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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE VISCOUS BURGERS EQUATION

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Abstract

We study the asymptotic behavior of solutions to the viscous Burgers equation by presenting a new asymptotic approximate solution. This approximate solution, called a diffusion wave approximate solution to the viscous Burgers equation of k-th order, is expanded in terms of the initial moments up to k-th order. Moreover, the spatial and time shifts are introduced into the leading order term to capture precisely the effect of the initial data on the long-time behavior of the actual solution. We also show the optimal convergence order in L^p-norm, $1 \leq p \leq \infty$, of the diffusion wave approximate solution of k-th order. These results allow us to obtain the convergence of any higher order in L^p-norm by taking such a diffusion wave approximate solution with order k large enough.

1. Introduction

We consider the viscous Burgers equation

$$u_t + uu_x = u_{xx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

with the initial condition

$$u(x, 0) = u_0(x) \quad \text{on} \quad \mathbb{R}. \tag{1.2}$$

Here $u = u(x, t)$ is an unknown function; the coefficient of viscosity is assumed to be 1, for simplicity; $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t > 0\}$. We assume that the initial data u_0 satisfies that, for a nonnegative integer $k \in \mathbb{N}_0$ ($\equiv \mathbb{N} \cup \{0\}$) and a small positive constant ϵ,

$$(1 + |x|)^{k+3+\epsilon} u_0 \in L^1(\mathbb{R}). \tag{1.3}$$

We introduce Cole-Hopf transformation of the solution u to (1.1)–(1.2) which is given by

$$H[u(t)](x) = \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u(y, t)\, dy\right) - 1, \tag{1.4}$$

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t being regarded as a parameter. Then we put $\phi(x,t) = (d/dx)H[u(t)](x)$. It is well-known that (1.1)–(1.2) is converted into the heat equation of ϕ

$$\phi_t = \phi_{xx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \tag{1.5}$$

with the initial condition

$$\begin{aligned} \phi(x,0) &= H[u_0]'(x) \\ &= -\frac{1}{2}u_0(x)\exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_0(y)\,dy\right) \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{1.6}$$

We notice that the condition (1.3) implies

$$(1+|x|)^{k+3+\epsilon} H[u_0]' \in L^1(\mathbb{R}), \tag{1.7}$$

thereby it is easy to see that there exists a unique smooth solution ϕ to (1.5)–(1.6). Hence, taking the inverse of Cole-Hopf transformation of ϕ, we obtain a unique global smooth solution to (1.1)–(1.2) (see (2.35) and (2.29)).

The purpose of this paper is to present a precise description of the long-time asymptotic behavior of the solution to the viscous Burgers equation (1.1) with the initial condition (1.2). For this purpose, we begin by introducing an asymptotic approximate solution to the heat equation (1.5) with (1.6) up to arbitrary finite order: First we define *the j-th moment of a function $f(x)$, $j \in \mathbb{N}_0$,* by

$$\mathcal{M}_j(f) = \int_{\mathbb{R}} x^j f(x)\,dx. \tag{1.8}$$

We note here that (1.7) ensures that $\mathcal{M}_j(H[u_0]') < \infty$ for $0 \le j \le k+2$. In what follows we shall assume, in addition, that

$$\mathcal{M}_k(H[u_0]') \neq 0. \tag{1.9}$$

We then introduce *an asymptotic approximate solution to* (1.5)–(1.6) *of k-th order* which is defined by

$$\begin{aligned} \phi^k(x,t) = &\sum_{j=0}^{k-1}(-1)^j \frac{\mathcal{M}_j(H[u_0]')}{j!}\left(\frac{\partial}{\partial x}\right)^j G_t(x) \\ &+ (-1)^k \frac{\mathcal{M}_k(H[u_0]')}{k!}\left(\frac{\partial}{\partial x}\right)^k G_{t+(t_k)_+}(x-\gamma_k) \quad \text{for} \quad t>0,\ x\in\mathbb{R}. \end{aligned} \tag{1.10}$$

Here $G_t(x)$ denotes 1-D heat kernel, i.e.,

$$G_t(x) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right). \tag{1.11}$$

Of course, in case that $k = 0$, RHS of (1.10) except for the last term should be dropped. The spatial shift γ_k and the time shift $(t_k)_+$, where $(t_k)_+ = \max(t_k, 0)$, appearing in the last term on RHS of (1.10) are specified as follows: The spatial shift γ_k is determined by the relation

$$\mathcal{M}_{k+1}(H[u_0]') - (k+1)\mathcal{M}_k(H[u_0]')\gamma_k = 0. \tag{1.12}$$

Then the time shift $(t_k)_+$ is given through t_k which is determined by the relation

$$\mathcal{M}_{k+2}(H[u_0]') - \frac{(k+2)(k+1)}{2}\mathcal{M}_k(H[u_0]')(\gamma_k^2 + 2t_k) = 0. \tag{1.13}$$

From (1.12), (1.13) we see that γ_k and $(t_k)_+$ are expressed as

$$\gamma_k = \frac{\mathcal{M}_{k+1}(H[u_0]')}{(k+1)\mathcal{M}_k(H[u_0]')}, \tag{1.14}$$

and

$$(t_k)_+ = \begin{cases} \dfrac{2(k+1)\mathcal{M}_{k+2}(H[u_0]')\mathcal{M}_k(H[u_0]') - (k+2)\{\mathcal{M}_{k+1}(H[u_0]')\}^2}{2(k+1)^2(k+2)\{\mathcal{M}_k(H[u_0]')\}^2} \\ \qquad \text{if} \quad \mathcal{M}_{k+2}(H[u_0]')\mathcal{M}_k(H[u_0]') > \dfrac{k+2}{2(k+1)}\{\mathcal{M}_{k+1}(H[u_0]')\}^2, \\ 0 \quad \text{if} \quad \mathcal{M}_{k+2}(H[u_0]')\mathcal{M}_k(H[u_0]') \le \dfrac{k+2}{2(k+1)}\{\mathcal{M}_{k+1}(H[u_0]')\}^2. \end{cases} \tag{1.15}$$

It should be remarked that the asymptotic approximate solution ϕ^k of k-th order defined by (1.10) can be also represented in terms of the Hermite polynomials as follows:

$$\begin{aligned} \phi^k(x,t) = {} & \sum_{j=0}^{k-1} \frac{\mathcal{M}_j(H[u_0]')}{j!}\left(\frac{1}{2\sqrt{t}}\right)^j H_j\left(\frac{x}{2\sqrt{t}}\right)\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right) \\ & + \frac{\mathcal{M}_k(H[u_0]')}{k!}\left(\frac{1}{2\sqrt{t+(t_k)_+}}\right)^k H_k\left(\frac{x-\gamma_k}{2\sqrt{t+(t_k)_+}}\right) \\ & \times \frac{1}{\sqrt{4\pi(t+(t_k)_+)}}\exp\left(-\frac{(x-\gamma_k)^2}{4(t+(t_k)_+)}\right). \end{aligned} \tag{1.16}$$

Here H_j is the j-th Hermite polynomial which is generated by the relation

$$H_j(\xi) = (-1)^j \exp(\xi^2)\frac{d^j}{d\xi^j}(\exp(-\xi^2)). \tag{1.17}$$

As seen from (1.10), the asymptotic approximate solution ϕ^k of k-th order consists of the j-th order spatial derivative of heat kernel with the strength given by the j-th

moments of $H[u_0]'$, $0 \le j \le k$. By taking the approximate solution like this form, we can make all the initial moments of the error term up to k-th order canceled. Furthermore, adjusting the center and width of the highest order derivative of heat kernel of ϕ^k by use of the shifts γ_k and $(t_k)_+$, we find that the $(k+1)$-th and $(k+2)$-th initial moments of the error term also vanish (see Remark 2.2 (i) in §2 and Lemma A.3 in Appendix). As a result, our asymptotic approximate solution ϕ^k controls the initial moments up to $k+2$-th order and this fact enables us to obtain the optimal decay rate of the error term in L^p-norm (see Proposition 2.1 in §2).

Once we obtain the asymptotic approximate solution ϕ^k to (1.5)–(1.6) of k-th order, by taking formally the inverse of Cole-Hopf transformation of ϕ^k, we finally reach *a diffusion wave approximate solution to* (1.1)–(1.2) *of k-th order* which is defined by

$$\chi^k(x,t) = -2\frac{\phi^k(x,t)}{1+\int_{-\infty}^{x}\phi^k(y,t)\,dy} \quad \text{for} \quad t>0,\ x\in\mathbb{R}. \tag{1.18}$$

The validity of taking the inverse of Cole-Hopf transformation of ϕ^k with $k \ge 1$ is ensured by the fact that the denominator in (1.18) is certainly uniformly positive after the time T_k which is determined only by k and the moments of $H[u_0]'$ up to k-th order (see Lemma 2.5 in §2).

Then, on the base of the optimal error estimates of the asymptotic approximate solution ϕ^k in L^p-norm, the following main theorem of this paper builds the precise convergence estimates of this diffusion wave approximate solution χ^k of k-th order.

Theorem 1.1. *Suppose that* $(1+|x|)^{k+3+\epsilon}u_0 \in L^1(\mathbb{R})$ *and* $\mathcal{M}_k(H[u_0]') \neq 0$ *for an integer* $k \ge 0$ *with* $\epsilon > 0$ *small. Let* u *be a solution to* (1.1)–(1.2) *and* χ^k *a diffusion wave approximate solution to* (1.1)–(1.2) *of k-th order defined by* (1.18). *Then, for any* $p \in [1,\infty]$:

1. *In case that* $k=0$, *the diffusion wave approximation solution* χ^0 *of* 0*-th order is well-defined on the interval* $(0,\infty)$ *and the following estimates hold for a constant* C_0:

$$\|u(t)-\chi^0(t)\|_{L^p(\mathbb{R})} \le C_0 t^{1/(2p)-2} \quad \textit{for} \quad t>0, \quad \textit{when} \quad (t_k)_+ > 0, \tag{1.19}$$

$$\|u(t)-\chi^0(t)\|_{L^p(\mathbb{R})} \le C_0 t^{1/(2p)-3/2} \quad \textit{for} \quad t>0, \quad \textit{when} \quad (t_k)_+ = 0. \tag{1.20}$$

Here C_0 *depends only on* $H[u_0]'$ *in* (1.6).

2. *In case that* $k \ge 1$, *there exists a constant* $T_k \ge 0$ *such that the diffusion wave approximate solution* χ^k *of k-th order is well-defined on the interval* (T_k,∞) *and the following estimates hold for a constant* C_1:

$$\|u(t)-\chi^k(t)\|_{L^p(\mathbb{R})} \le C_1 t^{1/(2p)-2-k/2} \quad \textit{for} \quad t>T_k, \quad \textit{when} \quad (t_k)_+ > 0, \tag{1.21}$$

$$\|u(t)-\chi^k(t)\|_{L^p(\mathbb{R})} \le C_1 t^{1/(2p)-3/2-k/2} \quad \textit{for} \quad t>T_k, \quad \textit{when} \quad (t_k)_+ = 0. \tag{1.22}$$

Here T_k *depends only on* k *and* $\mathcal{M}_j(H[u_0]')$, $1 \le j \le k$, *and* C_1 *depends only on* $H[u_0]'$.

REMARK 1.2. (i) It is easy to see that the condition $(1+|x|)^{k+3+\epsilon}u_0 \in L^1(\mathbb{R})$ can be replaced by a weaker condition $(1+|x|)^{k+2+\epsilon}u_0 \in L^1(\mathbb{R})$ when $(t_k)_+ = 0$. (See the proof of Lemma 2.3 in §2.)
(ii) We can readily observe from (1.4) that $\mathcal{M}_0(H[u_0]') = 0$ if and only if $\mathcal{M}_0(u_0) = 0$. In addition, if $\mathcal{M}_0(H[u_0]') = 0$, it is not hard to see that

$$\mathcal{M}_k(H[u_0]') = -\frac{1}{2}\mathcal{M}_k(u_0), \quad k \in \mathbb{N}.$$

Therefore, for any $k \in \mathbb{N}$, the condition that $\mathcal{M}_0(H[u_0]') = 0$ and $\mathcal{M}_k(H[u_0]') = 0$ is equivalent to the condition that $\mathcal{M}_0(u_0) = 0$ and $\mathcal{M}_k(u_0) = 0$.
(iii) Suppose that the initial data u_0 is a nontrivial continuous function with compact support in $\mathbb{R}$. Then, since $H[u_0]'$ also has the same property as supposed above, by virtue of well-known Hausdorff's moments theorem (see §6 in [1], for example), we find that there exists an integer $k_0 \in \mathbb{N}_0$ such that $\mathcal{M}_{k_0}(H[u_0]') \neq 0$. So, in this case we need not assume the condition (1.9).
(iv) We say that the moments of $f(x)$ degenerate up to l-th order, $l \in \mathbb{N}_0$, provided that

$$\mathcal{M}_j(f) = 0 \quad \text{for} \quad 0 \le j \le l, \quad \mathcal{M}_{l+1}(f) \neq 0. \tag{1.23}$$

Assume that the moments of $H[u_0]'$ degenerate up to $(k-1)$-th order, $k \in \mathbb{N}$. Then, it readily follows from the item (ii) above that

$$\mathcal{M}_j(H[u_0]') = 0 \quad \text{for} \quad 0 \le j \le k-1, \quad \mathcal{M}_k(H[u_0]') \neq 0. \tag{1.24}$$

We can now deduce from (1.24), (1.12) that the spatial shift γ_k is given by

$$\int_{\mathbb{R}} (x-\gamma_k)^{k+1} H[u_0]'(x)\,dx = 0, \tag{1.25}$$

which implies that the γ_k is regarded as a center of k-th moment of $H[u_0]'$. Similarly, we can see from (1.24), (1.13) that the t_k is expressed simply as

$$t_k = \frac{1}{2}\binom{k+2}{2}\mathcal{M}_k(H[u_0]')\int_{\mathbb{R}} (x-\gamma_k)^{k+2} H[u_0]'(x)\,dx. \tag{1.26}$$

It is clear that this expression is valid even when $k = 0$. Hence, we conclude from (1.26) that, if u_0 is nonnegative and the moments of $H[u_0]'$ degenerate up to $(k-1)$-th order, the time shift $(t_k)_+$ becomes strictly positive for any $k = 2m$, $m \in \mathbb{N}_0$.

(v) The diffusion wave approximate solution χ^k of order k is an explicit smooth solution to the viscous Burgers equation for $t > 0$, and is self-similar when $k = 0$. It can be also seen that

$$\int_{\mathbb{R}} \chi^k(x,t)\,dx = \mathcal{M}_0(u_0) \quad \text{for} \quad t > 0, \tag{1.27}$$

which means that the 0-th moment of χ^k is a conserved quantity.

In [2], the diffusion and N-wave approximate solutions to the viscous Burgers equation were introduced and their error estimates in L^p-norm were obtained. They further discuss the metastability phenomenon appearing in the long-time behavior of the solution to the viscous Burgers equation with small viscosity (see also [3]). However, they study the diffusion wave approximate solution of *only* 0*-th order without the time shift*. In [5], both the spatial and time shifts are taken into account for the diffusion wave approximate solution to (1.1)–(1.2) and the error estimate in L^p-norm was gained. But the diffusion wave approximate solution of *only* 0*-th order* is again considered in [5] for *the restricted nonnegative initial data*. On the other hand, the asymptotic approximate solution to the heat equation of higher order with both spatial and time shifts like (1.10) has already been introduced in [4], although *no error estimates were obtained* there. Moreover, in [6], an asymptotic approximate solution of 0-th order with the spatial and time shifts to the heat equation was introduced for the study of the asymptotic behavior of the damped wave equation. Curious investigation on the adequate choice of the time shift for an approximate self-similar solution to the nonlinear porous medium equation was presented in [7].

In brief, our Theorem 1.1 generalizes and refines the results in [2] and [5], by introducing a diffusion wave approximate solution to (1.1)–(1.2) *of any higher order with both spatial and time shifts* and by showing the optimal convergence rates of the error term, *under a weakened condition on the initial data*.

The plan of the paper is the following: In §2 we give the optimal error estimates in L^p-norm of the asymptotic approximate solution ϕ^k to the heat equation of k-th order. In this stage, checking the integrability of the iterated anti-derivatives of the initial error term of ϕ^k becomes most crucial (see Lemma 2.3). We then show the uniform pointwise estimates of the denominator appearing in the inverse of Cole-Hopf transformation of both the actual and asymptotic approximate solutions to the heat equation. Finally, combining the error estimates and the uniform pointwise estimates obtained in the preceding steps, we accomplish the proof of Theorem 1.1. The proof of several elementary lemmas which are used in §2 and the proof of equivalence of the relations (1.12)–(1.13) and the conditions (2.5) for $j = k + 1, k + 2$ are given in Appendix.

2. Proof of Theorem 1.1

We begin with studying the long-time behavior of the solution to the initial value problem of the heat equation (1.5)–(1.6) under the conditions (1.3), (1.9). The following proposition gives the optimal error estimates of the asymptotic approximate solution ϕ^k to (1.5)–(1.6) of order k defined by (1.10). This result seems to be new and of independent interest.

Proposition 2.1. *Let $1 \le p \le \infty$. Suppose that the conditions* (1.3) *and* (1.9) *hold. Let ϕ be a solution to* (1.5)–(1.6) *and ϕ^k an asymptotic approximate solution to* (1.5)–(1.6) *of order k defined by* (1.10). *Then the following estimates hold*:

$$\|\phi(t) - \phi^k(t)\|_{L^p(\mathbb{R})} \le C_2 t^{1/(2p)-2-k/2} \quad \text{for} \quad t > 0, \quad \text{when} \quad (t_k)_+ > 0, \tag{2.1}$$

$$\|\phi(t) - \phi^k(t)\|_{L^p(\mathbb{R})} \le C_2 t^{1/(2p)-3/2-k/2} \quad \text{for} \quad t > 0, \quad \text{when} \quad (t_k)_+ = 0, \tag{2.2}$$

and

$$\left\| \int_{-\infty}^{\bullet} \phi(y,t)\,dy - \int_{-\infty}^{\bullet} \phi^k(y,t)\,dy \right\|_{L^p(\mathbb{R})} \le C_3 t^{1/(2p)-3/2-k/2} \quad \text{for} \quad t > 0, \quad \text{when} \quad (t_k)_+ > 0, \tag{2.3}$$

$$\left\| \int_{-\infty}^{\bullet} \phi(y,t)\,dy - \int_{-\infty}^{\bullet} \phi^k(y,t)\,dy \right\|_{L^p(\mathbb{R})} \le C_3 t^{1/(2p)-1-k/2} \quad \text{for} \quad t > 0, \quad \text{when} \quad (t_k)_+ = 0. \tag{2.4}$$

Here the constants C_2 and C_3 depend only on $\|I_k(H[u_0]')\|_{L^1(\mathbb{R})}$, where $I_k(H[u_0]')$ is defined below in (2.6) *when $(t_k)_+ > 0$, and in* (2.7) *when $(t_k)_+ = 0$, respectively.*

REMARK 2.2. (i) The following claim reveals the reason why we specify the shifts γ_k, $(t_k)_+$ appearing in ϕ^k by the relations (1.12)–(1.13):

Suppose that the data u_0 satisfies

$$\mathcal{M}_{k+2}(H[u_0]')\mathcal{M}_k(H[u_0]') > \frac{k+2}{2(k+1)}\{\mathcal{M}_{k+1}(H[u_0]')\}^2,$$

which means that $(t_k)_+ > 0$. Then the relations (1.12)–(1.13) hold if and only if the following condition is fulfilled for $j = k+1, k+2$:

$$\lim_{t \to 0+} \int_{\mathbb{R}} x^j \{\phi(x,t) - \phi^k(x,t)\}\,dx = 0. \tag{2.5}$$

This claim implies that the choice of the shifts γ_k, $(t_k)_+$ as in (1.12)–(1.13) enables us to control the $(k+1)$-th and $(k+2)$-th moments of the initial error term. Moreover

we notice that the conditions (2.5) for $0 \le j \le k$ always hold even if the shifts γ_k, $(t_k)_+$ are not specified.

The equivalence of the relations (1.12)–(1.13) and the conditions (2.5) for $j = k+1, k+2$ is proved in Lemma A.3 in Appendix.

(ii) In view of (1.10), we can see that the fastest decay term of ϕ^k is $(-1)^k(\mathcal{M}_k(H[u_0]')/k!)(\partial/\partial x)^k G_{t+(t_k)_+}(x-\gamma_k)$ and its decay rate in L^p-norm is $t^{1/(2p)-1/2-k/2}$. Accordingly, we know that the error estimates (2.1)–(2.2) are effective. In a similar way, the error estimates (2.3)–(2.4) are also viewed as effective.

To prove Proposition 2.1 we shall make some auxiliary observations. First define a functional $I_k(H[u_0]')$ by: In case that $(t_k)_+ > 0$,

$$
\begin{aligned}
&I_k(H[u_0]')(x)\\
&= \int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{k+1}}\left(\int_{-\infty}^{x_{k+2}} H[u_0]'(x_{k+3})\,dx_{k+3}\right)dx_{k+2}\cdots\right)dx_1\\
&\quad -\sum_{j=0}^{k-1}(-1)^j\frac{\mathcal{M}_j(H[u_0]')}{j!}\\
&\qquad\times\int_{0}^{x}\left(\int_{0}^{x_1}\cdots\int_{0}^{x_{k-j}}\left(\int_{0}^{x_{k+1-j}} Y_0(x_{k+2-j})\,dx_{k+2-j}\right)dx_{k+1-j}\cdots\right)dx_1\\
&\quad -(-1)^k\frac{\mathcal{M}_k(H[u_0]')}{k!}\int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}\left(\int_{\mathbb{R}} G_{t_k}(x_2-y)Y_{\gamma_k}(y)\,dy\right)dx_2\right)dx_1;
\end{aligned}
\tag{2.6}
$$

In case that $(t_k)_+ = 0$,

$$
\begin{aligned}
&I_k(H[u_0]')(x)\\
&= \int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_k}\left(\int_{-\infty}^{x_{k+1}} H[u_0]'(x_{k+2})\,dx_{k+2}\right)dx_{k+1}\cdots\right)dx_1\\
&\quad -\sum_{j=0}^{k-1}(-1)^j\frac{\mathcal{M}_j(H[u_0]')}{j!}\int_{0}^{x}\left(\int_{0}^{x_1}\cdots\int_{0}^{x_{k-j}} Y_0(x_{k+1-j})\,dx_{k+1-j}\cdots\right)dx_1\\
&\qquad -(-1)^k\frac{\mathcal{M}_k(H[u_0]')}{k!}\int_{-\infty}^{x}\left(\int_{-\infty}^{x_1} Y_{\gamma_k}(x_2)\,dx_2\right)dx_1.
\end{aligned}
\tag{2.7}
$$

Here $Y_a(x)$ is the Heaviside function with the jump at $x = a$, that is,

$$
Y_a(x) = \begin{cases} 0 & \text{if} \quad x \le a,\\ 1 & \text{if} \quad x > a. \end{cases}
$$

We are going to prove the integrability of $I_k(H[u_0]')$ on $\mathbb{R}$ by making use of the relations (1.12)–(1.13) in essence.

Lemma 2.3. *Suppose that the relations* (1.12)–(1.12) *hold. Then, under the conditions* (1.7), *it follows that* $I_k(H[u_0]') \in L^1_{\mathrm{loc}}(\mathbb{R})$ *and*

$$|I_k(H[u_0]')(x)| = o(|x|^{-1-\epsilon}) \quad \textit{when} \quad (t_k)_+ > 0, \tag{2.8}$$

$$|I_k(H[u_0]')(x)| = o(|x|^{-2-\epsilon}) \quad \textit{when} \quad (t_k)_+ = 0, \tag{2.9}$$

as $|x| \to \infty$. *Here* ϵ *is the small positive constant appearing in* (1.3).

Proof. We give the proof only of the case that $(t_k)_+ > 0$. The proof of the case that $(t_k)_+ = 0$ is achieved in the similar way, so we omit it.

First we decompose the term $I_k(H[u_0]')(x)$ into the following three parts:

$$\begin{aligned}
B_1(x) &= \int_{-\infty}^{x} \left(\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{k+1}} \left(\int_{-\infty}^{x_{k+2}} H[u_0]'(x_{k+3})\, dx_{k+3} \right) dx_{k+2} \cdots \right) dx_1, \\
B_2(x) &= -\sum_{j=0}^{k-1} (-1)^j \frac{\mathcal{M}_j(H[u_0]')}{j!} \\
&\quad \times \int_0^x \left(\int_0^{x_1} \cdots \int_0^{x_{k-j}} \left(\int_{-\infty}^{x_{k+1-j}} Y_0(x_{k+2-j})\, dx_{k+2-j} \right) dx_{k+1-j} \cdots \right) dx_1, \\
B_3(x) &= -(-1)^k \frac{\mathcal{M}_k(H[u_0]')}{k!} \int_{-\infty}^{x} \left(\int_{-\infty}^{x_1} \left(\int_{\mathbb{R}} G_{t_k}(x_2 - y) Y_{\gamma_k}(y)\, dy \right) dx_2 \right) dx_1.
\end{aligned} \tag{2.10}$$

Then, applying Lemma A.1 in Appendix and recalling the condition (1.7), we readily have

$$B_1(x) = \frac{1}{(k+2)!} \sum_{j=0}^{k+2} (-1)^j \binom{k+2}{j} x^{k+2-j}\, \overline{\mathcal{M}}_j(H[u_0]')(x), \tag{2.11}$$

where $\overline{\mathcal{M}}_j(f)(x) = \int_{-\infty}^{x} y^j f(y)\, dy$. Furthermore, since

$$\begin{aligned}
&\int_0^x \left(\int_0^{x_1} \cdots \int_0^{x_{k-j}} \left(\int_0^{x_{k+1-j}} Y_0(x_{k+2-j})\, dx_{k+2-j} \right) dx_{k+1-j} \cdots \right) dx_1 \\
&= \frac{x_+^{k+2-j}}{(k+2-j)!} \quad \text{for} \quad 0 \le j \le k-1,
\end{aligned}$$

we obtain

$$B_2(x) = -\sum_{j=0}^{k-1} (-1)^j \frac{\mathcal{M}_j(H[u_0]')}{j!} \frac{x_+^{k+2-j}}{(k+2-j)!}, \tag{2.12}$$

where $x_+ = \max\{x, 0\}$.

As for $B_3(x)$, by the Fubini theorem and by integration by parts we have

$$
\begin{aligned}
B_3(x) &= -(-1)^k \frac{\mathcal{M}_k(H[u_0]')}{k!} \int_{\mathbb{R}} \left(\int_{-\infty}^{x} \left(\int_{-\infty}^{x_1} G_{t_k}(x_2 - y)\, dx_2 \right) dx_1 \right) Y_{\gamma_k}(y)\, dy \\
&= -(-1)^k \frac{\mathcal{M}_k(H[u_0]')}{k!} \int_{\mathbb{R}} \left\{ \left[x_1 \int_{-\infty}^{x_1} G_{t_k}(x_2 - y)\, dx_2 \right]_{x_1=-\infty}^{x_1=x} \right. \\
&\qquad \left. - \int_{-\infty}^{x} x_1 G_{t_k}(x_1 - y)\, dx_1 \right\} Y_{\gamma_k}(y)\, dy \qquad (2.13) \\
&= -(-1)^k \frac{\mathcal{M}_k(H[u_0]')}{k!} \left\{ x \int_{\mathbb{R}} \left(\int_{-\infty}^{x} G_{t_k}(x_1 - y)\, dx_1 \right) Y_{\gamma_k}(y)\, dy \right. \\
&\qquad \left. - \int_{\mathbb{R}} \left(\int_{-\infty}^{x} x_1 G_{t_k}(x_1 - y)\, dx_1 \right) Y_{\gamma_k}(y)\, dy \right\}.
\end{aligned}
$$

Then, the first and second terms of the braces on the last line of (2.13) can be treated as follows:

$$
\begin{aligned}
& x \int_{\mathbb{R}} \left(\int_{-\infty}^{x} G_{t_k}(x_1 - y)\, dx_1 \right) Y_{\gamma_k}(y)\, dy \\
&= x \int_{-\infty}^{x} \left(\int_{\gamma_k}^{\infty} G_{t_k}(x_1 - y)\, dy \right) dx_1 \\
&= x \int_{-\infty}^{x} \left(\int_{-\infty}^{x_1} G_{t_k}(y - \gamma_k)\, dy \right) dx_1 \\
&= x^2 \int_{-\infty}^{x} G_{t_k}(y - \gamma_k)\, dy - x \int_{-\infty}^{x} x_1 G_{t_k}(x_1 - \gamma_k)\, dx_1,
\end{aligned}
$$

and

$$
\begin{aligned}
& - \int_{\mathbb{R}} \left(\int_{-\infty}^{x} x_1 G_{t_k}(x_1 - y)\, dx_1 \right) Y_{\gamma_k}(y)\, dy \\
&= - \int_{-\infty}^{x} x_1 \left(\int_{\gamma_k}^{\infty} G_{t_k}(x_1 - y)\, dy \right) dx_1 \\
&= - \int_{-\infty}^{x} x_1 \left(\int_{-\infty}^{x_1} G_{t_k}(y - \gamma_k)\, dy \right) dx_1 \\
&= -\frac{x^2}{2} \int_{-\infty}^{x} G_{t_k}(y - \gamma_k)\, dy + \frac{1}{2} \int_{-\infty}^{x} x_1^2 G_{t_k}(x_1 - \gamma_k)\, dx_1,
\end{aligned}
$$

so that we arrive at

(2.14)
$$B_3(x) = -(-1)^k \frac{\mathcal{M}_k(H[u_0]')}{k!} \left\{ \frac{x^2}{2} \int_{-\infty}^{x} G_{t_k}(y-\gamma_k)\,dy \right. \\ \left. - x \int_{-\infty}^{x} y G_{t_k}(y-\gamma_k)\,dy + \frac{1}{2} \int_{-\infty}^{x} y^2 G_{t_k}(y-\gamma_k)\,dy \right\}.$$

Hence we promptly find from (2.11), (2.12), (2.14) that $I_k(H[u_0]') \in L^1_{\mathrm{loc}}(\mathbb{R})$.

We now look at the case when $x \to -\infty$. Recalling (1.7) again and applying (A.4) of Lemma A.2 with $s = k + 3 + \epsilon$ yield, for $0 \le j \le k+3$,

$$\overline{\mathcal{M}}_j(H[u_0]')(x) = o(|x|^{-k-3-\epsilon+j}) \quad \text{as} \quad x \to -\infty, \tag{2.15}$$

thereby it follows from (2.11), (2.12) and (2.15) that

$$B_1(x) + B_2(x) = o(|x|^{-1-\epsilon}) \quad \text{as} \quad x \to -\infty. \tag{2.16}$$

In view of (2.14), we also have

$$B_3(x) = O(1)(1 + \sqrt{t_k} + t_k) e^{-(x-\gamma_k)^2/32t_k} \quad \text{as} \quad x \to -\infty. \tag{2.17}$$

Accordingly, from (2.16), (2.17) we can conclude that (2.8) holds when $x \to -\infty$.

Next we turn to the case when $x \to \infty$. First, it is readily observed from (2.11), (2.12) that

$$\begin{aligned} &B_1(x) + B_2(x) \\ &= \frac{(-1)^{k+2}}{(k+2)!} \overline{\mathcal{M}}_{k+2}(H[u_0]')(x) + \frac{(-1)^{k+1}}{(k+1)!\,1!} \overline{\mathcal{M}}_{k+1}(H[u_0]')(x) x \\ &\quad + \frac{(-1)^k}{k!\,2!} \overline{\mathcal{M}}_k(H[u_0]')(x) x^2 \\ &\quad + \sum_{j=0}^{k-1} \frac{(-1)^j}{(k+2-j)!\,j!} \left\{ \overline{\mathcal{M}}_j(H[u_0]')(x) x^{k+2-j} - \mathcal{M}_j(H[u_0]') x_+^{k+2-j} \right\}. \end{aligned} \tag{2.18}$$

On the other hand, by referring to (2.14), we see that

$$
\begin{aligned}
B_3(x) = -(-1)^k \frac{\mathcal{M}_k(H[u_0]')}{k!} \Bigg\{ & \frac{x^2}{2} \int_{-\infty}^{x-\gamma_k} G_{t_k}(y)\, dy \\
& - x \int_{-\infty}^{x-\gamma_k} y G_{t_k}(y)\, dy - \gamma_k x \int_{-\infty}^{x-\gamma_k} G_{t_k}(y)\, dy \\
& + \frac{1}{2} \int_{-\infty}^{x-\gamma_k} y^2 G_{t_k}(y)\, dy \\
& + \gamma_k \int_{-\infty}^{x-\gamma_k} y G_{t_k}(y)\, dy + \frac{\gamma_k^2}{2} \int_{-\infty}^{x-\gamma_k} G_{t_k}(y)\, dy \Bigg\} \\
= -(-1)^k \frac{\mathcal{M}_k(H[u_0]')}{k!} & \left(\frac{x^2}{2} - \gamma_k x + t_k + \frac{\gamma_k^2}{2} \right) \\
& + O(1)(1 + \sqrt{t_k} + t_k) e^{-(x-\gamma_k)^2/32t_k},
\end{aligned}
\tag{2.19}
$$

as $x \to \infty$. Here we have used the fact that $\int_{\mathbb{R}} G_t(x)\, dx = 1$, $\int_{\mathbb{R}} x G_t(x)\, dx = 0$, and $\int_{\mathbb{R}} x^2 G_t(x)\, dx = 2t$ for $t > 0$.

Eventually, combining (2.18) and (2.19), then applying (A.3) of Lemma A.2 with $s = k + 3 + \epsilon$, we have

(2.20)

$$
\begin{aligned}
I_k(H[u_0]')(x) &= B_1(x) + B_2(x) + B_3(x) \\
&= \frac{(-1)^{k+2}}{(k+2)!} \left\{ \mathcal{M}_{k+2}(H[u_0]') - \frac{(k+2)(k+1)}{2} (\gamma_k^2 + 2t_k) \mathcal{M}_k(H[u_0]') \right\} \\
&\quad + \frac{(-1)^{k+1}}{(k+1)!} \{ \mathcal{M}_{k+1}(H[u_0]') - \gamma_k (k+1) \mathcal{M}_k(H[u_0]') \} x \\
&\quad + \sum_{j=0}^{k+2} \frac{(-1)^j}{(k+2-j)!\, j!} \left\{ \overline{\mathcal{M}}_j(H[u_0]') - \mathcal{M}_j(H[u_0]')(x) \right\} x^{k+2-j} \\
&\quad + O(1)(1 + \sqrt{t_k} + t_k) e^{-(x-\gamma_k)^2/32t_k} \\
&= \frac{(-1)^{k+2}}{(k+2)!} \left\{ \mathcal{M}_{k+2}(H[u_0]') - \frac{(k+2)(k+1)}{2} (\gamma_k^2 + 2t_k) \mathcal{M}_k(H[u_0]') \right\} \\
&\quad + \frac{(-1)^{k+1}}{(k+1)!} \{ \mathcal{M}_{k+1}(H[u_0]') - \gamma_k (k+1) \mathcal{M}_k(H[u_0]') \} x + o(|x|^{-1-\epsilon}),
\end{aligned}
$$

as $x \to \infty$. Consequently, in view of the relations (1.12)–(1.13), we conclude from (2.20) that (2.8) holds when $x \to \infty$. We have thus proved Lemma 2.3. □

Thanks to Lemma 2.3 just proved, we reach the conclusion that $I_k(H[u_0]') \in L^1(\mathbb{R})$. So we can now define $e^{t\Delta} I_k(H[u_0]')$ by

$$e^{t\Delta} I_k(H[u_0]')(x) = \int_{\mathbb{R}} G_t(x-y) I_k(H[u_0]')(y)\, dy. \tag{2.21}$$

The following lemma says that we can represent the error term of the asymptotic approximate solution ϕ^k in terms of the derivative of $e^{t\Delta} I_k(H[u_0]')$.

Lemma 2.4. *The following equalities hold*: *In case that* $(t_k)_+ > 0$,

$$\partial_x^{k+3}(e^{t\Delta} I_k(H[u_0]'))(x) = \phi(x,t) - \phi^k(x,t) \quad \textit{for} \quad t > 0,\ x \in \mathbb{R}; \tag{2.22}$$

In case that $(t_k)_+ = 0$,

$$\partial_x^{k+2}(e^{t\Delta} I_k(H[u_0]'))(x) = \phi(x,t) - \phi^k(x,t) \quad \textit{for} \quad t > 0,\ x \in \mathbb{R}. \tag{2.23}$$

Here $\phi(x,t)$ *is a solution to* (1.5)–(1.6) *and* $\phi^k(x,t)$ *is an asymptotic approximate solution to* (1.5)–(1.6) *of k-th order defined by* (1.10).

Proof. We shall give the proof only for the case that $(t_k)_+ > 0$ by the same reasoning in the preceding proof. We again use the same decomposition of $I_k(H[u_0]')$ as in (2.10) to obtain

$$\begin{aligned} &\partial_x^{k+3}(e^{t\Delta} I_k(H[u_0]'))(x) \\ &= \partial_x^{k+3} \int_{\mathbb{R}} G_t(x-y) B_1(y)\, dy + \partial_x^{k+3} \int_{\mathbb{R}} G_t(x-y) B_2(y)\, dy \\ &\quad + \partial_x^{k+3} \int_{\mathbb{R}} G_t(x-y) B_3(y)\, dy. \end{aligned} \tag{2.24}$$

Then, owing to the property of the convolution, we know from (1.5), (1.6) that

$$\begin{aligned} &\partial_x^{k+3} \int_{\mathbb{R}} G_t(x-y) B_1(y)\, dy \\ &= \int_{\mathbb{R}} G_t(x-y) \partial_y^{k+3} \left(\int_{-\infty}^{y} \left(\int_{-\infty}^{x_1} \left(\cdots \left(\int_{-\infty}^{x_{k+2}} H[u_0]'(x_{k+3})\, dx_{k+3} \right) \cdots dx_1 \right) dy \right. \right. \\ &= \int_{\mathbb{R}} G_t(x-y) H[u_0]'(y)\, dy \\ &= \phi(x,t). \end{aligned} \tag{2.25}$$

Moreover, it is observed similarly that

(2.26)
$$
\begin{aligned}
&\partial_x^{k+3}\int_{\mathbb{R}} G_t(x-y)B_2(y)\,dy\\
&=-\sum_{j=0}^{k-1}(-1)^j\frac{\mathcal{M}_j(H[u_0]')}{j!}\\
&\quad\times\partial_x^{j+1}\int_{\mathbb{R}} G_t(x-y)\partial_y^{k+2-j}\left(\int_0^y\left(\int_0^{x_1}\cdots\int_0^{x_{k+1-j}} Y_0(x_{k+2-j})\,dx_{k+2-j}\cdots\right)dx_1\right)dy\\
&=-\sum_{j=0}^{k-1}(-1)^j\frac{\mathcal{M}_j(H[u_0]')}{j!}\int_0^\infty \partial_x^{j+1}G_t(x-y)\,dy\\
&=\sum_{j=0}^{k-1}(-1)^j\frac{\mathcal{M}_j(H[u_0]')}{j!}\int_0^\infty \partial_y(\partial_x^j G_t(x-y))\,dy\\
&=-\sum_{j=0}^{k-1}(-1)^j\frac{\mathcal{M}_j(H[u_0]')}{j!}\partial_x^j G_t(x).
\end{aligned}
$$

Whereas, since we have by utilizing the Fubini theorem and the semigroup property of the heat kernel,

$$
\begin{aligned}
&\partial_x^{k+3}\int_{\mathbb{R}} G_t(x-y)\left(\int_{-\infty}^y\left(\int_{-\infty}^{x_1}\left(\int_{\mathbb{R}} G_{t_k}(x_2-z)Y_{\gamma_k}(z)\,dz\right)dx_2\right)dx_1\right)dy\\
&=\partial_x^{k+1}\int_{\mathbb{R}} G_t(x-y)\partial_y^2\left(\int_{-\infty}^y\left(\int_{-\infty}^{x_1}\left(\int_{\mathbb{R}} G_{t_k}(x_2-z)Y_{\gamma_k}(z)\,dz\right)dx_2\right)dx_1\right)dy\\
&=\partial_x^{k+1}\int_{\mathbb{R}} G_t(x-y)\left(\int_{\mathbb{R}} G_{t_k}(y-z)Y_{\gamma_k}(z)\,dz\right)dy\\
&=\partial_x^{k+1}\int_{\mathbb{R}} G_{t+t_k}(x-z)Y_{\gamma_k}(z)\,dz\\
&=\partial_x^{k+1}\int_{-\infty}^x G_{t+t_k}(z-\gamma_k)\,dz\\
&=\partial_x^k G_{t+t_k}(x-\gamma_k),
\end{aligned}
$$

we can see that

(2.27)
$$
\begin{aligned}
&\partial_x^{k+3}\int_{\mathbb{R}} G_t(x-y)B_3(y)\,dy\\
&=-(-1)^k\frac{\mathcal{M}_k(H[u_0]')}{k!}\\
&\quad\times\partial_x^{k+3}\int_{\mathbb{R}} G_t(x-y)\left(\int_{-\infty}^y\left(\int_{-\infty}^{x_1}\left(\int_{\mathbb{R}} G_{t_k}(x_2-z)Y_{\gamma_k}(z)\,dz\right)dx_2\right)dx_1\right)dy\\
&=-(-1)^k\frac{\mathcal{M}_k(H[u_0]')}{k!}\partial_x^k G_{t+t_k}(x-\gamma_k).
\end{aligned}
$$

Combining (2.25), (2.26) and (2.27), and recalling the form of ϕ^k in (1.10), we finally arrive at (2.22). We complete the proof of Lemma 2.4. □

We now give

Proof of Proposition 2.1. By virtue of the well-known $L^p - L^q$ estimate of solutions to the heat equation, we have, for $1 \le p \le \infty$,

$$
\begin{aligned}
&\|\partial_x^j e^{t\Delta} I_k(H[u_0]')\|_{L^p(\mathbb{R})} \\
&= \left\| \partial_x^j \int_{\mathbb{R}} G_t(x-y) I_k(H[u_0]')(y)\, dy \right\|_{L^p(\mathbb{R})} \\
&\le C t^{1/(2p)-1/2-j/2} \|I_k(H[u_0]')\|_{L^1(\mathbb{R})} \quad \text{for all} \quad t > 0.
\end{aligned} \tag{2.28}
$$

Here C is a universal constant.

Now, from Lemma 2.4 and the estimates (2.28) with $j = k+3, k+2, k+1$, and k, we readily obatin the desired estimates (2.1)–(2.4). The proof of Proposition 2.1 is completed. □

We next show the uniform pointwise estimates for the denominator appearing in the inverse of Cole-Hopf transformation for both a solution ϕ to (1.5)–(1.6) and an asymptotic approximate solution ϕ^k.

Lemma 2.5. *Let ϕ be a solution to* (1.5)–(1.6) *on* $[0,\infty)$ *and ϕ^k an asymptotic approximate solution to* (1.5)–(1.6) *of order k with $k \in \mathbb{N}_0$. Then the estimates*

$$
\begin{aligned}
\min_{x\in\mathbb{R}} \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_0(y)\, dy\right) &\le 1 + \int_{-\infty}^{x} \phi(y,t)\, dy \\
&\le \max_{x\in\mathbb{R}} \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_0(y)\, dy\right) < \infty
\end{aligned} \tag{2.29}
$$

hold for any $x \in \mathbb{R}$, $t > 0$.

Furthermore, we have:

In case that $k = 0$, the estimate

$$
\min\left\{1, \exp\left(-\frac{1}{2}\int_{\mathbb{R}} u_0(y)\, dy\right)\right\} < 1 + \int_{-\infty}^{x} \phi^0(y,t)\, dy \tag{2.30}
$$

holds for any $x \in \mathbb{R}$, $t > 0$;

In case that $k \ge 1$, there exists a constant $T_k \ge 0$ such that the estimate

$$
\frac{1}{2}\min\left\{1, \exp\left(-\frac{1}{2}\int_{\mathbb{R}} u_0(y)\, dy\right)\right\} < 1 + \int_{-\infty}^{x} \phi^k(y,t)\, dy \tag{2.31}
$$

holds for any $x \in \mathbb{R}$, $t > T_k$. Here T_k depends only on k and $\mathcal{M}_j(H[u_0]')$, $1 \le j \le k$.

Proof. Since $1+\int_{-\infty}^{x}\phi(y,t)\,dy$ is still a solution of the heat equation (1.5) with the initial data $\exp\bigl(-(1/2)\int_{-\infty}^{x}u_0(y)\,dy\bigr)$, the estimate (2.29) immediately follows from the maximum principle. Next, noticing that $\mathcal{M}_0(H[u_0]')\exp\bigl((1/2)\int_{\mathbb{R}}u_0(y)\,dy\bigr)-1$, we have

$$1+\mathcal{M}_0(H[u_0]')\int_{-\infty}^{x}G_{t+(t_0)_+}(y)\,dy>\min\left\{1,\exp\left(-\frac{1}{2}\int_{\mathbb{R}}u_0(y)\,dy\right)\right\}, \tag{2.32}$$

for any $x\in\mathbb{R}$, $t>0$. Accordingly, since

$$1+\int_{-\infty}^{x}\phi^0(y,t)\,dy=1+\mathcal{M}_0(H[u_0]')\int_{-\infty}^{x}G_{t+(t_0)_+}(y-\gamma_0)\,dy,$$

it is easy to derive from (2.30) the estimate (2.30).

On the other hand, in view of (1.17), we obtain, for $j\in\mathbb{N}$,

$$\begin{aligned}\int_{-\infty}^{x}\left(\frac{\partial}{\partial y}\right)^{j}G_t(y)\,dy&=\left(\frac{\partial}{\partial x}\right)^{j-1}G_t(x)\\&=\frac{1}{2^j\sqrt{\pi}}(-1)^{j-1}t^{-j/2}H_{j-1}\left(\frac{x}{2\sqrt{t}}\right)\exp\left(-\frac{x^2}{4t}\right).\end{aligned} \tag{2.33}$$

Remark that similar equality to (2.33) remains valid even when $G_t(y)$ is shifted with respect to t and y. Therefore, when $k\ge 1$, we can see from (1.10), (2.33) and the remark just above that

$$\begin{aligned}&1+\int_{-\infty}^{x}\phi^k(y,t)\,dy\\&=1+\mathcal{M}_0(H[u_0]')\int_{-\infty}^{x}G_t(y)\,dy\\&\quad-\sum_{j=1}^{k-1}\frac{\mathcal{M}_j(H[u_0]')}{j!}\frac{1}{2^j\sqrt{\pi}}t^{-j/2}H_{j-1}\left(\frac{x}{2\sqrt{t}}\right)\exp\left(-\frac{x^2}{4t}\right)\\&\quad-\frac{\mathcal{M}_k(H[u_0]')}{k!}\frac{1}{2^k\sqrt{\pi}}(t+(t_k)_+)^{-k/2}H_{k-1}\left(\frac{x-\gamma_k}{2\sqrt{t+(t_k)_+}}\right)\exp\left(-\frac{(x-\gamma_k)^2}{4(t+(t_k)_+)}\right).\end{aligned} \tag{2.34}$$

Consequently, since $\sup_{\xi\in\mathbb{R}}|\xi^l\exp(-\xi^2)|<C_l$ holds for any $l\in\mathbb{N}_0$ with a constant $C_l>0$, we easily find from (2.32) with $(t_0)_+=0$, (2.34) that there exists a constant $T_k\ge 0$ depending only on k and $\mathcal{M}_j(H[u_0]')$, $1\le j\le k$, such that the desired estimate (2.31) holds for any $x\in\mathbb{R}$, $t>T_k$. This completes the proof of Lemma 2.5. □

We are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By virtue of (2.29) of Lemma 2.5, we find that the unique solution u to (1.1)–(1.2) is indeed given by the inverse of Cole-Hopf transformation of

the solution ϕ to (1.5)–(1.6), that is,

$$u(x,t) = -2\frac{\phi(x,t)}{1+\int_{-\infty}^{x}\phi(y,t)\,dy}. \tag{2.35}$$

We consider the case that $k \geq 1$. Since the estimate (2.31) of Lemma 2.5 holds for any $x \in \mathbb{R}$, $t > T_k$, we can show that the diffusion wave approximate solution χ^k defined by (1.18) is well-defined on the interval (T_k, ∞). Accordingly, owing to Proposition 2.1 and Lemma 2.5, we have the following estimates: for any $1 \leq p \leq \infty$,

$$\begin{aligned}
&\|u(t) - \chi^k(t)\|_{L^p(\mathbb{R})} \\
&= \left\| -2\frac{\phi(\cdot,t)}{1+\int_{-\infty}^{\cdot}\phi(y,t)\,dy} + 2\frac{\phi^k(\cdot,t)}{1+\int_{-\infty}^{\cdot}\phi^k(y,t)\,dy} \right\|_{L^p(\mathbb{R})} \\
&\leq \frac{2\|\phi(t)-\phi^k(t)\|_{L^p(\mathbb{R})}\left\|1+\int_{-\infty}^{\cdot}\phi(y,t)\,dy\right\|_{L^\infty(\mathbb{R})}}{\inf_{x\in\mathbb{R}}\left\{\left|1+\int_{-\infty}^{x}\phi(y,t)\,dy\right|\left|1+\int_{-\infty}^{x}\phi^k(y,t)\,dy\right|\right\}} \\
&\quad + \frac{2\left\|\int_{-\infty}^{\cdot}\phi(y,t)\,dy - \int_{-\infty}^{\cdot}\phi^k(y,t)\,dy\right\|_{L^p(\mathbb{R})}\|\phi(t)\|_{L^\infty(\mathbb{R})}}{\inf_{x\in\mathbb{R}}\left\{\left|1+\int_{-\infty}^{x}\phi(y,t)\,dy\right|\left|1+\int_{-\infty}^{x}\phi^k(y,t)\,dy\right|\right\}} \\
&\leq \begin{cases} Ct^{1/(2p)-2-k/2} & \text{for} \quad t > T_k, \quad \text{when} \quad (t_k)_+ > 0, \\ Ct^{1/(2p)-3/2-k/2} & \text{for} \quad t > T_k, \quad \text{when} \quad (t_k)_+ = 0, \end{cases}
\end{aligned} \tag{2.36}$$

where C depends only on the constants C_2, C_3 in Proposition 2.1 and the lower and upper bounds in Lemma 2.5. Note that we used an elementary estimate $\|\phi(t)\|_{L^\infty(\mathbb{R})} \leq (1/\sqrt{4\pi})\|H[u_0]'\|_{L^1(\mathbb{R})}t^{-1/2}$, $t > 0$, in deriving (2.36). The estimates in (2.36) imply (1.21) and (1.22). The case that $k = 0$ can be treated likewise and the proof in that case is omitted. We complete the proof of Theorem 1.1. □

3. Appendix

In this appendix, we first give the proof of several elementary lemmas which were used in the preceding section.

Lemma A.1. *Suppose that* $(1+|x|)^k f \in L^1(\mathbb{R})$, $k \in \mathbb{N}_0$. *Then the equality*

$$\begin{aligned}
&\int_{-\infty}^{x_0}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_k} f(x_{k+1})\,dx_{k+1}\cdots\right)dx_1 \\
&= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}x_0^j\,\overline{\mathcal{M}}_{k-j}(f)(x_0)
\end{aligned} \tag{A.1}$$

holds. Here $\overline{\mathcal{M}}_j(f)(x) = \int_{-\infty}^{x} y^j f(y)\,dy$ *for* $j \in \mathbb{N}_0$.

Proof. We use induction on k. The claim is trivial for $k = 0$; assume that (A.1) with $k = k_0$, $k_0 \in \mathbb{N}_0$, holds for any f such that $(1+|x|)^{k_0} f \in L^1(\mathbb{R})$. Then, for any f such that $(1+|x|)^{k_0+1} f \in L^1(\mathbb{R})$, we have by integration by parts

$$\int_{-\infty}^{x} y^l \, \overline{\mathcal{M}}_j(f)(y)\, dy = \frac{x^{l+1}}{l+1}\overline{\mathcal{M}}_j(f)(x) - \frac{1}{l+1}\overline{\mathcal{M}}_{j+l+1}(f)(x) \tag{A.2}$$

with $0 \le j + l + 1 \le k_0 + 1$ and $j, l \in \mathbb{N}_0$, so that we find from the induction assumption that

$$\begin{aligned}
&\int_{-\infty}^{x_0}\left(\int_{-\infty}^{x_1}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{k_0+1}} f(x_{k_0+2})\, dx_{k_0+2}\cdots\right) dx_1\right) dx_0 \\
&= \frac{1}{k_0!}\sum_{j=0}^{k_0}(-1)^{k_0-j}\binom{k_0}{j}\int_{-\infty}^{x_0} x_1^j\, \overline{\mathcal{M}}_{k_0-j}(f)(x_1)\, dx_1 \\
&= \frac{1}{k_0!}\sum_{j=0}^{k_0}(-1)^{k_0-j}\binom{k_0}{j}\left(\frac{x_0^{j+1}}{j+1}\overline{\mathcal{M}}_{k_0-j}(f)(x_0) - \frac{1}{j+1}\overline{\mathcal{M}}_{k_0+1}(f)(x_0)\right) \\
&= \frac{1}{(k_0+1)!}\sum_{j=1}^{k_0+1}(-1)^{k_0+1-j}(k_0+1)\binom{k_0}{j-1}\frac{x_0^j}{j}\,\overline{\mathcal{M}}_{k_0+1-j}(f)(x_0) \\
&\quad - \frac{1}{(k_0+1)!}\left\{\sum_{j=1}^{k_0+1}(-1)^{k_0+1-j}(k_0+1)\binom{k_0}{j-1}\frac{1}{j}\right\}\overline{\mathcal{M}}_{k_0+1}(f)(x_0) \\
&= \frac{1}{(k_0+1)!}\sum_{j=1}^{k_0+1}(-1)^{k_0+1-j}\binom{k_0+1}{j} x_0^j\,\overline{\mathcal{M}}_{k_0+1-j}(f)(x_0) \\
&\qquad - \frac{1}{(k_0+1)!}\left\{\sum_{j=1}^{k_0+1}(-1)^{k_0+1-j}\binom{k_0+1}{j}\right\}\overline{\mathcal{M}}_{k_0+1}(f)(x_0), \\
&= \frac{1}{(k_0+1)!}\sum_{j=0}^{k_0+1}(-1)^{k_0+1-j}\binom{k_0+1}{j} x_0^j\,\overline{\mathcal{M}}_{k_0+1-j}(f)(x_0).
\end{aligned}$$

This shows that (A.1) holds when $k = k_0+1$. So we complete the proof of Lemma A.1. □

Lemma A.2. *Suppose that* $(1+|x|)^s f \in L^1(\mathbb{R})$ *for* $s > 0$. *Then,*

$$\mathcal{M}_j(f) - \overline{\mathcal{M}}_j(f)(x) = o((1+|x|)^{-s+j}) \quad \text{as} \quad x \to \infty \tag{A.3}$$

and

$$\overline{\mathcal{M}}_j(f)(x) = o((1+|x|)^{-s+j}) \quad \text{as} \quad x \to -\infty \tag{A.4}$$

hold for $j = 0, 1, \ldots, [s]$. *Here* $[s]$ *denotes the greatest integer not exceeding* s.

Proof. Since

$$\begin{aligned}\left|\mathcal{M}_j(f) - \overline{\mathcal{M}}_j(f)(x)\right| &= \left|\int_x^\infty y^j f(y)\,dy\right| \\ &\le \frac{\int_x^\infty (1+|y|)^s |f(y)|\,dy}{(1+|x|)^{s-j}} \quad \text{for} \quad x > 0,\end{aligned}$$

and

$$\left|\overline{\mathcal{M}}_j(f)(x)\right| \le \frac{\int_{-\infty}^x (1+|y|)^s |f(y)|\,dy}{(1+|x|)^{s-j}} \quad \text{for} \quad x < 0,$$

we can obtain easily (A.3), (A.4) from the assumption $(1+|x|)^s f(x) \in L^1(\mathbb{R})$. □

We next give the proof of equivalence of the relations (1.12)–(1.13) and the conditions (2.5) for $j = k+1, k+2$, which was cited in Remark 2.2 (i).

Lemma A.3. *Let* ϕ *be a smooth solution to* (1.5)–(1.6) *on* $\mathbb{R} \times [0, T')$, $T' > 0$, *and let* ϕ^k *be an asymptotic approximate solution to* (1.5)–(1.6) *of k-th order defined by* (1.10) *with* $k \in \mathbb{N}_0$. *Then it holds that*

$$\begin{aligned}&\lim_{t\to 0+} \int_{\mathbb{R}} x^{k+1}\{\phi(x,t) - \phi^k(x,t)\}\,dx \\ &= \mathcal{M}_{k+1}(H[u_0]') - (k+1)\mathcal{M}_k(H[u_0]')\gamma_k\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}&\lim_{t\to 0+} \int_{\mathbb{R}} x^{k+2}\{\phi(x,t) - \phi^k(x,t)\}\,dx \\ &= \mathcal{M}_{k+1}(H[u_0]') - \frac{(k+2)(k+1)}{2}\mathcal{M}_k(H[u_0]')(2(t_k)_+ + \gamma_k^2).\end{aligned} \tag{A.6}$$

Proof. We first prove (A.5). By integration by parts, we obtain

$$\lim_{t\to 0+} \int_{\mathbb{R}} x^j \left(\frac{\partial}{\partial x}\right)^l G_t(x)\,dx = (-1)^l j!\,\delta_{lj} \quad \text{for any} \quad l, j \in \mathbb{N}_0, \tag{A.7}$$

where δ_{lj} denotes Kronecker's delta. Moreover, we have by integration by parts

$$\begin{aligned}
&\int_{\mathbb{R}} x^{k+1}\left(\frac{\partial}{\partial x}\right)^k G_{t+(t_k)_+}(x-\gamma_k)\,dx \\
&= (-1)^k(k+1)!\int_{\mathbb{R}} xG_{t+(t_k)_+}(x-\gamma_k)\,dx \\
&= (-1)^k(k+1)!\left\{\int_{\mathbb{R}} xG_{t+(t_k)_+}(x)\,dx + \gamma_k\int_{\mathbb{R}} G_{t+(t_k)_+}(x)\,dx\right\} \\
&= (-1)^k(k+1)!\,\gamma_k.
\end{aligned} \tag{A.8}$$

Therefore, in view of (1.5)–(1.6) and (1.10), it follows from (A.7)–(A.8) that

$$\begin{aligned}
&\lim_{t\to 0+}\int_{\mathbb{R}} x^{k+1}\{\phi(x,t)-\phi^k(x,t)\}\,dx \\
&= \mathcal{M}_{k+1}(H[u_0]') - (-1)^k\frac{\mathcal{M}_k(H[u_0]')}{k!}(-1)^k(k+1)!\,\gamma_k \\
&= \mathcal{M}_{k+1}(H[u_0]') - (k+1)\mathcal{M}_k(H[u_0]')\gamma_k,
\end{aligned} \tag{A.9}$$

which proves (A.5).

Next, recalling the fact that

$$\int_{\mathbb{R}} x^2 G_{t+(t_k)_+}(x)\,dx = 2(t+(t_k)_+),$$

we see that

$$\begin{aligned}
&\int_{\mathbb{R}} x^{k+2}\left(\frac{\partial}{\partial x}\right)^k G_{t+(t_k)_+}(x-\gamma_k)\,dx \\
&= (-1)^k\frac{(k+2)!}{2!}\int_{\mathbb{R}} x^2 G_{t+(t_k)_+}(x-\gamma_k)\,dx \\
&= (-1)^k\frac{(k+2)!}{2!}\left\{\int_{\mathbb{R}} x^2 G_{t+(t_k)_+}(x)\,dx\right. \\
&\qquad\qquad \left. + 2\gamma_k\int_{\mathbb{R}} xG_{t+(t_k)_+}(x)\,dx + \gamma_k^2\int_{\mathbb{R}} G_{t+(t_k)_+}(x)\,dx\right\} \\
&= (-1)^k\frac{(k+2)!}{2!}\{2(t+(t_k)_+)+\gamma_k^2\}.
\end{aligned} \tag{A.10}$$

Accordingly, we can derive from (A.7) and (A.10) that

$$\begin{aligned}
&\lim_{t\to 0+}\int_{\mathbb{R}} x^{k+2}\{\phi(x,t)-\phi^k(x,t)\}\,dx \\
&= \mathcal{M}_{k+2}(H[u_0]') - (-1)^k\frac{\mathcal{M}_k(H[u_0]')}{k!}(-1)^k\frac{(k+2)!}{2!}(2(t_k)_+ + \gamma_k^2) \\
&= \mathcal{M}_{k+2}(H[u_0]') - \frac{(k+2)(k+1)}{2}\mathcal{M}_k(H[u_0]')(2(t_k)_+ + \gamma_k^2),
\end{aligned}$$

which shows (A.7). The proof of Lemma A.3 is now completed. □

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