Kim, S. and Kim, Y.R.
Osaka J. Math.
44 (2007), 685–690

PROJECTIVE NORMALITY OF ALGEBRAIC CURVES AND ITS APPLICATION TO SURFACES

SEONJA KIM and YOUNG ROCK KIM

(Received February 2, 2006, revised October 24, 2006)

Abstract

Let L be a very ample line bundle on a smooth curve C of genus g with $(3g+3)/2 < \deg L \leq 2g-5$. Then L is normally generated if $\deg L > \max\{2g+2-4h^1(C, L), 2g-(g-1)/6-2h^1(C, L)\}$. Let C be a triple covering of genus p curve C' with $C \xrightarrow{\phi} C'$ and D a divisor on C' with $4p < \deg D < (g-1)/6-2p$. Then $K_C(-\phi^* D)$ becomes a very ample line bundle which is normally generated. As an application, we characterize some smooth projective surfaces.

1. Introduction

We work over the algebraically closed field of characteristic zero. Specially the base field is the complex numbers in considering the classification of surfaces. A smooth irreducible algebraic variety V in $\mathbb{P}^r$ is said to be projectively normal if the natural morphisms $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(V, \mathcal{O}_V(m))$ are surjective for every non-negative integer m. Let C be a smooth irreducible algebraic curve of genus g. We say that a base point free line bundle L on C is normally generated if C has a projectively normal embedding via its associated morphism $\phi_L\colon C \to \mathbb{P}(H^0(C, L))$.

Any line bundle of degree at least $2g+1$ on a smooth curve of genus g is normally generated but a line bundle of degree at most $2g$ might fail to be normally generated ([8], [9], [10]). Green and Lazarsfeld showed a sufficient condition for L to be normally generated as follows ([5], Theorem 1): If L is a very ample line bundle on C with $\deg L \geq 2g+1-2h^1(C, L)-\mathrm{Cliff}(C)$ (and hence $h^1(C, L) \leq 1$), then L is normally generated. Using this, we show that a line bundle L on C with $(3g+3)/2 < \deg L \leq 2g-5$ is normally generated for $\deg L > \max\{2g+2-4h^1(C, L), 2g-(g-1)/6-2h^1(C, L)\}$. As a corollary, if C is a triple covering of a genus p curve C' with $C \xrightarrow{\phi} C'$ then it has a very ample $K_C(-\phi^* D)$ which is normally generated for any divisor D on C' with $4p < \deg D < (g-1)/6-2p$. It is a kind of generalization of the result that $K_C(-rg^1_3)$ on a trigonal curve C is normally generated for $3r \leq g/2-1$ ([7]).

2000 Mathematics Subject Classification. 14H45, 14H10, 14C20, 14J10, 14J27, 14J28.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (2005-070-C00005) for the first author. This work was supported by Hankuk University of Foreign Studies Research Fund of 2007 for the second author.

As an application to nondegenerate smooth surface $S \subset \mathbb{P}^r$ of degree $2\Delta - e$ with $g(H) = \Delta + f$, $\max\{e/2, 6e - \Delta\} < f - 1 < (\Delta - 2e - 6)/3$ for some $e, f \in \mathbb{Z}_{\geq 1}$, we obtain that S is projectively normal with $p_g = f$ and $-2f - e + 2 \leq K_S^2 \leq (2f + e - 2)^2/(2\Delta - e)$ if its general hyperplane section H is linearly normal, where $\Delta := \deg S - r + 1$. We derive this application using the methods in Akahori's paper ([2]).

We follow most notations in [1], [4], [6]. Let C be a smooth irreducible projective curve of genus $g \geq 2$. The Clifford index of C is taken to be $\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(L) \mid h^0(C, L) \geq 2,\ h^1(C, L) \geq 2\}$, where $\mathrm{Cliff}(L) = \deg L - 2(h^0(C, L) - 1)$ for a line bundle L on C. By abuse of notation, we sometimes use a divisor D on a smooth variety V instead of $\mathcal{O}_V(D)$. We also denote $H^i(V, \mathcal{O}_V(D))$ by $H^i(V, D)$ and $h^0(V, L) - 1$ by $r(L)$ for a line bundle L on V. We denote K_V the canonical line bundle on a smooth variety V.

2. Main results

Any line bundle of degree at least $2g + 1$ on a smooth curve of genus g is normally generated. If the degree is at most $2g$, then there are curves which have a non normally generated line bundle of given degree ([8], [9], [10]). In this section, we investigate the normal generation of a line bundle with given degree on a smooth curve under some condition about the speciality of the line bundle.

Theorem 2.1. *Let L be a very ample line bundle on a smooth curve C of genus g with $(3g+3)/2 < \deg L \leq 2g - 5$. Then L is normally generated if $\deg L > \max\{2g + 2 - 4h^1(C, L), 2g - (g-1)/6 - 2h^1(C, L)\}$.*

Proof. We have $h^1(C, L) \geq 2$, since $2g - 5 \geq \deg L > 2g + 2 - 4h^1(C, L)$. Suppose L is not normally generated. Then there exists a line bundle $A \simeq L(-R)$, $R > 0$, such that (i) $\mathrm{Cliff}(A) \leq \mathrm{Cliff}(L)$, (ii) $\deg A \geq (g-1)/2$, (iii) $h^0(C, A) \geq 2$ and $h^1(C, A) \geq h^1(C, L) + 2$ by the proof of Theorem 3 in [5]. Assume $\deg K_C L^{-1} = 3$. Then $|K_C L^{-1}| = g^1_3$. On the other hand, $L = K_C(-g^1_3)$ is normally generated. So we may assume $\deg K_C L^{-1} \geq 4$ and then $r(K_C L^{-1}) \geq 2$ since $\deg L > 2g + 2 - 4h^1(C, L)$. Let B_1 (resp. B_2) be the base locus of $K_C L^{-1}$ (resp. $K_C A^{-1}$). And let $N_1 := K_C L^{-1}(-B_1)$, $N_2 := K_C A^{-1}(-B_2)$. Then $N_1 \lneq N_2$ since $A \cong L(-R)$, $R > 0$ and $h^1(C, A) \geq h^1(C, L) + 2$. Hence we have the following diagram,

$$\begin{tikzcd} C \arrow[r, "\phi_{N_2}"] \arrow[dr, "\phi_{N_1}"'] & C_2 \arrow[d, "\pi:\ \text{projection}"] \\ & C_1 \end{tikzcd}$$

where $C_i = \phi_{N_i}(C)$.

If we set $m_i := \deg \phi_{N_i}$, $i = 1, 2$, then we have $m_2 | m_1$. If N_1 is birationally very ample, then by Lemma 9 in [8] and $\deg K_C L^{-1} < (g-1)/2$ we have $r(N_1) \leq [(\deg N_1 -$

$1)/5]$. It is a contradiction to $\deg L > 2g + 2 - 4h^1(C, L)$ that is equivalent to $\deg K_C L^{-1} < 4(h^0(C, K_C L^{-1}) - 1)$. Therefore N_1 is not birationally very ample, and then we have $m_1 \le 3$ since $\deg K_C L^{-1} < 4(h^0(C, K_C L^{-1}) - 1)$.

Let H_1 be a hyperplane section of C_1. If $|H_1|$ on a smooth model of C_1 is special, then $r(N_1) \le (\deg N_1)/4$, which is absurd. Thus $|H_1|$ is nonspecial. If $m_1 = 2$, then

$$r(K_C L^{-1}(-B_1 + P + Q)) \ge r(K_C L^{-1}(-B_1)) + 1$$

for any pairs (P, Q) such that $\phi_{N_1}(P) = \phi_{N_1}(Q)$ since $|H_1|$ is nonspecial. Therefore we have $r(L(-P-Q)) \ge r(L)-1$ for (P, Q) such that $\phi_{N_1}(P) = \phi_{N_1}(Q)$, which contradicts that L is very ample. Therefore we get $m_1 = 3$. Suppose B_1 is nonzero. Set $P \le B_1$ for some $P \in C$. Consider Q, R in C such that $\phi_{N_1}(P) = \phi_{N_1}(Q) = \phi_{N_1}(R) = P'$ for some $P' \in C_1$. Since $|H_1|$ is nonspecial, we have

$$\begin{aligned} r(K_C L^{-1}(Q + R)) &\ge r(N_1(P + Q + R)) = r(H_1 + P') \\ &= r(H_1) + 1 = r(K_C L^{-1}) + 1 \end{aligned}$$

which is a contradiction to the very ampleness of L. Hence $K_C L^{-1}$ is base point free, i.e., $K_C L^{-1} = N_1$. On the other hand, we have $m_2 = 1$ or 3 for $m_2 | m_1$. Since $K_C A^{-1}(-B_2) = N_2 \gneq N_1 = K_C L^{-1}$, we may set $N_1 = N_2(-G)$ for some $G > 0$.

Assume $m_2 = 1$, i.e. $K_C A^{-1}(-B_2) = N_2$ is birationally very ample. On the other hand we have $r(N_2) \ge r(N_1) + (\deg G)/2$, since $N_2(-G) \cong N_1$ and $\operatorname{Cliff}(N_2) \le \operatorname{Cliff}(A) \le \operatorname{Cliff}(L) = \operatorname{Cliff}(N_1)$. In case $\deg N_2 \ge g$ we have $r(N_2) \le (2\deg N_2 - g + 1)/3$ by Castelnuovo's genus bound and hence

$$\operatorname{Cliff}(L) \ge \operatorname{Cliff}(N_2) \ge \deg N_2 - \frac{4\deg N_2 - 2g + 2}{3} = \frac{2g - 2 - \deg N_2}{3} \ge \frac{g-1}{6},$$

since $N_2 = K_C A^{-1}(-B_2)$ and $\deg A \ge (g-1)/2$. If we observe that the condition $\deg L > 2g - (g-1)/6 - 2h^1(C, L)$ is equivalent to $\operatorname{Cliff}(K_C L^{-1}) < (g-1)/6$, then we meet an absurdity. Thus we have $\deg N_2 \le g - 1$, and then Castelnuovo's genus bound produces $\deg N_2 \ge 3r(N_2) - 2$. Note that the Castelnuovo number $\pi(d, r)$ has the property $\pi(d, r) \le \pi(d-2, r-1)$ for $d \ge 3r - 2$ and $r \ge 3$, where $\pi(d, r) = (m(m-1)/2)(r-1) + m\epsilon$, $d - 1 = m(r-1) + \epsilon$, $0 \le \epsilon \le r - 2$ (Lemma 6, [8]). Hence

$$\pi(\deg N_2, r(N_2)) \le \cdots \le \pi\left(\deg N_2 - \deg G, r(N_2) - \frac{\deg G}{2}\right) \le \pi(\deg N_1, r(N_1)),$$

because of $2 \le r(N_1) \le r(N_2) - (\deg G)/2$. Since $r(N_1) \ge (\deg N_1)/4$ and $\deg N_1 < (g-1)/2$, we can induce a strict inequality $\pi(\deg N_1, r(N_1)) < g$ as only the number regardless of birational embedding from the proof of Lemma 9 in [8]. It is absurd. Hence $m_2 = 3$, which yields $C_1 \cong C_2$.

Let H_2 be a hyperplane section of C_2. If $|H_2|$ on a smooth model of C_2 is special, then $r(N_2) \leq (\deg N_2)/6$. Thus the condition $\deg K_C L^{-1} < 4(h^0(C, K_C L^{-1}) - 1)$ yields the following inequalities:

$$\frac{2\deg N_2}{3} \leq \mathrm{Cliff}(N_2) \leq \mathrm{Cliff}(N_1) \leq \frac{\deg N_1}{2},$$

which contradicts to $N_1 \lneqq N_2$. Accordingly $|H_2|$ is also nonspecial.

Now we have $r(N_i) = (\deg N_i)/3 - p$, $i = 1, 2$ where p is the genus of a smooth model of $C_1 \cong C_2$. Therefore

$$\frac{\deg N_1}{3} + 2p = \mathrm{Cliff}(N_1) \geq \mathrm{Cliff}(N_2) = \frac{\deg N_2}{3} + 2p$$

which is a contradiction that $\deg N_1 < \deg N_2$. This contradiction comes from the assumption that L is not normally generated, thus the result follows. □

Using the above theorem, we obtain the following corollary under the same assumption:

Corollary 2.2. *Let C be a triple covering of a genus p curve C' with $C \xrightarrow{\phi} C'$ and D a divisor on C' with $4p < \deg D < (g-1)/6 - 2p$. Then $K_C(-\phi^* D)$ becomes a very ample line bundle which is normally generated.*

Proof. Set $d := \deg D$ and $L := K_C(-\phi^* D)$. Suppose L is not base point free, then there is a $P \in C$ such that $|K_C L^{-1}(P)| = g^{r+1}_{3d+1}$. Note that g^{r+1}_{3d+1} cannot be composed with ϕ by degree reason. Therefore we have $g \leq 6d + 3p$ due to the Castelnuovo-Severi inequality. Hence it cannot occur by the condition $d < (g-1)/6 - 2p$. Suppose L is not very ample, then there are $P, Q \in C$ such that $|K_C L^{-1}(P + Q)| = g^{r+1}_{3d+2}$. By the same method as above, we get a similar contradiction. Thus L is very ample. The condition $d < (g-1)/6 - 2p$ produces $\mathrm{Cliff}(K_C L^{-1}) = d + 2p < (g-1)/6$ since $\deg K_C L^{-1} = 3d$ and $h^0(C, K_C L^{-1}) = h^0(C', D) = d - p + 1$. Whence $\deg L > 2g - (g-1)/6 - 2h^1(C, L)$ is satisfied. The condition $4p < d$ induces $\deg K_C L^{-1} > 4(h^0(C, K_C L^{-1}) - 1)$, i.e., $\deg L > 2g + 2 - 4h^1(C, L)$. Consequently L is normally generated by Theorem 2.1. □

REMARK 2.3. In fact, we have a similar result in [8] for trigonal curve $C\colon K_C(-r g^1_3)$ is normally generated if $3r < g/2 - 1$ ([7]). Thus our result could be considered as a generalization which deals with triple coverings under the some condition.

Let $S \subseteq \mathbb{P}^r$ be a nondegenerate smooth surface and H a smooth hyperplane section of S. If H is projectively normal and $h^1(H, \mathcal{O}_H(2)) = 0$, then $q = h^1(S, \mathcal{O}_S) =$

0, $p_g = h^2(S, \mathcal{O}_S) = h^1(H, \mathcal{O}_H(1))$ and $h^1(S, \mathcal{O}_S(t)) = 0$ for all nonnegative integer t ([2], Lemma 2.1, Lemma 3.1). Using Theorem 2.1, we can characterize smooth projective surfaces with the wider range of degrees and sectional genera. Recall the definition of Δ-genus given by $\Delta := \deg S - r + 1$.

Theorem 2.4. *Let $S \subset \mathbb{P}^r$ be a nondegenerate smooth surface of degree $2\Delta - e$ with $g(H) = \Delta + f$, $\max\{e/2, 6e - \Delta\} < f - 1 < (\Delta - 2e - 6)/3$ for some $e, f \in \mathbb{Z}_{\geq 1}$ and its general hyperplane section H is linearly normal. Then S is projectively normal with $p_g = f$ and $-2f - e + 2 \leq K_S^2 \leq (2f + e - 2)^2/(2\Delta - e)$.*

Proof. From the linear normality of H, we get $h^0(H, \mathcal{O}_H(1)) = r$ and hence

$$\begin{aligned} h^1(H, \mathcal{O}_H(1)) &= -\deg \mathcal{O}_H(1) - 1 + g(H) + h^0(H, \mathcal{O}_H(1)) \\ &= -2\Delta + e - 1 + g(H) + h^0(H, \mathcal{O}_H(1)) \\ &= g(H) - \Delta = f. \end{aligned}$$

Therefore we have $h^1(H, \mathcal{O}_H(1)) > \deg((K_H \otimes \mathcal{O}_H(-1))/4) + 1$ since $f > e/2 + 1$ and $\deg \mathcal{O}_H(1) = 2\Delta - e = 2g(H) - 2 - (2f + e - 2)$. Thus $\mathcal{O}_H(1)$ satisfies $\deg \mathcal{O}_H(1) > 2g(H) + 2 - 4h^1(H, \mathcal{O}_H(1))$. The condition $f - 1 > 6e - \Delta$ implies $\deg \mathcal{O}_H(1) > 2g - (g - 1)/6 - 2h^1(H, \mathcal{O}_H(1))$. Also the condition $f - 1 < (\Delta - 2e - 6)/3$ yields $\deg \mathcal{O}_H(1) > (3g + 3)/2$. Hence $\mathcal{O}_H(1)$ is normally generated by Theorem 2.1, and thus its general hyperplane section H is projectively normal since it is linearly normal. Therefore S is projectively normal with $q = 0$, $p_g = h^0(S, K_S) = h^1(H, \mathcal{O}_H(1)) = f > 1$ since $h^1(H, \mathcal{O}_H(2)) = 0$ from $\deg \mathcal{O}_H(1) > (3g + 3)/2$.

If we consider the adjunction formula then $K_S.H = 2f + e - 2$ and $0 \to K_S \to K_S + H \to K_H \to 0$. Thus we have $0 \to H^0(S, K_S) \to H^0(S, K_S + H) \to H^0(H, K_H) \to 0$, since $H^1(S, K_S) = q = 0$. Assume $|K_S + H|$ has a fixed component B. Set $p \in B \cap H$, then p becomes a base point of $|K_H|$ since $H^0(S, K_S + H) \to H^0(H, K_H)$ is surjective, which cannot occur. Therefore $K_S + H$ is free from fixed components. Thus for any irreducible curve C in S, we can choose effective $D \in |H + K_S|$ such that D does not contain C and then $D.C \geq 0$, which implies $H + K_S$ is nef. Hence we get $K_S.(H + K_S) \geq 0$ and then

$$K_S^2 \geq -K_S.H = -2f - e + 2.$$

Thus $-2f - e + 2 \leq K_S^2 \leq (2f + e - 2)^2/(2\Delta - e)$ by the Hodge index theorem $K_S^2 H^2 \leq (K_S.H)^2$. Hence the theorem is proved. $\square$

ACKNOWLEDGEMENT. We would like to thank referee for valuable comments which improve our paper.

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris: Geometry of Algebraic Curves I, Springer, New York, 1985.

[2] K. Akahori: *Classification of projective surfaces and projective normality*, Tsukuba J. Math. **22** (1998), 213–225.

[3] A. Beauville: Complex Algebraic Surfaces, Cambridge Univ. Press, Cambridge, 1983.

[4] P. Griffiths and J. Harris: Principles of Algebraic Geometry, Wiley-Intersci., New York, 1978.

[5] M. Green and R. Lazarsfeld: *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83** (1986), 73–90.

[6] R. Hartshorne: Algebraic Geometry, Graduate Text in Math. **52**, Springer, New York, 1977.

[7] S. Kim and Y. Kim: *Projectively normal embedding of a k-gonal curve*, Comm. Algebra **32** (2004), 187–201.

[8] S. Kim and Y. Kim: *Normal generation of line bundles on algebraic curves*, J. Pure Appl. Algebra **192** (2004), 173–186.

[9] H. Lange and G. Martens: *Normal generation and presentation of line bundles of low degree on curves*, J. Reine Angew. Math. **356** (1985), 1–18.

[10] D. Mumford: *Varieties defined by quadric equations*; in Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969), Ed. Cremonese, Rome, 1970, 29–100.

Seonja Kim
Department of Digital Broadcasting & Electronics
Chungwoon University
Chungnam, 350–701
Korea
e-mail: sjkim@chungwoon.ac.kr

Young Rock Kim
Department of Mathematics Education
Graduate School of Education
Hankuk University of Foreign Studies
Seoul, 130–791
Korea
e-mail: rocky777@hufs.ac.kr