Ng, S.P.K. and Tan, S.P.
Osaka J. Math.
44 (2007), 247–254

THE COMPLEMENT OF THE BOWDITCH SPACE IN THE $\mathrm{SL}(2,\mathbb{C})$ CHARACTER VARIETY

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(Received April 24, 2006)

Abstract

Let $\mathcal{X}$ be the space of type-preserving $\mathrm{SL}(2,\mathbb{C})$ characters of the punctured torus T. The Bowditch space $\mathcal{X}_{\mathrm{BQ}}$ is the largest open subset of $\mathcal{X}$ on which the mapping class group acts properly discontinuously, this is characterized by two simple conditions called the BQ-conditions. In this note, we show that $[\rho] \in \mathrm{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$ if there exists an essential simple closed curve X on T such that $|\mathrm{tr}\,\rho(X)| < 0.5$.

1. Introduction

Let T be the punctured torus and $\pi := \pi_1(T) = \langle X, Y \rangle$ be its fundamental group which is free on the generators X, Y. The relative $\mathrm{SL}(2,\mathbb{C})$ character variety of *type-preserving* characters is the set

$$\mathcal{X} := \{[\rho] \in \mathrm{Hom}(\pi, \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})\colon \mathrm{tr}(XYX^{-1}Y^{-1}) = -2\},$$

where the equivalence is by the conjugation action. The Bowditch space is the subset $\mathcal{X}_{\mathrm{BQ}} \subset \mathcal{X}$ of characters which satisfy two simple conditions (see Definition 2.1), this is the largest open subset of $\mathcal{X}$ on which the mapping class group of T acts properly discontinuously. It is conjectured by Bowditch to be precisely the quasi-Fuchsian space $\mathcal{X}_{\mathrm{QF}}$ (Conjecture A, [1]). To attempt to verify or disprove the conjecture, and also to study the dynamics of the action of the mapping class group on the non-discrete characters, it is useful to have an effective sufficient condition for $[\rho]$ to be inside $\mathrm{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$. We have the following:

Theorem 1.1 (Main theorem). *For $[\rho] \in \mathcal{X}$, $[\rho] \in \mathrm{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$ if there exists $X \in \mathscr{C}$ such that $|\mathrm{tr}\,\rho(X)| < 0.5$, where $\mathscr{C}$ is the set of free homotopy classes of essential simple closed curves on T.*

REMARK 1.2. (a) The bound 0.5 in the theorem is not optimal, and can be improved, but for computational purposes, it is quite effective.

2000 Mathematics Subject Classification. Primary 57M05; Secondary 30F60, 20H10, 37F30.

(b) Jorgensen's inequality implies that if there exists $X \in \mathscr{C}$ such that $0 < |\operatorname{tr} \rho(X)| < 1$, then $[\rho]$ corresponds to a non-discrete representation. Rough computer experiments have shown that in fact, in many examples considered (no counterexamples were detected), if $|\operatorname{tr} \rho(X)| < 1$ for some $X \in \mathscr{C}$, with $|\operatorname{tr} \rho(X)| \notin (-1, 1)$, then by a trace reduction algorithm, one can find some $Y \in \mathscr{C}$ such that $|\operatorname{tr} \rho(Y)| < 0.5$, that is, $[\rho] \in \operatorname{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$. This can be regarded as supporting evidence towards Bowditch's conjecture as experiments with the Wada's OPTi program [9] has shown that in almost all cases where $[\rho]$ is non-discrete, there exists $X \in \mathscr{C}$ with $|\operatorname{tr} \rho(X)| < 1$.
(c) The theorem quantifies the result of Bowditch in [1] (Theorem 5.5) by giving an explicit bound for the constant ε_0 in his theorem, and hence generalizes Corollary 5.6 there, that $[\rho_0] \in \operatorname{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$, where $[\rho_0]$ is the quaternionic character with $\operatorname{tr} \rho_0(X) = \operatorname{tr} \rho_0(Y) = \operatorname{tr} \rho_0(XY) = 0$ (and hence $\operatorname{tr} \rho_0(X) = 0$ for all $X \in \mathscr{C}$).
(d) The set $\mathcal{X}_{\mathrm{BQ}}$ can be expected to have a very interesting and complicated geometry, especially at the boundary, as evidenced by pictures and studies of various slices of deformation spaces of discrete, faithful representations including the Maskit slice, Earle slice, Riley slice, Bers slices (obtained using Wada's Opti program [9]), and also the bumping phenomena on the boundary of the quasi-Fuchsian space, as studied by various authors. In particular, we have the recent results of Bromberg that states that the closure of $\mathcal{X}_{\mathrm{QF}}$ is not locally connected. Theorem 1.1 can be used in a computer program to draw the Bowditch space and its complement and this should prove useful in studying the geometry of these spaces and various related conjectures.
(e) More generally, as studied in [6], [7] and [8], we can study the relative character varieties $\mathcal{X}_\kappa$, where

$$\operatorname{tr} \rho(XYX^{-1}Y^{-1}) = \kappa$$

with $\kappa \neq 2$, and the Bowditch space can be defined similarly for these relative character varieties. If κ is close to -2, our methods can be modified to give similar conditions for when $[\rho] \in \operatorname{int}(\mathcal{X}_\kappa \setminus \mathcal{X}_{\mathrm{BQ}})$ and this can be used together with the BQ-conditions to draw the Bowditch space and complement. Note that in this case, the Jorgensen inequality may no longer apply, for example if $\kappa \in (-2, 2)$, since in this case the image may never be discrete.

The rest of this paper is organized as follows. In Section 2, we set up the notation and definitions to be used and in Section 3, we give the proof of the theorem.

2. Preliminaries: Notation and definitions

As in the introduction, let T be the punctured torus, X, Y a pair of simple closed curves on T with geometric intersection number one so that $\pi := \pi_1(T) = \langle X, Y \rangle$. The relative character variety of *type-preserving* characters is the set (denoted by $\mathcal{X}$) of equivalence classes of representations from π to $SL(2,\mathbb{C})$ satisfying

$$\mathrm{tr}\,\rho(XYX^{-1}Y^{-1}) = -2, \tag{1}$$

where two representations are equivalent if they are conjugate by an element of $SL(2,\mathbb{C})$. By classical results of Nielsen [5], (see for example [2] for background and references) it does not matter which pair of generators is used for π in the definition. Fixing a pair of generators X, Y of T, by results of Fricke, see [3] for an exposition, the map

$$\iota\colon \mathcal{X} \mapsto \{(x,y,z) \in \mathbb{C}^3 : x^2+y^2+z^2 = xyz\}, \tag{2}$$

given by

$$\iota[\rho] = (\mathrm{tr}\,\rho(X), \mathrm{tr}\,\rho(Y), \mathrm{tr}\,\rho(XY))$$

is a bijection. Henceforth we shall identify $\mathcal{X}$ with the cubic variety given in (2), and the topology on $\mathcal{X}$ will be that induced by this identification. The character $[\rho]$ such that $\iota[\rho] = \mathbf{0} = (0,0,0)$ is the quaternionic character, denoted by $[\rho_0]$.

The outer automorphism group of π,

$$\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi),$$

is isomorphic to the mapping class group of T

$$\pi_0(\mathrm{Homeo}(T)) \cong GL(2,\mathbb{Z})$$

by results of Nielsen [5], and it acts on $\mathcal{X}$, via the action

$$\phi([\rho]) = [\rho \circ \phi^{-1}], \quad \text{where} \quad \phi \in \mathrm{Out}(\pi), \quad [\rho] \in \mathcal{X}. \tag{3}$$

This action is not effective, the kernel is generated by the automorphism ϕ_{inv}, where $\phi_{\mathrm{inv}}(X) = X^{-1}$, $\phi_{\mathrm{inv}}(Y) = Y^{-1}$, corresponding to the elliptic involution on T. Denote by $\Gamma \cong PGL(2,\mathbb{Z})$ the quotient of $\pi_0(\mathrm{Homeo}(T))$ (equivalently, $\mathrm{Out}(\pi)$) by the elliptic involution, Γ now acts effectively on $\mathcal{X}$.

The set $\mathscr{C}$ of free homotopy classes of essential (non-trivial and non-peripheral) simple closed curves on T forms the vertices of the pants graph $\mathscr{C}(T)$ of T, where two vertices are connected by an edge if and only if the corresponding curves have geometric intersection number one. $\mathscr{C}(T)$ is isomorphic to the Farey graph of the hyperbolic plane, and every vertex has infinite valence (see for example [7]). $X, Y \in \mathscr{C}$ are called *neighbors* if they are joined by an edge in $\mathscr{C}(T)$. This is equivalent to saying that X and Y generate π. Note that for any $X \in \mathscr{C}$ and $[\rho] \in \mathcal{X}$, $\mathrm{tr}[\rho](X)$ is well-defined. To simplify notation, we shall use the notationally simpler $\mathrm{tr}\,\rho(X)$ henceforth.

Γ acts on $\mathscr{C}(T)$, and is transitive on the set of vertices $\mathscr{C}$, in fact, it is transitive on the set of neighbors (X, Y), and the set of triples of mutual neighbors (X, Y, Z).

DEFINITION 2.1. The Bowditch space is the subset $\mathcal{X}_{\mathrm{BQ}} \subset \mathcal{X}$ consisting of all characters $[\rho] \in \mathcal{X}$ satisfying the following two conditions, called the *BQ-conditions*:
(i) $\operatorname{tr} \rho(X) \notin [-2, 2]$ for any $X \in \mathscr{C}$; and
(ii) $|\operatorname{tr} \rho(X)| \leq 2$ for only finitely many (possibly none) $X \in \mathscr{C}$.

In [1], Bowditch showed that $\mathcal{X}_{\mathrm{BQ}}$ is open in $\mathcal{X}$, and that Γ acts properly discontinuously on $\mathcal{X}_{\mathrm{BQ}}$. It is also not difficult to see that in fact, $\mathcal{X}_{\mathrm{BQ}}$ is the largest open subset of $\mathcal{X}$ for which the action is properly discontinuous, (see for example [7] and [6] for details, and generalizations to not necessarily type-preserving characters). Furthermore, the subset $\mathcal{X}_{\mathrm{QF}}$ of characters corresponding to the quasi-Fuchsian representations of π is contained in $\mathcal{X}_{\mathrm{BQ}}$ as a connected component. Bowditch has conjectured that in fact, $\mathcal{X}_{\mathrm{QF}} = \mathcal{X}_{\mathrm{BQ}}$.

The dynamics of the action of Γ on $\operatorname{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$ is also very interesting, and some natural questions arise. The first (see [4]), is whether there exists $[\rho] \in \operatorname{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$ such that the closure of its orbit contains $[\rho_0]$ and intersects $\partial \mathcal{X}_{\mathrm{BQ}}$. More generally one can ask if there is a dense orbit under this action, or if most orbits are dense, and finally, if this action is ergodic. Another natural question is whether $\operatorname{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$ is dense in $\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}}$.

Our main theorem can be considered as a first step towards the study of these questions as it gives an effective way of determining if $[\rho] \in \operatorname{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$. In fact, the proof, which is based on a trace reduction algorithm gives in many cases a way of constructing a sequence of elements in the orbit of $[\rho]$ which converges to $[\rho_0]$. (In particular, it can be modified to give an effective constant $\varepsilon > 0$ such that if there exists neighbors (X, Y) such that $|\operatorname{tr} \rho(X)| < \varepsilon$ and $|\operatorname{tr} \rho(Y)| < \varepsilon$, then there exists a sequence of elements in the orbit of $[\rho]$ which converges to $[\rho_0]$. Our result is also useful for attacking the conjecture in [8] that the set of ends of a character $[\rho]$ should be a Cantor set if it contains at least three points and is not the entire projective lamination space, since the trace reduction algorithm given produces lots of ends of the character when there exists $X \in \mathscr{C}$ with $|\operatorname{tr} \rho(X)| < 0.5$.

3. Proof of Main Theorem: A trace reduction algorithm

Our proof of Theorem 1.1 is similar in spirit to that given by Bowditch in [1] that $[\rho_0] \in \operatorname{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$, although somewhat more geometric. The key lemma is the following:

Lemma 3.1. *Let $[\rho] \in \mathcal{X}$ and suppose that there exists $X \in \mathscr{C}$ such that $|\operatorname{tr} \rho(X)| < 0.5$, with $\operatorname{tr} \rho(X) \notin \mathbb{R}$. Then there exists a neighbor Y of X in $\mathscr{C}$ such that $|\operatorname{tr} \rho(Y)| < |\operatorname{tr} \rho(X)|$.*

The theorem now follows from the lemma since if $|\operatorname{tr} \rho(X)| < 0.5$ and $\operatorname{tr} \rho(X) \in \mathbb{R}$, then $[\rho] \notin \mathcal{X}_{\mathrm{BQ}}$, otherwise, we can construct a sequence (of neighbors) $\{X_n\}$ in $\mathscr{C}$

such that $X_0 = X$, and furthermore, either (i) the sequence is infinite and $|\mathrm{tr}\,\rho(X_{j+1})| < |\mathrm{tr}\,\rho(X_j)|$ for all j, or (ii) the sequence is finite and terminates at X_N with $\mathrm{tr}\,\rho(X_N) \in (-2, 2)$. In either case, $[\rho] \notin \mathcal{X}_{\mathrm{BQ}}$. Note that the condition is an open condition, so $[\rho] \in \mathrm{int}(\mathcal{X} \setminus \mathcal{X}_{\mathrm{BQ}})$.

Proof of Lemma 3.1. Let Y_n, $n \in \mathbb{Z}$ denote the (successive) neighbors of X, that is, $Y_n = X^n Y_0$ for some neighbor Y_0 of X. For simplicity of notation, we use the lower case letters x, y_n to denote $\mathrm{tr}\,\rho(X)$, $\mathrm{tr}\,\rho(Y_n)$ respectively. The condition in the lemma is then

$$|x| < 0.5, \quad x \notin \mathbb{R}. \tag{4}$$

By conjugating the representation so that $\rho(X)$ is diagonal and ∞ is its attracting fixed point, that is,

$$\rho(X) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \rho(Y_0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we see that

$$x = \lambda + \lambda^{-1}, \quad \text{where} \quad |\lambda| > 1, \tag{5}$$

and

$$y_n = A\lambda^n + D\lambda^{-n}, \tag{6}$$

where

$$AD = \frac{x^2}{x^2 - 4}, \tag{7}$$

by the commutator relation (1).

Write $\lambda = re^{i\theta}$, so $|\lambda| = r > 1$, $\arg\lambda = \theta \in (-\pi, \pi]$. By renaming Y_n as Y_0, and interchanging A and D if necessary, we may assume that

$$1 \le \left|\frac{D}{A}\right| \le |\lambda| = r. \tag{8}$$

The idea now is that if $|x|$ is small, then $r \sim 1$ and $|\theta| \sim \pi/2$. Hence $|A| \sim |D| \sim |x|/2$, so that either $|y_0| < |x|$ (if $\arg A \not\sim \arg D$), or $|y_1| < |x|$ (if $\arg A \sim \arg D$). We make these arguments precise in the following estimates.

From (4), we have the following bounds for r and θ:

$$1 < r = |\lambda| < \frac{0.5 + \sqrt{4.25}}{2} \cong 1.281, \tag{9}$$

$$-0.25 < \cos\theta < 0.25, \quad 0.419\pi < |\theta| < 0.581\pi. \tag{10}$$

From (4), (7) and (8), we have

$$|AD| < \frac{|x|^2}{3.75} \Longrightarrow |A|^2 < \frac{|x|^2}{3.75} \Longrightarrow |A| < \frac{|x|}{\sqrt{3.75}}. \tag{11}$$

Hence,

$$|y_0| = |A + D| = |A|\left|1 + \frac{D}{A}\right| < \frac{|x|}{\sqrt{3.75}}\left|1 + \frac{D}{A}\right|. \tag{12}$$

Now we claim that either

$$\left|1 + \frac{D}{A}\right| < \sqrt{3.75}, \tag{13}$$

or

$$\left|\lambda + \frac{D}{A\lambda}\right| < \sqrt{3.75} \Longleftrightarrow \left|1 + \frac{D\lambda^{-1}}{A\lambda}\right| < \frac{\sqrt{3.75}}{|\lambda|}. \tag{14}$$

Proof of Claim. Suppose that the first statement is not true, that is, $|1+(D/A)| \geq \sqrt{3.75}$. Let $D/A = r_0 e^{i\theta_0}$, where $\theta_0 \in (-\pi, \pi]$, and write $a := |1+(D/A)|$, $\alpha := \pi - \theta_0$. So our assumption is equivalent to

$$a^2 \geq 3.75. \tag{15}$$

Applying the cosine rule to the triangle with vertices at the complex numbers 0, D/A and $1 + (D/A)$, we get

$$\cos(\pi - \theta_0) = \cos\alpha = \frac{1 + |D/A|^2 - a^2}{2|D/A|}. \tag{16}$$

Now applying the bounds for a and $|D/A|$ from (15), (8) and (9) to (16) and rounding off, we get

$$\cos\alpha \leq \frac{1 + r^2 - 3.75}{2r} < -0.432. \tag{17}$$

In particular,

$$|\theta_0| < 0.36\pi, \quad 0.64\pi < \alpha < 1.36\pi. \tag{18}$$

Now write $b := |1 + (D\lambda^{-1}/(A\lambda))|$ and as before, apply the cosine rule to the triangle with vertices at 0, $D\lambda^{-1}/(A\lambda)$ and $1 + (D\lambda^{-1}/(A\lambda))$ to get

$$b^2 = 1 + \frac{1}{r^4}\left|\frac{D}{A}\right|^2 - \frac{2}{r^2}\left|\frac{D}{A}\right|\cos(\alpha + 2\theta). \tag{19}$$

Using the bounds for θ and α in (10) and (18), we get that

$$|\alpha + 2\theta| < 0.522\pi \Longrightarrow \cos(\alpha + 2\theta) > -0.07. \tag{20}$$

Applying (20) and $|D/A| \leq r < r^2$ to (19), we have

$$b^2 < 1 + 1 + 2(0.07) = 2.14 \Longrightarrow b < 1.463 < \frac{\sqrt{3.75}}{1.281}\ (\cong 1.512) < \frac{\sqrt{3.75}}{|\lambda|} \tag{21}$$

where the last inequality follows from (9). This proves the claim as the second statement of the claim holds in this case. $\square$

To complete the proof of the lemma, we see that if the first part of the claim holds, we have $|y_0| = |A + D| < |A|\sqrt{3.75} < |x|$, otherwise, $|y_1| = |A\lambda + D\lambda^{-1}| = |A|\,|\lambda|b < |A|\sqrt{3.75} < |x|$, where the last part of the inequalities in both cases follow from (11). $\square$

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