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A NOTE ON STABLE CLIFFORD EXTENSIONS OF MODULES

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Abstract

Let H be a normal subgroup of G. Let W be a G-invariant indecomposable RH-module with vertex Q. Let V be an indecomposable direct summand of the induced module W^G. Let W' and V' be the Green correspondents of W and V in $N_H(Q)$ and $N_G(Q)$ respectively. Then we prove that $\operatorname{rank}_R V/\operatorname{rank}_R W = \operatorname{rank}_R V'/\operatorname{rank}_R W'$.

Let $\mathcal{O}$ be a complete discrete valuation ring, and let F be the residual field of $\mathcal{O}$ of characteristic $p > 0$. We assume that $\mathcal{O}$ and F are big enough. Let R be $\mathcal{O}$ or F.

Let G be a finite group. Let H be a normal subgroup of G. Let W be a G-invariant indecomposable RH-module with vertex Q. Let W' be the Green correspondent of W with respect to $(H, Q, N_H(Q))$. Then $G = HN_G(Q)$, and W' is $N_G(Q)$-invariant. We write W^G for the induction of W to G. Set $E = \operatorname{End}_{RG}(W^G)$ and $\Lambda = \operatorname{End}_{RH}(W)$. We can write E in the form $E = \sum_{\bar{x}\in X} \bigoplus E_{\bar{x}}$ where $X = G/H$ and $E_{\bar{x}}$ is the R-submodule of E mapping $W = W \otimes 1$ to $W \otimes x$ inside W^G, and $\operatorname{Hom}_{RH}(W, Wx) \cong E_{\bar{x}}$ (as R-module) by [4, Chap. 4, Lemma 6.4]. Clearly $E_{\bar{x}}E_{\bar{y}} \subset E_{\overline{xy}}$, for $\bar{x}, \bar{y} \in X$. Also we can use the stability hypothesis to choose an element $\varphi_{\bar{x}} \in E_{\bar{x}}$ mapping $W \otimes 1$ isomorphically onto $W \otimes x$; it follows that $\varphi_{\bar{x}}$ is a unit in E. Since $E_{\bar{1}}$ can be identified with Λ, we have $E_{\bar{x}} = \Lambda\varphi_{\bar{x}} = \varphi_{\bar{x}}\Lambda$. Thus $E_{\bar{x}}$ can also be identified with $\operatorname{Hom}_{RH}(W, Wx)$. We do so in this paper. Set $E' = \operatorname{End}_{RN_G(Q)}((W')^{N_G(Q)})$, and $\Lambda' = \operatorname{End}_{RN_H(Q)}(W')$. It is well-known that $J(\Lambda)E \subset J(E)$ (resp. $J(\Lambda')E' \subset J(E')$), and $E/J(\Lambda)E$ (resp. $E'/J(\Lambda')E'$) is a twisted group algebra.

In [3], the outhor proves that $E'/J(\Lambda')E'$ is isomorphic to $E/J(\Lambda)E$, which already appears without proof in Cline [2]. Here we will give an application of the above isomorphism. The following is our main result.

Theorem. *Keep notation and assumptions as above. Then the Green correspondence gives a one-to-one correspondence between the set of non-isomorphic indecomposable direct summands of W^G and that of $(W')^{N_G(Q)}$, and keeps multiplicities. Let*

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V be an indecomposable direct summand of W^G. Let V' be the Green correspondent of V in $N_G(Q)$. Then $\mathrm{rank}_R V/\mathrm{rank}_R W = \mathrm{rank}_R V'/\mathrm{rank}_R W'$.

Proof. From the proof of the main theorem in [3], we do not know whether the isomorphism given there is compatible with the Green correspondence. Here we will first construct an isomorphism from $E'/J(\Lambda')E'$ to $E/J(\Lambda)E$, which is compatible with the Green correspondence.

Let L be a subgroup of G. For RL-modules W_1 and W_2, there is a homomorphism

$$\mathrm{Tr}_L^G\colon \mathrm{Hom}_{RL}(W_1, W_2) \to \mathrm{Hom}_{RG}((W_1)^G, (W_2)^G),$$

such that the following holds: for $f \in \mathrm{Hom}_{RL}(W_1, W_2)$ and $\sum_{x\in L\backslash G} u_x \otimes x \in (W_1)^G$,

$$\mathrm{Tr}_L^G(f)\left(\sum_{x\in L\backslash G} u_x \otimes x\right) = \sum_{x\in L\backslash G} f(u_x)\otimes x.$$

Assume that $(W')^H = W \oplus M$ for some RH-module M. Then $(W')^G = W^G \oplus M^G$. Let $\Omega = \{P : P < Q \cap Q^x,$ fo some $x \in H - N_H(Q)\}$. Then M and M^G are Ω-projective. Let $\iota_{W^G}\colon W^G \to (W')^G$ and $\pi_{W^G}\colon (W')^G \to W^G$ be the inclusion map and the projection map, respectively. We have the following algebra homomorphism:

$$\begin{aligned}\beta\colon \mathrm{End}_{RN_G(Q)}((W')^{N_G(Q)}) &\to \mathrm{End}_{RG}(W^G)\\ f &\mapsto \pi_{W^G}\cdot \mathrm{Tr}_{N_G(Q)}^G(f)\cdot \iota_{W^G}.\end{aligned}$$

Since M^G is Ω-projective,

$$\mathrm{End}_{RG}((W')^G) = \mathrm{End}_{RG}(W^G) + \mathrm{Tr}_\Omega^G(\mathrm{End}_R((W')^G)).$$

It is easy to see that

$$\mathrm{End}_{RG}(W^G)\cap \mathrm{Tr}_\Omega^G(\mathrm{End}_R((W')^G)) = \mathrm{Tr}_\Omega^G(\mathrm{End}_R(W^G)).$$

By [4, Chapter 4, Lemma 5.3], the map $\mathrm{Tr}_{N_G(Q)}^G$ induces the following algebra isomorphism:

$$\mathrm{End}_{RN_G(Q)}((W')^{N_G(Q)})/\mathrm{Tr}_\Omega^{N_G(Q)}(\mathrm{End}_R((W')^{N_G(Q)})) \to \mathrm{End}_{RG}((W')^G)/\mathrm{Tr}_\Omega^G(\mathrm{End}_R((W')^G),$$

so β induces the following algebra isomorphism:

$$\beta\colon \mathrm{End}_{RN_G(Q)}((W')^{N_G(Q)})/\mathrm{Tr}_\Omega^{N_G(Q)}(\mathrm{End}_R((W')^{N_G(Q)})) \to \mathrm{End}_{RG}(W^G)/\mathrm{Tr}_\Omega^G(\mathrm{End}_R(W^G)).$$

By [4, Chapter 4, Lemma 5.1 (Dade)], the following isomorphism holds:

$$\sum_{x\in H\backslash G}\bigoplus \mathrm{Tr}_\Omega^H(\mathrm{Hom}_R(W, Wx)) = \mathrm{Tr}_\Omega^H(\mathrm{Hom}_R(W, W^G)) \cong \mathrm{Tr}_\Omega^G(\mathrm{End}_R(W^G)).$$

Let $I = \mathrm{Tr}_{\Omega}^{H}(\mathrm{End}_R(W))$ and $I' = \mathrm{Tr}_{\Omega}^{N_H(Q)}(\mathrm{End}_R(W'))$. It is easy to see that $\mathrm{Tr}_{\Omega}^{H}(\mathrm{Hom}_R(W, Wx_i)) = \mathrm{Hom}_{RH}(W, Wx_i) \cdot I$. Thus by identification, $\mathrm{Tr}_{\Omega}^{G}(\mathrm{End}_R(W^G))$ is just the graded ideal EI of E generated by I. Let $E'I'$ be the graded ideal of E' generated by I'. Then $\mathrm{Tr}_{\Omega}^{G}(\mathrm{End}_R(W^G)) = EI$ and $\mathrm{Tr}_{\Omega}^{N_G(Q)}(\mathrm{End}_R(W'^{N_G(Q)})) = E'I'$. Thus β gives an isomorphism from $E'/E'I'$ to E/EI. Since both W and W' are not Ω-projective, $I \subseteq J(\Lambda)$ and $I' \subseteq J(\Lambda')$. Note that by restriction β induces an algebra isomorphism from Λ'/I' to Λ/I. Thus β induces an isomorphism from $\Lambda'/J(\Lambda')$ to $\Lambda/J(\Lambda)$. So β sends $J(\Lambda')$ to $J(\Lambda)$. As β is an isomorphism from $E'/E'I'$ to E/EI, we have that β is an isomorphism from $E'/J(\Lambda')E'$ to $E/J(\Lambda)E$.

The first statement of the Theorem is obvious. Thus V' is an indecomposable direct summand of $(W')^{N_G(Q)}$. Let e' be a (primitive) idempotent of E' correspondent to V'. Let e be a primitive idempotent of E such that $\beta(\overline{e'}) = \overline{e}$. Set $\tilde{V} = eW^G$. Then $\tilde{V}$ is of vertex Q, and is a direct summand of $\mathrm{Tr}_{N_G(Q)}^{G}(e')(W')^G = (e'(W')^{N_G(Q)})^G = (V')^G$. We must have that $\tilde{V}$ is the Green correspondent of V'. Thus $V \cong \tilde{V}$. By Cline [1, Corollary 3.15], $V \cong W \otimes_{\Lambda} eE$ and $V' \cong W' \otimes_{\Lambda'} e'E'$. Since $eE/J(\Lambda)eE = \beta(e'E'/J(\Lambda')e'E')$, we have $\dim_F e'E'/J(\Lambda')e'E' = \dim_F eE/J(\Lambda)eE$. So

$$\mathrm{rank}_R V/\mathrm{rank}_R W = \dim_F eE/J(\Lambda)eE = \dim_F e'E'/J(\Lambda')e'E' = \mathrm{rank}_R V'/\mathrm{rank}_R W',$$

as desired. □

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