

ORDINARY INDUCTION FROM A SUBGROUP AND FINITE GROUP BLOCK THEORY

MORTON E. HARRIS

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Abstract

The first step in the fundamental Clifford Theoretic Approach to General Block Theory of Finite Groups reduces to: H is a subgroup of the finite group G and b is a block of H such that $b({}^g b) = 0$ for all $g \in G - H$. We extend basic results of several authors in this situation and place these results into current categorical and character theoretic equivalences frameworks.

1. Introduction and statements of results

Let G be a finite group, let p be a prime integer and let $(\mathcal{O}, \mathcal{K}, k)$ be a p -modular system for G that is “large enough” for all subgroups of G (i.e., \mathcal{O} is a complete discrete valuation ring, $k = \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of characteristic p and the field of fractions \mathcal{K} of \mathcal{O} is of characteristic zero and is a splitting field for all subgroups of G).

Let N be a normal subgroup of G and let γ be a block (a primitive) idempotent of $Z(\mathcal{O}N)$. Set $H = \text{Stab}_G(\gamma)$ so that $N \leq H \leq G$. Also let $Bl(\mathcal{O}H|\gamma)$ and $Bl(\mathcal{O}G|\gamma)$ denote the set of blocks of $\mathcal{O}H$ and $\mathcal{O}G$ that cover γ , resp. Then it is well-known that if $b \in Bl(\mathcal{O}H|\gamma)$, then $b({}^g b) = 0$ for all $g \in G - H$ and the trace map from H to G , Tr_H^G , induces a bijection $\text{Tr}_H^G: Bl(\mathcal{O}H|\gamma) \rightarrow Bl(\mathcal{O}G|\gamma)$ such that corresponding blocks are “equivalent.” This basic analysis pioneered by P. Fong and W. Reynolds (cf. [5, V, Theorem 2.5]) is the first step in the fundamental Clifford theoretic approach to general block theory: the reduction to the case of a stable block of a normal subgroup.

Consider the more general situation: (P) H is a subgroup of G and e is an idempotent of $Z(\mathcal{O}H)$ is such that $e({}^g e) = 0$ for all $g \in G - H$.

Note that if β is an idempotent of $Z(\mathcal{O}H)$ such that $e\beta = \beta$, then $\beta({}^g \beta) = 0$ for all $g \in G - H$.

Fundamental contributions to this context appear in [9, Theorem 1] and in [11, Theorem 1].

The purpose of this paper is to put the significant results of [9, Theorem 1] and [11, Theorem 1] into current categorical and character theoretic equivalences context and to extend these basic results in this context.

It is also well-known that if H is a subgroup of G and if $\chi \in \text{Irr}_{\mathcal{K}}(H)$ is such that $\text{Ind}_H^G(\chi) \in \text{Irr}_{\mathcal{K}}(G)$ and if $e_\chi = (\chi(1)/|H|)(\sum_{h \in H} \chi(h^{-1})h)$ denotes the primitive idempotent of $Z(\mathcal{K}H)$ associated to χ , then $e_\chi({}^g e_\chi) = 0$ for all $g \in G - H$ and $\text{Tr}_H^G(e_\chi)$ is the primitive idempotent of $Z(\mathcal{K}G)$ associated to $\text{Ind}_H^G(\chi)$ (cf. Corollary 1.5).

In this article, we shall generally follow the (standard) notation and terminology of [5] and [10].

All rings have identities and are Noetherian and all modules over a ring are unitary and finitely generated left modules. If R is a ring, then $R\text{-mod}$ will denote the category of left R -modules and R^0 denotes the ring opposite to R .

The required proofs of the following main results will be presented in Section 3. Section 2 contains basic results that are needed in our proofs. We shall assume that H is a subgroup of the finite group G in the remainder of this section and we shall let T be a left transversal of H in G with $1 \in T$. Thus $G = \bigcup_{t \in T} tH$ is disjoint.

For our first three results, \mathcal{O} will denote a commutative Noetherian ring.

Our first two results are well-known and easy to prove (cf. [10, Sections 9 and 16]).

Lemma 1.1. *Let B be a unitary \mathcal{O} -algebra that is an interior H -algebra (as in [10, Section 16]). Then:*

(a)

$$\begin{aligned} \text{Ind}_H^G(B) &= \mathcal{O}G \otimes_{\mathcal{O}H} B \otimes_{\mathcal{O}H} \mathcal{O}G = \bigoplus_{s,t \in T} (s(\mathcal{O}H) \otimes_{\mathcal{O}H} B \otimes_{\mathcal{O}H} (\mathcal{O}H)t^{-1}) \\ &\cong \bigoplus_{s,t \in T} (s \otimes_{\mathcal{O}} B \otimes_{\otimes} t^{-1}) \end{aligned}$$

is a unitary interior G -algebra with $1_{\text{Ind}_H^G(B)} = \sum_{t \in T} (t \otimes_{\mathcal{O}} 1_B \otimes_{\mathcal{O}} t^{-1})$ and with $\phi: G \rightarrow \text{Ind}_H^G(B)^\times$ such that $g \mapsto \sum_{t \in T} (gt \otimes_{\mathcal{O}} 1_B \otimes_{\mathcal{O}} t^{-1})$ for all $g \in G$. Moreover $\{t \otimes_{\mathcal{O}} 1_B \otimes_{\mathcal{O}} t^{-1} \mid t \in T\}$ is a set of orthogonal idempotents of $\text{Ind}_H^G(B)$; and

(b) The map $\alpha: Z(B) \rightarrow Z(\text{Ind}_H^G(B))$ such that $z \mapsto \sum_{t \in T} (t \otimes_{\mathcal{O}} z \otimes_{\mathcal{O}} t^{-1})$ for all $z \in Z(B)$ is an \mathcal{O} -algebra isomorphism.

Proposition 1.2. *Let e be an idempotent of $Z(\mathcal{O}H)$ such that $e({}^g e) = 0$ for all $g \in G - H$ and set $E = \text{Tr}_H^G(e) = \sum_{t \in T} ({}^t e)$, so that E is an idempotent of $Z(\mathcal{O}G)$. Then:*

(a)

$$(\mathcal{O}G)E = (\mathcal{O}G)e(\mathcal{O}G), \quad e(\mathcal{O}G)e = e(\mathcal{O}G)Ee = (\mathcal{O}H)e$$

and the \mathcal{O} -linear map

$$f: \text{Ind}_H^G((\mathcal{O}H)e) \rightarrow (\mathcal{O}G)E$$

such that $x \otimes_{\mathcal{O}H} b \otimes_{\mathcal{O}H} y \mapsto xby$ for all $x, y \in G$ and all $b \in (\mathcal{O}H)e$ is an interior G -algebra isomorphism. Also the \mathcal{O} -linear map

$$\phi: Z((\mathcal{O}H)e) \rightarrow Z(\text{Ind}_H^G((\mathcal{O}H)e))$$

such that $z \mapsto \sum_{t \in T} (t \otimes_{\mathcal{O}} z \otimes_{\mathcal{O}} t^{-1})$ for all $z \in Z((\mathcal{O}H)e)$ is an \mathcal{O} -algebra isomorphism;

- (b) The inclusion map $\iota: (\mathcal{O}H)e \rightarrow (\mathcal{O}G)E$ is an embedding of interior H -algebras;
 (c) The functors

$$\text{Ind}_H^G(*) = (\mathcal{O}G)e \otimes_{(\mathcal{O}H)e} (*): (\mathcal{O}H)e\text{-mod} \rightarrow (\mathcal{O}G)E\text{-mod}$$

and

$$e \cdot \text{Res}_{\mathcal{O}H}^{\mathcal{O}G} (*) = e(\mathcal{O}G) \otimes_{(\mathcal{O}G)E} (*): (\mathcal{O}G)E\text{-mod} \rightarrow (\mathcal{O}H)e\text{-mod}$$

exhibit a Morita equivalence between the Abelian categories $(\mathcal{O}H)e\text{-mod}$ and $(\mathcal{O}G)E\text{-mod}$ with associated $((\mathcal{O}H)e, (\mathcal{O}G)E)$ -bimodule $e(\mathcal{O}G)$; and

- (d) Let M be an $(\mathcal{O}H)e$ -module. Then

$$\text{Ind}_H^G(M) = (\mathcal{O}G)e \otimes_{(\mathcal{O}H)e} M = \bigoplus_{t \in T} (t \otimes_{\mathcal{O}} M)$$

and

$$\alpha(g \otimes_{(\mathcal{O}H)e} m) = \begin{cases} 0 & \text{if } g \notin H \\ 1 \otimes_{\mathcal{O}} (\alpha g)m & \text{if } g \in H, \text{ for all } \alpha \in (\mathcal{O}H)e, \text{ all } m \in M \text{ and all } g \in G. \end{cases}$$

Let e be an idempotent of $Z(\mathcal{O}H)$.

REMARK 1.3. Let $g \in G$. The following three conditions are equivalent:

- (i) $e(\mathcal{O}(HgH))e = (0)$;
 (ii) $e({}^g e) = 0$; and
 (iii) $e(\mathcal{O}(HgH) \otimes_{\mathcal{O}H} V) = (0)$ for any module V in $(\mathcal{O}H)e\text{-mod}$.

Indeed, it is clear that (i) implies (ii) and (iii). Let $h_1, h_2 \in H$. Then $e(h_1gh_2)e = h_1e({}^g e)gh_2$, so that (ii) implies (i). Also if $V = (\mathcal{O}H)e$ in (iii), then

$$e(\mathcal{O}(HgH) \otimes_{\mathcal{O}H} (\mathcal{O}H)e) \cong e(\mathcal{O}(HgH))e$$

in $(\mathcal{O}H)e\text{-mod}$ and so (iii) implies (i).

Lemma 1.4 (E.C. Dade [4]). *Let \mathcal{K} be a field and let e be an idempotent in $Z(\mathcal{K}H)$. Suppose that*

$$\dim(\mathrm{Hom}_{\mathcal{K}G}(\mathrm{Ind}_H^G(X), \mathrm{Ind}_H^G(Y))/\mathcal{K}) = \dim(\mathrm{Hom}_{\mathcal{K}H}(X, Y)/\mathcal{K})$$

for any irreducible modules X, Y in $(\mathcal{K}H)e$ -mod. Then $e({}^g e) = 0$ for all $g \in G - H$.

An immediate implication of Lemma 1.4 is:

Corollary 1.5. *Assume that \mathcal{K} is a splitting field for G and H and that e is an idempotent of $Z(\mathcal{K}H)$ such that Ind_H^G defines an injective map $\mathrm{Irr}_{\mathcal{K}}(e) \rightarrow \mathrm{Irr}_{\mathcal{K}}(G)$. Then $e({}^g e) = 0$ for all $g \in G - H$.*

For the remainder of this section, we assume that $(\mathcal{O}, \mathcal{K}, k)$ is a p -modular system that is “large enough” for all subgroups of G . As is standard, the natural ring epimorphism $- : \mathcal{O} \rightarrow k = \mathcal{O}/J(\mathcal{O})$ induces an epimorphism on all \mathcal{O} -algebras that is also denoted by $-$. Similarly for \mathcal{O} -modules.

Theorem 1.6 (cf. [5, V, Theorem 2.5], [9, Proposition 1] and [11, Theorem 1]). *Assume that $b \in \mathrm{Bl}(\mathcal{O}H)$ is such that $b({}^g b) = 0$ for all $g \in G - H$ (as in Proposition 1.2) and let D be a defect group of b in H . Then:*

- (a) *Proposition 1.2 applies (with $R = \mathcal{O}$), $B = \mathrm{Tr}_H^G(b) \in \mathrm{Bl}(\mathcal{O}G)$ and D is a defect group of B in G ;*
- (b) *The functors $\mathrm{Ind}_H^G(*) = (\mathcal{O}G) \otimes_{\mathcal{O}H} (*) = (\mathcal{O}G)b \otimes_{(\mathcal{O}H)b} (*)$:*

$$(\mathcal{O}H)b\text{-mod} \rightarrow (\mathcal{O}G)B\text{-mod} \quad \text{and} \quad b \cdot \mathrm{Res}_H^G(*) : (\mathcal{O}G)B\text{-mod} \rightarrow (\mathcal{O}H)b\text{-mod}$$

exhibit a Morita equivalence between the Abelian categories $(\mathcal{O}H)b\text{-mod}$ and $(\mathcal{O}G)B\text{-mod}$. On the character level, this Morita equivalence induces the bijections:

$$\mathrm{Ind}_H^G : \mathrm{Irr}_{\mathcal{K}}(b) \rightarrow \mathrm{Irr}_{\mathcal{K}}(B), \quad \mathrm{Ind}_H^G : \mathrm{Irr}_k(b) \rightarrow \mathrm{Irr}_k(B)$$

and

$$\mathrm{Ind}_H^G : \mathrm{Irr} \mathrm{Br}_{\mathcal{K}}(b) \rightarrow \mathrm{Irr} \mathrm{Br}_{\mathcal{K}}(B).$$

Moreover, this Morita equivalence has associated bimodules:

$$(\mathcal{O}G)b \text{ in } (\mathcal{O}G)B\text{-mod}-(\mathcal{O}H)b \text{ and } b(\mathcal{O}G) \text{ in } (\mathcal{O}H)b\text{-mod}-(\mathcal{O}G)B.$$

Here $(\mathcal{O}G)b$ when viewed as an $\mathcal{O}(G \times H)$ -module is indecomposable with $\Delta D = \{(d, d) \mid d \in D\}$ and trivial ΔD -source and a similar fact holds for $b(\mathcal{O}G)$;

- (c) Let M be an indecomposable $(\mathcal{O}H)b$ -module with vertex Q and Q -source V . Then $\text{Ind}_H^G(M) = \mathcal{O}G \otimes_{\mathcal{O}H} M = (\mathcal{O}G)b \otimes_{(\mathcal{O}H)b} M$ in $(\mathcal{O}G)B\text{-mod}$ is an indecomposable $(\mathcal{O}G)$ -module with vertex Q and Q -source V ;
- (d) The above conditions hold over k for $\bar{b} \in \text{Bl}(kH)$ and $\bar{B} = \text{Tr}_H^G(\bar{b}) \in \text{Bl}(kG)$, etc;
- (e) The inclusion map $i: (\mathcal{O}H)b \rightarrow (\mathcal{O}G)B$ is an embedding of interior H -algebras so that i induces injective maps ([10, Proposition 15.1])

$$i_*: \mathcal{P}\mathcal{G}((\mathcal{O}H)b) \rightarrow \mathcal{P}\mathcal{G}((\mathcal{O}G)B) \quad \text{and} \quad i_*: \mathcal{L}\mathcal{P}\mathcal{G}((\mathcal{O}H)b) \rightarrow \mathcal{L}\mathcal{P}\mathcal{G}((\mathcal{O}G)B).$$

Let D_γ be a defect pointed group of $(\mathcal{O}H)b$ as an H -algebra. Thus $i_*(D_\gamma) = D_{i(\gamma)}$, where $i(\gamma) = \{\gamma^{((\mathcal{O}G)E)^*}\}$, is a defect pointed group of $(\mathcal{O}G)B$ as a G -algebra. Thus if $j \in \gamma$, then $j \in i(\gamma)$ and $j(\mathcal{O}G)Bj = jb(\mathcal{O}G)bj = j(\mathcal{O}H)bj$, so that these source algebras of b and B are equal as interior D -algebras; and

- (f) The Puig category of local pointed groups of b in $\mathcal{O}H$ and of B in $\mathcal{O}G$ are equivalent.

The next result illuminates the hypothesis of [11, Theorem 1].

Proposition 1.7. *Let b be a block idempotent of $Z(\mathcal{O}H)$. The following four conditions are equivalent:*

- (a) Ind_H^G induces an injective map of $\text{Irr}_k(\bar{b}) \rightarrow \text{Irr}_k(G)$;
- (b) Ind_H^G induces an injective map of $\text{Irr}_K(b) \rightarrow \text{Irr}_K(G)$; and
- (c) Ind_H^G induces an injective map of $\text{Irr } Br_K(b) \rightarrow \text{Irr } Br_K(G)$; and
- (d) $b({}^g b) = 0$ for all $g \in G - H$.

In which case, Theorem 1.6 applies so that $B = \text{Tr}_H^G(b) \in \text{Bl}(\mathcal{O}G)$, the functor

$$\text{Ind}_H^G = (\mathcal{O}G)b \otimes_{(\mathcal{O}H)b} (*): (\mathcal{O}H)b\text{-mod} \rightarrow (\mathcal{O}G)B\text{-mod}$$

induces a (Morita) categorical equivalence, the maps of (a), (b) and (c) are bijections, etc.

In our final result, (a), (b), (c) and (d) are presented in [9, Theorem 1] without proof. For the convenience of the reader, we shall include a proof of these items.

Theorem 1.8 (cf. [9, Theorem 1]). *Assume that $b \in \text{Bl}(\mathcal{O}H)$ is such that $b({}^g b) = 0$ for all $g \in G - H$ (as in Theorem 1.6). Set $\Omega = \{{}^g b \mid g \in G\}$ so that $B = (\sum_{\omega \in \Omega} \omega) \in \text{Bl}(\mathcal{O}G)$, etc.*

- (a) Let (P, \bar{b}_P) be a b -subpair of H . Then $\bar{b}_P({}^x \bar{b}_P) = 0$ for all $x \in C_G(P) - C_H(P)$, Theorem 1.6 (d) applies $s(\bar{b}_P) = \text{Tr}_{C_H(P)}^{C_G(P)}(\bar{b}_P) \in \text{Bl}(kC_G(P))$, $(P, s(\bar{b}_P))$ is a B -subpair of G and the k -linear map

$$\begin{aligned} \mu: \text{Ind}_{C_H(P)}^{C_G(P)}(kC_H(P)\bar{b}_P) &= kC_G(P) \otimes_{kC_H(P)} kC_H(P)\bar{b}_P \otimes_{kC_H(P)} kC_G(P) \\ &\rightarrow kC_G(P)s(\bar{b}_P) \end{aligned}$$

such that $x \otimes_{kC_H(P)} \alpha \otimes_{kC_H(P)} y \rightarrow x\alpha y$ for all $x, y \in C_G(P)$ and all $\alpha \in kC_H(P)\overline{b}_P$ is an interior $C_G(P)$ -algebra isomorphism. Also $\text{Ind}_H^G: \text{Irr}_k(\overline{b}_P) \rightarrow \text{Irr}_k(s(\overline{b}_P))$ is a bijection;

(b) The map $(P, \overline{b}_P) \mapsto (P, s(\overline{b}_P))$ from the set of b -subpairs of H into the set of B -subpairs of G is injective;

(c) Let (Q, \overline{b}_Q) and (P, \overline{b}_P) be b -subpairs of H . Then:

(i) $\{g \in G \mid {}^g(Q, s(\overline{b}_Q)) = (P, s(\overline{b}_P))\} = C_G(P)\{h \in H \mid {}^h(Q, \overline{b}_Q) = (P, \overline{b}_P)\}$ so that (Q, \overline{b}_Q) and (P, \overline{b}_P) are conjugate in H if and only if $(Q, s(\overline{b}_Q))$ and $(P, s(\overline{b}_P))$ are conjugate in G , and

(ii) $(Q, \overline{b}_Q) \leq (P, \overline{b}_P)$ in H if and only if $(Q, s(\overline{b}_Q)) \leq (P, s(\overline{b}_P))$ in G ;

(d) For any B -subpair $(P', \overline{B}_{P'})$ of G there is an $x \in G$ and a b -subpair (P, \overline{b}_P) of H such that ${}^x(P', \overline{B}_{P'}) = (P, s(\overline{b}_P))$; consequently the Brauer category of b in H is equivalent to the Brauer category of B in G ;

(e) Let (Q, \overline{b}_Q) be a b -subpair of H . The injective map $i_*: \mathcal{LPG}((\mathcal{O}H)b) \rightarrow \mathcal{LPG}((\mathcal{O}G)B)$ of Theorem 1.6 induces a bijection

$$i_*^{(Q, \overline{b}_Q)}: \{Q_\gamma \in \mathcal{LPG}((\mathcal{O}H)b) \mid Q_\gamma \text{ is associated with } (Q, \overline{b}_Q)\} \\ \rightarrow \{Q_\delta \in \mathcal{LPG}((\mathcal{O}G)B) \mid Q_\delta \text{ is associated with } (Q, s(\overline{b}_Q))\}$$

in which $Q_\gamma \mapsto Q_{i_*(\gamma)}$ for all $Q_\gamma \in \mathcal{LPG}((\mathcal{O}H)b)$ such that Q_γ is associated with (Q, \overline{b}_Q) ;

(f) Let (P, \overline{b}_P) be a b -subpair of H and let $(P, s(\overline{b}_P))$ be the corresponding B -subpair of G . Let b_P be the unique block idempotent of $Z(\mathcal{O}C_H(P))$ that “lifts” \overline{b}_P . Then $b_P({}^x b_P) = 0$ for all $x \in C_G(P) - C_H(P)$, $s(b_P) = \text{Tr}_{C_H(P)}^{C_G(P)}(b_P)$ is a block idempotent of $\mathcal{O}C_G(P)$ that “lifts” $s(\overline{b}_P)$ and Theorem 1.6 applies to $b_P \in \text{Bl}(\mathcal{O}C_H(P))$ where $C_H(P) \leq C_G(P)$; and

(g) Let (D, \overline{b}_D) be a maximal b -subpair of H . Let $P \leq D$ and let (P, \overline{b}_P) be the unique b -subpair of H such that $(P, \overline{b}_P) \leq (D, \overline{b}_D)$. Then

$$\text{Ind}_{C_H(P)}^{C_G(P)}(*): R_{\mathcal{K}}(C_H(P), b_P) \rightarrow R_{\mathcal{K}}(C_G(P), s(b_P))$$

is a perfect isometry and consequently induces the linear map

$$\text{Ind}_{C_H(P)}^{C_G(P)}(*)' : CF_{P'}(C_H(P), b_P, \mathcal{K}) \rightarrow CF_{P'}(C_G(P), s(b_P), \mathcal{K}).$$

Let $u \in D$ and set $P = \langle \mu \rangle$. Then

$$d_G^{(u, s(b_P))} \circ \text{Ind}_H^G(*) = \text{Ind}_{C_H(P)}^{C_G(P)}(*)' \circ d_H^{(u, b_P)} : CF(H, b, \mathcal{K}) \rightarrow CF_{P'}(C_G(P), s(b_P), \mathcal{K}).$$

Consequently the perfect isometry $\text{Ind}_H^G(*): R_{\mathcal{K}}(H, b) \rightarrow R_{\mathcal{K}}(G, B)$ is part of an isotopy between b and B with local system the family $\{\text{Ind}_{C_H(P)}^{C_G(P)}(*) \mid P \leq D, P \text{ cyclic}\}$.

REMARK 1.9. In the situation of Theorem 1.8 and after Theorem 1.6 (a) has been established, the more general investigations of [6] apply (cf. [6, Remark 1.3 (a)]).

2. Preliminary results

Let G be a finite group and let $(\mathcal{O}, \mathcal{K}, k = \mathcal{O}/J(\mathcal{O}))$ be a p -modular system that is “large enough” for all subgroups of G . We shall, as in [3], set $CF_{p'}(G, \mathcal{K}) = \{f \in CF(G, \mathcal{K}) \mid f(G - G_{p'}) = (0)\}$.

Let u be a p -element of G and set $P = \langle u \rangle$. Let $\chi \in \text{Irr}_{\mathcal{K}}(G)$ and let $\phi \in \text{Irr } Br_{\mathcal{K}}(C_G(P)) \subseteq CF_{p'}(C_G(P), \mathcal{K})$. We shall let $d_u(\chi, \phi)$ denote the generalized decomposition number associated to $u \in G_p$, $\chi \in \text{Irr}_{\mathcal{K}}(G)$ and $\phi \in \text{Irr } Br_{\mathcal{K}}(C_G(P))$, cf. [5, IV, Section 6]. Thus $d_G^u(\chi)(*) \in CF_{p'}(C_G(P), \mathcal{K})$ where $d_G^u(\chi)(s) = \chi(us) = \sum_{\phi \in \text{Irr } Br_{\mathcal{K}}(C_G(P))} d_u(\chi, \phi)\phi(s)$ for all $s \in C_G(P)_{p'}$. Moreover, as in [3, Section 4A], if $b \in \text{Bl}(\mathcal{O}G)$ and $b_P \in \text{Bl}(\mathcal{O}C_G(P))$, then $d_G^{(u, b_P)}: CF(G, b, \mathcal{K}) \rightarrow CF_{p'}(C_G(P), b_P, \mathcal{K})$ is defined by: if $\alpha \in CF(G, b, \mathcal{K})$ and $s \in C_G(P)_{p'}$, then $(d_G^{(u, b_P)}(\alpha))(s) = (b_P \cdot d_G^u(\alpha))(s) = \alpha(usb_P)$.

Since $\text{Irr}_{\mathcal{K}}(b)$ is a basis of $CF(G, b, \mathcal{K})$, the \mathcal{K} -linear map $d_G^{(u, b_P)}$ is characterized by the well-known:

Lemma 2.1. *Let $\chi \in \text{Irr}_{\mathcal{K}}(b)$. If $Br_P(b)\bar{b}_P = 0$, then $d_G^{(u, b_P)}(\chi) = 0$. If $Br_P(b)\bar{b}_P = \bar{b}_P$, then $d_G^{(u, b_P)}(\chi) = \sum_{\phi \in \text{Irr } Br_{\mathcal{K}}(b_P)} d_u(\chi, \phi)\phi$*

Proof. With the notation and hypotheses of this lemma, the first statement is a consequence of Brauer’s Second Main Theorem on Blocks ([5, IV, Theorem 6.1]) and the second statement is a consequence of [2, Theorem A2.1]. □

REMARK 2.2. As above, if $\phi \in \text{Irr } Br_{\mathcal{K}}(C_G(P))$ corresponds to $\gamma \in \mathcal{LP}((\mathcal{O}G)^P)$ (i.e., ϕ is the irreducible Brauer character obtained from the irreducible $kC_G(P)$ -module $kC_G(P)Br_P(j)/J(kC_G(P)Br_P(j))$ for any $j \in \gamma$), then, by [10, Theorem 43.4] $d_u(\chi, \phi) = \chi(uj)$ for any $j \in \gamma$.

3. Proofs

As noted above, Lemma 1.1 and Proposition 1.2 are well-known and easy to prove.

Proof of Lemma 1.4. Assume the hypotheses of Lemma 1.4. Let S be a set of double (H, H) -coset representatives in G such that $1 \in S$ and let X, Y be irreducible modules in $(\mathcal{K}H)e$ -mod. Here

$$\begin{aligned} \text{Hom}_{\mathcal{K}G}(\text{Ind}_H^G(X), \text{Ind}_H^G(Y)) &\cong \text{Hom}_{\mathcal{K}H} \left(X, \bigoplus_{s \in S} (\mathcal{K}(HsH) \otimes_{\mathcal{K}H} Y) \right) \\ &\cong \bigoplus_{s \in S} \text{Hom}_{\mathcal{K}H}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}H} Y) \end{aligned}$$

in \mathcal{K} -mod. Thus $\text{Hom}_{\mathcal{K}H}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}H} Y) = (0)$ for all $1 \neq s \in S$.

Fix $1 \neq s \in S$ and an irreducible module X in $(\mathcal{K}H)e$ -mod.

We assert: (*) $\text{Hom}_{\mathcal{K}H}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}H} V) = (0)$ for all V in $(\mathcal{K}H)e$ -mod.

Indeed, we may assume that V is reducible in $(\mathcal{K}H)e$ -mod and we proceed by induction on $\dim(V/\mathcal{K})$. Let V_1 be a maximal submodule of V . Then

$$(0) \rightarrow V_1 \rightarrow V \rightarrow V/V_1 \rightarrow (0)$$

is a short exact sequence in $(\mathcal{K}H)e$ -mod. Thus, since $\mathcal{K}(HsH)|(\mathcal{K}G)$ in $\mathcal{K}H$ -mod- $\mathcal{K}H$,

$$(0) \rightarrow \mathcal{K}(HsH) \otimes_{\mathcal{K}H} V_1 \rightarrow \mathcal{K}(HsH) \otimes_{\mathcal{K}H} V \rightarrow \mathcal{K}(HsH) \otimes_{\mathcal{K}H} (V/V_1) \rightarrow (0)$$

is a short exact sequence in $\mathcal{K}H$ -mod. Consequently

$$\begin{aligned} \text{Hom}_{\mathcal{K}H}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}H} V_1) \\ \rightarrow \text{Hom}_{\mathcal{K}H}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}H} V) \\ \rightarrow \text{Hom}_{\mathcal{K}H}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}H} (V/V_1)) \end{aligned}$$

is exact in \mathcal{K} -mod and we conclude from the induction hypothesis that

$$\text{Hom}_{\mathcal{K}H}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}H} V) = (0).$$

This establishes (*).

Since X can be any irreducible $(\mathcal{K}H)e$ -module, (*) implies that $\text{Soc}(e\mathcal{K}(HsH) \otimes_{\mathcal{K}H} V) = (0)$ for any module V in $(\mathcal{K}H)e$ -mod. Thus $e\mathcal{K}(HsH)e \otimes_{\mathcal{K}H} V = (0)$ for any module V in $(\mathcal{K}H)e$ -mod and we are done. \square

Proof of Theorem 1.6. Assume the hypotheses of Theorem 1.6. Applying Proposition 1.2, [5, V, Lemma 1.2] and [5, III, Lemma 9.6], D is contained in a defect group of $B \in \text{Bl}(\mathcal{O}G)$. Since $\text{Ind}_H^G: \text{Irr}_{\mathcal{K}}(b) \rightarrow \text{Irr}_{\mathcal{K}}(B)$ is bijective, [5, IV, Theorem 4.5] and degree considerations complete a proof of (a). Clearly $(\mathcal{O}G)b$ is indecomposable in $\mathcal{O}(G \times H)(B \otimes_{\mathcal{O}} b^0)$ -mod and $D \times D$ is a defect group of $B \otimes_{\mathcal{O}} b^0 \in \text{Bl}(\mathcal{O}(G \times H))$. Also $(\mathcal{O}H)b|_{\text{Res}_{H \times H}^{G \times H}}((\mathcal{O}G)b)$ in $\mathcal{O}(H \times H)$ -mod and $(\mathcal{O}H)b$ is indecomposable in $\mathcal{O}(H \times H)$ -mod with ΔD as a vertex and trivial ΔD -source. Then [5, III, Lemma 4.6 (ii) and Corollary 6.8] implies the last part of (b). Thus (b) holds, [5, III, Corollary 4.7] yields (c) and (d) and (e) are clear. Finally (e) and [5, Theorem 47.10 (b)] yield (f). \square

Proof of Proposition 1.7. Assume the situation of this proposition. Let V and W be irreducible $(kH)\bar{b}$ -modules with irreducible characters ϕ_V, ϕ_W in $\text{Irr}_k(\bar{b})$ and irreducible Brauer characters β_V, β_W in $\text{Irr } Br_{\mathcal{K}}(b)$.

Assume that (a) holds. Then $\text{Ind}_H^G(V)$ is an irreducible kG -module and $\text{Ind}_H^G(\beta_V) = \beta_{\text{Ind}_H^G(V)} \in \text{Irr } Br_{\mathcal{K}}(G)$. Similarly $\text{Ind}_H^G(W)$ is an irreducible kG -module and $\text{Ind}_H^G(\beta_W) = \beta_{\text{Ind}_H^G(W)} \in \text{Irr } Br_{\mathcal{K}}(G)$. Suppose that $\beta_{\text{Ind}_H^G(V)} = \beta_{\text{Ind}_H^G(W)}$. Then

$$\phi_{\text{Ind}_H^G(V)} = \overline{\beta_{\text{Ind}_H^G(V)}} = \overline{\beta_{\text{Ind}_H^G(W)}} = \phi_{\text{Ind}_H^G(W)}, \quad \text{Ind}_H^G(V) \cong \text{Ind}_H^G(W) \quad \text{in } kG\text{-mod}$$

and hence $\text{Ind}_H^G(\phi_V) = \text{Ind}_H^G(\phi_W)$. But then $\phi_V = \phi_W$, $V \cong W$ in $(kH)\bar{b}$ -mod and $\beta_V = \beta_W$, so that (c) follows.

Assume that (c) holds. Then $\bar{\beta}_V = \phi_V \in \text{Irr}_k(\bar{b})$ and $\text{Ind}_H^G(\beta_V) = \beta_{\text{Ind}_H^G(V)} \in \text{Irr } Br_{\mathcal{K}}(G)$. Thus $\text{Ind}_H^G(\phi_V) = \phi_{\text{Ind}_H^G(V)} \in \text{Irr}_{\mathcal{K}}(G)$. Similarly $\beta_{\text{Ind}_H^G(W)} \in \text{Irr } Br_{\mathcal{K}}(G)$ and $\text{Ind}_H^G(\phi_W) = \phi_{\text{Ind}_H^G(W)} \in \text{Irr}_k(G)$. Suppose that $\text{Ind}_H^G(\phi_V) = \text{Ind}_H^G(\phi_W)$. Then $\beta_{\text{Ind}_H^G(V)} = \beta_{\text{Ind}_H^G(W)} = \text{Ind}_H^G(\beta_V) = \text{Ind}_H^G(\beta_W)$, so that $\beta_V = \beta_W$, $\phi_V = \phi_W$ and (a) holds. Consequently (a) and (c) are equivalent.

That (d) implies (a), (b) and (c) is a consequence of Theorem 1.6 (b). Assume (a) and let $g \in G - H$. Then $M = b\mathcal{O}(HgH)b$ is an \mathcal{O} -lattice where $\bar{M} = \bar{b}k(HgH)\bar{b} = (0)$ by Corollary 1.5. Consequently $b\mathcal{O}(HgH)b = (0)$ and (d) holds.

Assume (b) and for each $\chi \in \text{Irr}_{\mathcal{K}}(b)$, let $e_\chi = (\chi(1)/|H|)(\sum_{h \in H} \chi(h^{-1})h)$ be the primitive idempotent of $Z(\mathcal{K}H)$ corresponding to χ . Then $e_\chi({}^g e_\chi) = 0$ for all $g \in G - H$ by Corollary 1.5 and $\text{Tr}_H^G(e_\chi) = e_{\text{Ind}_H^G(\chi)}$ is the primitive idempotent of $Z(\mathcal{K}G)$ corresponding to $\text{Ind}_H^G(\chi)$. Let $\chi, \psi \in \text{Irr}_{\mathcal{K}}(b)$ and let $g \in G - H$. Then $e_\chi({}^g e_\psi) = (e_\chi \text{Tr}_H^G(e_\chi))(\text{Tr}_H^G(e_\chi)^g e_\psi) = 0$, (d) holds and our proof is complete. \square

Proof of Theorem 1.8. For (a), note that $Br_P(b)\bar{b}_P = \bar{b}_P$. Let $x \in C_G(P) - C_H(P)$; then

$$\bar{b}_P({}^x \bar{b}_P) = \bar{b}_P Br_P(b) Br_P({}^x b)({}^x \bar{b}_P) = \bar{b}_P Br_P(b({}^x b))({}^x \bar{b}_P) = 0.$$

Thus $\text{Stab}_{C_G(P)}(\bar{b}_P) = C_H(P)$ and, since $Br_P(B)\bar{b}_P = \bar{b}_P$, we conclude that

$$Br_P(B) \text{Tr}_{C_H(P)}^{C_G(P)}(\bar{b}_P) = \text{Tr}_{C_H(P)}^{C_G(P)}(\bar{b}_P).$$

Then Proposition 1.2 and Theorem 1.6 yield (a). Since $Br_P(b)s(\bar{b}_P) = \bar{b}_P$, (b) holds.

Let (Q, \bar{b}_Q) and (P, \bar{b}_P) be b -subpairs of H and let S be a left transversal of $C_H(Q)$ in $C_G(Q)$ with $1 \in S$, so that $C_G(Q) = \bigcup_{s \in S} sC_H(Q)$ is disjoint. Let $h \in H$ be such that ${}^h(Q, \bar{b}_Q) = (P, \bar{b}_P)$. Then

$${}^h s(\bar{b}_Q) = \sum_{s \in S} (hs)\bar{b}_Q = \sum_{s \in S} (hsh^{-1})(h\bar{b}_Q) = s(\bar{b}_P)$$

and hence

$$C_G(P)\{h \in H \mid {}^h(Q, \bar{b}_Q) = (P, \bar{b}_P)\} \leq \{g \in G \mid {}^g(Q, s(\bar{b}_Q)) = (P, s(\bar{b}_P))\}.$$

Conversely, let $g \in G$ be such that ${}^g(Q, s(\bar{b}_Q)) = (P, s(\bar{b}_P))$. Then ${}^gB = B$ and ${}^g\Omega = \Omega$. Let U be a left transversal of $C_H(P)$ in $C_G(P)$ with $1 \in U$, so that $C_G(P) = \bigcup_{u \in U} uC_H(P)$ is disjoint. Here ${}^gs(\bar{b}_Q) = s(\bar{b}_P) = \sum_{u \in U} Br_P(u)b({}^u\bar{b}_P)$, $Br_Q(b)s(\bar{b}_Q) = \bar{b}_Q$ and $Br_P(u)b({}^u\bar{b}_P) = {}^u\bar{b}_P$ for all $u \in U$. Thus

$$0 \neq {}^g\bar{b}_Q = Br_P({}^gb)s(\bar{b}_P) = \sum_{u \in U} Br_P({}^gb)({}^u\bar{b}_P).$$

We conclude that ${}^gb = {}^bu$ for some $u \in U$ and so $g = uh$ for some $h \in H$. But then ${}^g(Q, s(\bar{b}_Q)) = {}^u({}^hQ, {}^hs(\bar{b}_Q)) = (P, s(\bar{b}_P))$ and $({}^hQ, {}^hs(\bar{b}_Q)) = (P, s(\bar{b}_P))$. Since $Br_Q(b)s(\bar{b}_Q) = \bar{b}_Q$, we have $Br_P(b)s(\bar{b}_P) = {}^h\bar{b}_Q$ and then ${}^h\bar{b}_Q = \bar{b}_P$, which completes a proof of (c) (i).

For a proof of (c) (ii), it suffices to assume that $Q \trianglelefteq P$. First suppose that $(Q, \bar{b}_Q) \leq (P, \bar{b}_P)$. Thus \bar{b}_Q is P -stable and $Br_P(b_Q)\bar{b}_P = \bar{b}_P$. As $P \leq N_G(Q)$, we conclude that P stabilizes $s(\bar{b}_Q)$. Let U be a left transversal of $C_H(P)$ in $C_G(P)$ with $1 \in U$. Here

$$Br_P(s(\bar{b}_Q))\bar{b}_P = Br_P(s(\bar{b}_Q)) Br_P(\bar{b}_Q)\bar{b}_P = Br_P(\bar{b}_Q)\bar{b}_P = \bar{b}_P$$

and, since $C_G(P) \leq C_G(Q)$, we have $Br_P(s(\bar{b}_Q)){}^u\bar{b}_P = {}^u\bar{b}_P$ for all $u \in U$. Thus $Br_P(s(\bar{b}_Q))s(\bar{b}_P) = s(\bar{b}_P)$. Conversely, suppose that $(Q, s(\bar{b}_Q)) \leq (P, s(\bar{b}_P))$. Then $s(\bar{b}_Q) \in (kC_G(Q))^P$ and $Br_P(s(\bar{b}_Q))s(\bar{b}_P) = s(\bar{b}_P)$. Utilizing [10, Lemma 40.2],

$$\begin{aligned} s(\bar{b}_P) Br_P(b) &= \bar{b}_P = Br_P(s(\bar{b}_Q))s(\bar{b}_P) Br_P(b) \\ &= Br_{P/Q}(s(b_Q) Br_Q(b))\bar{b}_P = Br_{P/Q}(\bar{b}_Q)\bar{b}_P = Br_P(\bar{b}_Q)\bar{b}_P. \end{aligned}$$

Since $s(\bar{b}_Q) Br_Q(b) = \bar{b}_Q$, \bar{b}_Q is P -stable and so $(Q, \bar{b}_Q) \leq (P, \bar{b}_P)$ which completes a proof of (c) (ii).

Let $(P', \bar{B}_{P'})$ be a B -subpair of G . Let (D, b_D) be a maximal b -subpair of H ; thus $(D, s(\bar{b}_D))$ is a maximal B -subpair of G . Then there is an $x \in G$ such that ${}^x(P', \bar{B}_{P'}) \leq (D, s(\bar{b}_D))$. Thus $({}^xP', {}^x\bar{B}_{P'}) \leq (D, s(\bar{b}_D))$ and setting $Q = {}^xP'$, we have $(Q, \bar{b}_Q) \leq (D, \bar{b}_D)$ for a unique $\bar{b}_Q \in Bl(kC_H(Q))$. But then $(Q, s(\bar{b}_Q)) \leq (D, s(\bar{b}_D))$; consequently ${}^x\bar{B}_{P'} = s(\bar{b}_Q)$ and ${}^x(P', \bar{B}_{P'}) = (Q, s(\bar{b}_Q))$, which completes a proof of (d).

For (e), let (Q, \bar{b}_Q) be a b -subpair of H . By (a), $kC_H(Q)\bar{b}_Q$ -mod and $kC_G(Q)\bar{B}_Q$ -mod are Morita equivalent. Thus $|\mathcal{P}(kC_H(Q)\bar{b}_Q)| = |\mathcal{P}(kC_G(Q)\bar{B}_Q)|$. Clearly

$$|\{Q_\gamma \in \mathcal{LPG}((OH)b) \mid Q_\gamma \text{ is associated with } (Q, \bar{b}_Q)\}| = |\mathcal{P}(kC_H(Q)\bar{b}_Q)|$$

and

$$|\{Q_\delta \in \mathcal{LPG}((OG)B) \mid Q_\delta \text{ is associated with } (Q, s(\bar{b}_Q))\}| = |\mathcal{P}(kC_G(Q)s(\bar{b}_Q))|.$$

Also if $Q_\gamma \in \mathcal{LPG}((\mathcal{O}H)b)$ and Q_γ is associated with (Q, \bar{b}_Q) and $j \in \gamma$, then

$$Br_Q(j)\bar{b}_Q = Br_Q(j) = Br_Q(j)\bar{b}_Q s(\bar{b}_Q) = Br_Q(j)s(\bar{b}_Q).$$

Thus $\iota_*(Q_\gamma) \in \mathcal{LPG}((\mathcal{O}G)B)$ and $i_*(Q_\gamma)$ is associated with $(Q, s(\bar{b}_Q))$. The desired conclusion now follows from Theorem 1.6 (e).

Let (P, \bar{b}_P) , $(P, s(\bar{b}_P))$ and b_P be as in (f). Note that $\text{Ind}_H^G: \text{Irr}_k(\bar{b}_P) \rightarrow \text{Irr}_k(s(\bar{b}_P))$ is a bijection by (a). Then Proposition 1.2 and Theorem 1.6 yield (f).

Let (D, \bar{b}_D) and (P, b_P) be as in (g). Then Theorem 1.6 (b) and [3, Proposition 1.2] imply that $\text{Ind}_{C_H(P)}^{C_G(P)}(*): \mathcal{RK}(C_H(P), b_P) \rightarrow \mathcal{RK}(C_G(P), s(b_P))$ is a perfect isometry that induces the linear map

$$\text{Ind}_{C_H(P)}^{C_G(P)}(*)' : CF_{p'}(C_H(P), b_P, \mathcal{K}) \rightarrow CF_{p'}(C_G(P), s(b_P), \mathcal{K}).$$

Let $u \in D$, set $P = \langle u \rangle$ and let $\psi \in \text{Irr}_{\mathcal{K}}(b)$. Then, by Lemma 2.1,

$$\text{Ind}_{C_H(P)}^{C_G(P)}(d^H(u, b_P)(\psi)) = \sum_{\phi \in \text{Irr } Br_{\mathcal{K}}(b_P)} d_u(\psi, \phi)(\text{Ind}_{C_H(P)}^{C_G(P)}(\phi))$$

and

$$d_G^{(u, s(b_P))}(\text{Ind}_H^G(\psi)) = \sum_{\phi \in \text{Irr } Br_{\mathcal{K}}(b_P)} (d_u(\text{Ind}_H^G(\psi), \text{Ind}_{C_H(P)}^{C_G(P)}(\phi)) \text{Ind}_{C_H(P)}^{C_G(P)}(\phi)).$$

The desired conclusion now follows from [11, Theorem 1 (iv)]. An alternate proof can be obtained from [10, Theorem 43.4]. Indeed, let $\phi \in \text{Irr } Br_{\mathcal{K}}(b_P)$ and let $\gamma \in \mathcal{LP}(((\mathcal{O}H)b)^P)$ correspond as in Remark 2.2. It is easy to see that $\text{Ind}_{C_H(P)}^{C_G(P)}(\phi) \in \text{Irr } Br_{\mathcal{K}}(s(b_P))$ corresponds $i(\gamma) \in \mathcal{LP}((\mathcal{O}G)B)^P$. Let $j \in \gamma$. Here Proposition 1.2 (d) implies that $\text{Ind}_H^G(\psi)(uj) = \psi(uj)$ and the desired conclusion follows from Remark 2.2. □

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Department of Mathematics
Statistics and Computer Science (M/C 249)
University of Illinois at Chicago
851 South Morgan Street
Chicago, IL 60607–7045
USA
e-mail: harris@math.uic.edu