian isometric immersion $x\colon M^2 \to S^2 \times S^2$ and the function φ is the angle function for x. The second fundamental form σ of x is then given by (2.25).

Proof. If the Gauss equation (4.5) and the Codazzi equations (4.6), (4.7) hold, then the left hands of the Ricci identities (3.9) and (3.10) are written as the same form

$$\{T_2(T_2 - T_0) + T_1(T_1 - T_3)\}\sin 2\varphi + 2\{\omega(e_1)T_2 - \omega(e_2)T_1\}\cos 2\varphi.$$

Hence (4.4) implies that this term is equal to zero. By Proposition 5, Theorem 1 (i) and Theorem 2, we can construct frame field $p \mapsto (\xi_1(p), \xi_2(p))$ of (3.1) on M^2. Consequently the Lagrangian isometric immersion

$$x\colon M^2 \to S^2 \times S^2, \quad p \mapsto (x_1(p), x_2(p))$$

is constructed by the above frame field and the projection

$$(\xi_1(p), \xi_2(p)) \mapsto (x_1(p), x_2(p)). \qquad \square$$

5. Minimal Lagrangian surfaces in $S^2 \times S^2$

In this section, we study minimal Lagrangian immersions $x\colon M^2 \to S^2 \times S^2$ with which the angle function satisfies $\varphi \in (-\pi/4, \pi/4)$. The Lagrangian immersion x is minimal if and only if the second fundamental form σ of x satisfies $\sum_{i=1,2}\langle \sigma(e_i, e_i), \tilde{J}e_j\rangle = 0$ for $j = 1, 2$, or equivalently

$$T_0 + T_2 = 0 \quad \text{and} \quad T_1 + T_3 = 0,$$

where T_j $(j = 0, 1, 2, 3)$ are the components of second fundamental form of x with respect to the orthonormal frame field $\{e_1, e_2\}$ of (2.16). By (2.26), T_1 and T_2 are determined by φ and, in the case of minimal Lagrangian immersions, Theorem 2 is described as:

Theorem 4. *Let M^2 be an oriented 2-dimensional Riemannian surface and $x^1, x^2\colon M^2 \to S^2 \times S^2$ be minimal Lagrangian immersions. Let $\varphi^i\colon M^2 \to (-\pi/4, \pi/4)$ be the angle function of x^i $(i = 1, 2)$. Then there is an isometry $g \in SO(3) \times SO(3)$ such that $x^2 = g \circ x^1$ if and only if $\varphi^1 = \varphi^2$.*

Using (2.26) and $\varphi \in (-\pi/4, \pi/4)$, we see that (2.27) is equivalent to

$$\begin{cases} \omega(e_1) = -(e_2\varphi)\tan 2\varphi, \\ \omega(e_2) = (e_1\varphi)\tan 2\varphi, \end{cases} \tag{5.1}$$

where ω is the connection form with respect to $\{e_1, e_2\}$. Then by (2.26) and (5.1), the Gauss equation (4.5) is

$$K = \frac{\sin^2 2\varphi}{2} - 2\|\mathrm{grad}\,\varphi\|^2. \tag{5.2}$$

According to (5.1), we get that the Codazzi equations (2.30) and (2.31) are written as the single equation

$$e_1(e_1\varphi) + e_2(e_2\varphi) - 3\omega(e_1)(e_2\varphi) + 3\omega(e_2)(e_1\varphi) = -\frac{\sin 4\varphi}{4}. \tag{5.3}$$

Then using the definition of Gauss curvature (3.8) and (5.1), we can see that (5.2) and (5.3) are equivalent. By the definition of the Laplacian

$$\triangle\varphi = e_1(e_1\varphi) + e_2(e_2\varphi) - \omega(e_1)e_2\varphi + \omega(e_2)e_1\varphi,$$

and (5.1), (5.3) is written as

$$\triangle\varphi + 2\|\mathrm{grad}\,\varphi\|^2 \tan 2\varphi = -\frac{\sin 4\varphi}{4}. \tag{5.4}$$

To show the existence of Lagrangian isometric minimal immersion $M^2 \to S^2 \times S^2$, we want to find desirable orthonormal frame field $\{e_1, e_2\}$ on M^2 in Theorem 3. Let $\{\overline{e}_1, \overline{e}_2\}$ be a given oriented orthonormal frame field on M^2 with connection form $\overline{\omega}$. Let ρ be a function on M^2 and put

$$\begin{cases} \overline{e}_1(\rho) = \cos\rho\,\overline{e}_1 + \sin\rho\,\overline{e}_2, \\ \overline{e}_2(\rho) = -\sin\rho\,\overline{e}_1 + \cos\rho\,\overline{e}_2. \end{cases}$$

Then the connection form $\overline{\omega}_\rho$ with respect to $\{\overline{e}_1, \overline{e}_2\}$ is written as

$$\overline{\omega}_\rho = \langle \nabla\overline{e}_1(\rho), \overline{e}_2(\rho)\rangle = d\rho + \overline{\omega}.$$

Hence $\{\overline{e}_1(\rho), \overline{e}_2(\rho)\}$ and $\overline{\omega}_\rho$ satisfy (5.1) if and only if

$$\begin{cases} d\rho(\overline{e}_1) = -\overline{\omega}(\overline{e}_1) - (\overline{e}_2\varphi)\tan 2\varphi, \\ d\rho(\overline{e}_2) = -\overline{\omega}(\overline{e}_2) + (\overline{e}_1\varphi)\tan 2\varphi \end{cases}$$

hold with given (angle) function φ. Consequently integrability condition of these equations are

$$\begin{aligned} 0 &= (\nabla d\rho)(\overline{e}_1, \overline{e}_2) - (\nabla d\rho)(\overline{e}_2, \overline{e}_1) \\ &= \overline{e}_2(d\rho(\overline{e}_1)) - \overline{e}_1(d\rho(\overline{e}_2)) - d\rho(\nabla_{\overline{e}_2}\overline{e}_1) + d\rho(\nabla_{\overline{e}_1}\overline{e}_2) \\ &= -\overline{e}_2\big(\overline{\omega}(\overline{e}_1)\big) - \overline{e}_2(\overline{e}_2\varphi)\tan 2\varphi - \frac{2(\overline{e}_2\varphi)^2}{\cos^2 2\varphi} \\ &\quad + \overline{e}_1(\overline{\omega}(\overline{e}_2)) - \overline{e}_1(\overline{e}_1\varphi)\tan 2\varphi - \frac{2(\overline{e}_1\varphi)^2}{\cos^2 2\varphi} \\ &\quad + \overline{\omega}(\overline{e}_2)^2 - \overline{\omega}(\overline{e}_2)(\overline{e}_1\varphi)\tan 2\varphi + \overline{\omega}(\overline{e}_1)^2 + \overline{\omega}(\overline{e}_1)(\overline{e}_2\varphi)\tan 2\varphi \\ &= -K - \triangle\varphi\tan 2\varphi - \frac{2\|\mathrm{grad}\,\varphi\|^2}{\cos^2 2\varphi}. \end{aligned}$$

If the Gauss equation (5.2) and the Codazzi equation (5.4) hold, then this term vanishes, and the existence of orthonormal frame field $\{e_1, e_2\}$ on M^2 satisfying (5.1) are guaranteed. If we put functions T_1 and T_2 on M^2 as $\operatorname{grad}\varphi = T_1 e_1 + T_2 e_2$, then the assumptions of Theorem 3 are satisfied in this case. Hence for existence of Lagrangian isometric minimal immersions, we obtain the following:

Theorem 5. *Let M^2 be a simply connected oriented Riemannian manifold of dimension 2 and let K be the Gauss curvature of M^2. Suppose there exists a function $\varphi\colon M^2 \to (-\pi/4, \pi/4)$ such that*

$$K = \frac{\sin^2 2\varphi}{2} - 2\|\operatorname{grad}\varphi\|^2,$$
$$\triangle\varphi + 2\|\operatorname{grad}\varphi\|^2 \tan 2\varphi = -\frac{\sin 4\varphi}{4}.$$

Then there exists a Lagrangian isometric minimal immersion $x\colon M^2 \to S^2 \times S^2$ such that φ is the angle function of x.

Next, we rewrite Theorem 5 in the case when M^2 is a domain U in $\mathbf{R}^2$ with an isothermal coordinate. Let (x, y) be an orthogonal coordinate system of U and suppose U has a metric

$$ds^2 = g(x, y)^2(dx^2 + dy^2) \tag{5.5}$$

for some function $g = g(x, y) > 0$, $(x, y) \in U$ and let $\varphi\colon U \to (-\pi/4, \pi/4)$ be a function. We use the notation that $g_x = \partial g/\partial x$, $g_y = \partial g/\partial y$ and also $\varphi_x = \partial\varphi/\partial x$, $\varphi_y = \partial\varphi/\partial y$. If we put $e_1 = (1/g)(\partial/\partial x)$, $e_2 = (1/g)(\partial/\partial y)$, then $\{e_1, e_2\}$ is an orthonormal frame field on U. The connection form ω with respect to the basis $\{e_1, e_2\}$ are written as

$$\omega(e_1) = \langle\nabla_{e_1} e_1, e_2\rangle = -\langle e_1, [e_1, e_2]\rangle = -\frac{g_y}{g^2},$$
$$\omega(e_2) = \langle\nabla_{e_2} e_1, e_2\rangle = -\langle e_2, [e_1, e_2]\rangle = \frac{g_x}{g^2}.$$

So the Gauss curvature K on U is

$$\begin{aligned}K &= e_2(\omega(e_1)) - e_1(\omega(e_2)) - \omega(e_1)^2 - \omega(e_2)^2\\ &= -\frac{g_{xx} + g_{yy}}{g^3} + \frac{(g_x)^2 + (g_y)^2}{g^4}\\ &= -\frac{\triangle_0 \log g}{g^2}\end{aligned}$$

where $\triangle_0 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. From

$$\|\operatorname{grad} \varphi\|^2 = (e_1\varphi)^2 + (e_2\varphi)^2 = \frac{(\varphi_x)^2 + (\varphi_y)^2}{g^2},$$

the Gauss equation (5.2) is

$$\triangle_0 \log g = 2((\varphi_x)^2 + (\varphi_y)^2) - \frac{g^2 \sin^2 2\varphi}{2} \tag{5.6}$$

and from

$$\begin{aligned}\triangle\varphi &= e_1(e_1\varphi) + e_2(e_2\varphi) - \omega(e_1)e_2\varphi + \omega(e_2)e_1\varphi \\ &= \frac{\varphi_{xx} + \varphi_{yy}}{g^2} \\ &= \frac{\triangle_0\varphi}{g^2},\end{aligned}$$

we get for the Codazzi equation (5.4)

$$\triangle_0\varphi = -2((\varphi_x)^2 + (\varphi_y)^2)\tan 2\varphi - \frac{g^2 \sin 4\varphi}{4}. \tag{5.7}$$

Since any 2-dimensional Riemannian manifold is conformally flat, the metric is locally isometric to (5.5). Thus the Gauss equation (5.2) and the Codazzi equation (5.4) for any minimal Lagrangian surface of $S^2 \times S^2$ are locally written as (5.6) and (5.7). Hence Theorem 5 is written as follows:

Theorem 6. *Let U be a simply connected domain in $\mathbf{R}^2$. Suppose $g\colon U \to (0,\infty)$ and $\varphi\colon U \to (-\pi/4, \pi/4)$ are solutions of two equations*

$$\begin{gathered}\triangle_0 \log g = 2((\varphi_x)^2 + (\varphi_y)^2) - \frac{g^2 \sin^2 2\varphi}{2}, \\ \triangle_0\varphi = -2((\varphi_x)^2 + (\varphi_y)^2)\tan 2\varphi - \frac{g^2 \sin 4\varphi}{4} \\ (\triangle_0 = \partial^2/\partial x^2 + \partial^2/\partial y^2).\end{gathered}$$

Then there exists a minimal Lagrangian immersion $x\colon U \to S^2 \times S^2$ such that the induced metric to x on U satisfies (5.5) *and φ is the angle function of the immersion x.*

Finally, we consider Theorem 6 in the case when g and the angle function φ are rotationally symmetric on U.

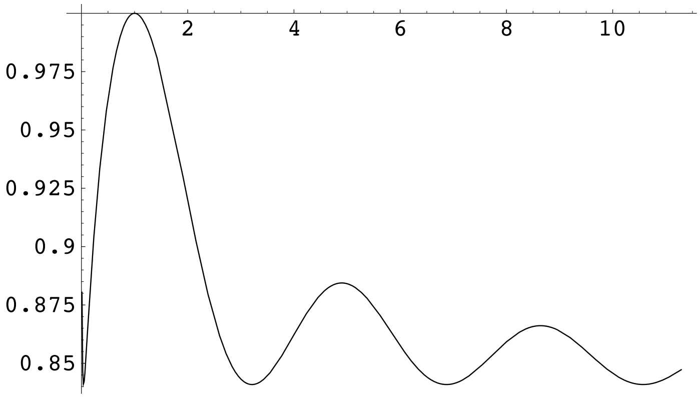

Fig. 1. solution curve in (r, g)-plane

Theorem 7. *Let $I_1, I_2 \subset [0, \infty)$ be intervals with $I_1 \cap I_2 \neq \emptyset$. Suppose $g\colon I_1 \to (0, \infty)$ and $\varphi\colon I_2 \to (-\pi/4, \pi/4)$ are solutions of the system of ordinary differential equations*

$$
\begin{cases} g'' = \dfrac{(g')^2}{g} - \dfrac{g'}{r} + 2g(\varphi')^2 - \dfrac{g^3 \sin^2 2\varphi}{2}, \\ \varphi'' = -2(\varphi')^2 \tan 2\varphi - \dfrac{\varphi'}{r} - \dfrac{g^2 \sin 4\varphi}{4}. \end{cases} \tag{5.8}
$$

Then there exists a minimal Lagrangian immersion x from a simply connected domain U in $\mathbf{R}^2$ to $S^2 \times S^2$ such that φ is the angle function of x for the solution $(g(r), \varphi(r))$ of (5.8).

Proof. It is well known that

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}, \\
\frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}, \\
\Delta_0 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta},
\end{aligned}
$$

where (x, y) and (r, θ) denote the orthogonal coordinates and the polar coordinates of $\mathbf{R}^2$, respectively. Then the two equations in Theorem 6 are written as (5.8). $\square$

Figs. 1, 2, 3 are numerical solution curves of (5.8) with initial conditions $g(1) = 1$, $\varphi(1) = \pi/8$ and $g'(1) = \varphi'(1) = 0$ with $0.01 \leq r \leq 11.3$.

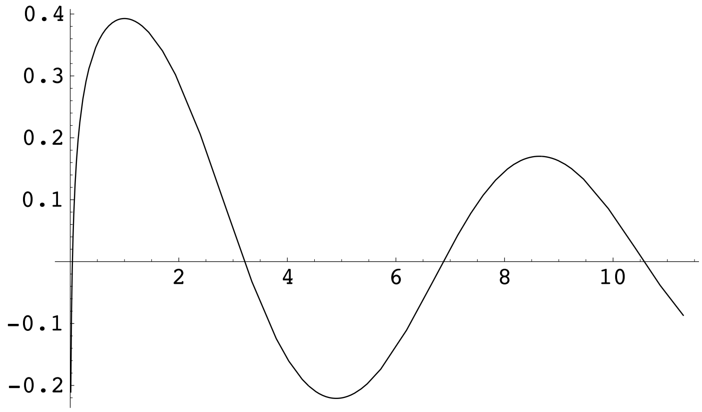

Fig. 2. solution curve in (r, φ)-plane

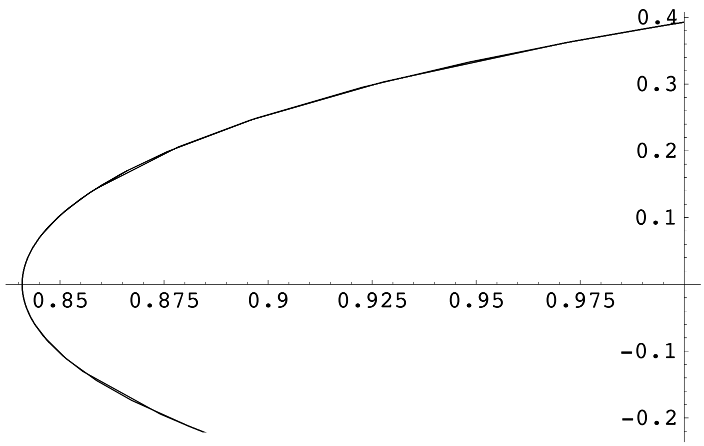

Fig. 3. solution curve in (g, φ)-plane

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