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COHOMOLOGY OF VECTOR BUNDLES FROM A DOUBLE COVER OF THE PROJECTIVE PLANE

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Abstract

The paper deals with locally free sheaves $\mathcal{F}_{p,q}$ on $\mathbb{P}^2$ obtained from a morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$. Bases of $\mathrm{H}^i(\mathbb{P}^2, \mathcal{F}_{p,q})$ are explicitly given in terms of elements of certain local cohomology modules, which built up canonically a complex for computing cohomology modules of locally free sheaves on $\mathbb{P}^2$.

1. Introduction

Let $\mathbb{P}^n = \operatorname{Proj} \kappa[X_0, X_1, \dots, X_n]$ be the projective n-space over a field κ and $\mathcal{F}$ be a locally free sheaf of finite rank on $\mathbb{P}^n$. In [4], a new method is introduced to compute cohomology modules of $\mathcal{F}$. The method involves a complex of κ-vector spaces

$$0 \to \mathcal{F}^{(0)} \xrightarrow{d^{(0)}} \mathcal{F}^{(1)} \xrightarrow{d^{(1)}} \mathcal{F}^{(2)} \to \cdots \to \mathcal{F}^{(n)} \to 0,$$

in which $\mathcal{F}^{(i)}$ depends only on the rank of $\mathcal{F}$ and $d^{(i)}$ is determined by the transition functions of $\mathcal{F}$. It is shown that the ith cohomology of the complex $\mathcal{F}^{(\bullet)}$ is isomorphic to the ith cohomology of $\mathcal{F}$. With computations of kernels and quotients of $d^{(i)}$, the problem of algebraic geometry on computing cohomology becomes a problem of linear algebra. In terms of elements of $\mathcal{F}^{(i)}$, one may ask *what a basis of the κ-vector space* $\mathrm{H}^i(\mathbb{P}^n, \mathcal{F})$ *looks like*. For twisted differentials $\Omega^p_{\mathbb{P}^n/\kappa}(m)$, this project is carried out [4]. A basis of the κ-vector space $\mathrm{H}^q\big(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m)\big)$ is exhibited, from which the Bott formula

$$\dim_\kappa \mathrm{H}^q\big(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m)\big) = \begin{cases} \dbinom{m-1}{p}\dbinom{m+n-p}{m}, & \text{for} \quad q=0,\ 0 \le p \le n,\ \ p < m; \\ 1, & \text{for} \quad m=0,\ 0 \le p = q \le n; \\ \dbinom{-m-1}{n-p}\dbinom{-m+p}{-m}, & \text{for} \quad q=n,\ 0 \le p \le n,\ m < p-n; \\ 0, & \text{otherwise} \end{cases}$$

is recovered by counting the cardinality of the basis. Invoking elaborated computations, our approach to the Bott formula interprets the combinatorial numbers in the formula.

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In this paper, we work on the project for some rank two locally free sheaves of modules on the projective plane $\mathbb{P}^2$. Let Q be the quadric surface in $\mathbb{P}^3$ defined by the equation $X_0X_1 - X_2X_3$. Via the Segre embedding, Q is identifies with $\mathbb{P}^1 \times \mathbb{P}^1$, whose invertible sheaves are classified as $\mathcal{L}_{p,q}$, $p, q \in \mathbb{Z}$. We consider a projection from a point of $\mathbb{P}^3$ to a plane, whose restriction to Q is denoted by π. It is known that

$$\dim_\kappa \mathrm{H}^r\left(\mathbb{P}^2, \pi_*\mathcal{L}_{p,q}\right) = (-1)^r(p+1)(q+1)$$

if $r = 0$ and $p, q \geq 0$; or if $r = 1$ and $p \geq 0$, $q < 0$ or $p < 0$, $q \geq 0$; or if $r = 2$ and $p, q < 0$; and is zero otherwise [6, Proposition 12]. The module structure of injective complexes defining sheaf cohomology is subtle. Our goal is to analyze $\mathrm{H}^r(\mathbb{P}^2, \pi_*\mathcal{L}_{p,q})$ in terms of elements of $(\pi_*\mathcal{L}_{p,q})^{(r)}$ to reveal its combinatorial nature.

Usually, the word "basis" stands for a minimal generating set of a free module. However, a set may have different module structures. To avoid confusion, we reserve the term only for a minimal generating set of a κ-vector space in this paper.

This paper is organized as follows.

- Section 2 recalls the construction of $\mathcal{F}^{(\bullet)}$ for a locally free sheaf $\mathcal{F}$ on the projective plane.
- Section 3 describes locally free sheaves $\mathcal{F}_{p,q}$ obtained from a double cover of the projective plane.
- Section 4 applies the construction of Section 2 to $\mathcal{F}_{p,q}$.
- Section 5 analyzes the module structure of $\mathcal{F}^{(2)}_{p,q}$.
- Section 6 gives bases of $\mathrm{H}^i(\mathbb{P}^2, \mathcal{F}_{p,q})$.

2. Complex for computing cohomology

Let $\mathcal{F}$ be a locally free sheaf of finite rank on $\mathbb{P}^n$. We recall the construction of the complex $\mathcal{F}^{(\bullet)}$ for the case $n = 2$. Given $\mathfrak{p} \in \mathbb{P}^2 = \mathrm{Proj}(\kappa[T_0, T_1, T_2])$, the local cohomology module

$$M(\mathfrak{p}) := \mathrm{H}^{\mathrm{ht}\,\mathfrak{p}}_{\mathfrak{m}_\mathfrak{p}}\left(\bigwedge^2 \Omega_{\mathcal{O}_{\mathbb{P}^2,\mathfrak{p}}/\kappa}\right)$$

supported at the maximal ideal $\mathfrak{m}_\mathfrak{p}$ of $\mathcal{O}_{\mathbb{P}^2,\mathfrak{p}}$ is an injective hull of the residue field $\kappa_\mathfrak{p}$ of $\mathcal{O}_{\mathbb{P}^2,\mathfrak{p}}$. Elements of $M(\mathfrak{p})$ can be written as generalized fractions, which we referred to [2, Chapter 2] or [5, §7]. We recall three special cases of $M(\mathfrak{p})$ needed for defining $\mathcal{F}^{(i)}$.

EXAMPLE 1.

- If $\mathfrak{p}$ is the generic point of $\mathbb{P}^2$, we write $M(\mathbb{P}^2)$ for $M(\mathfrak{p})$. Elements of $M(\mathbb{P}^2)$ are of the form

$$\frac{f}{g}\, d\frac{T_0}{T_2}\, d\frac{T_1}{T_2},$$

where $f \in \kappa[T_0/T_2, T_1/T_2]$ and $g \in \kappa[T_0/T_2, T_1/T_2] \setminus (0)$.

- If $\mathfrak{p}$ is the generic point of the line $T_2 = 0$, we write $M(\mathbb{P}^1)$ for $M(\mathfrak{p})$. Elements of $M(\mathbb{P}^1)$ are of the form

$$\begin{bmatrix} \dfrac{f}{g}\, d\dfrac{T_2}{T_1}\, d\dfrac{T_0}{T_1} \\ \left(\dfrac{T_2}{T_1}\right)^i \end{bmatrix}, \tag{1}$$

where $f \in \kappa[T_2/T_1, T_0/T_1]$ and $g \in \kappa[T_2/T_1, T_0/T_1] \setminus (T_2/T_1)$.

- If $\mathfrak{p}$ is the closed point $T_2 = T_1 = 0$, we write $M(\mathbb{P}^0)$ for $M(\mathfrak{p})$. Elements of $M(\mathbb{P}^0)$ are of the form

$$\begin{bmatrix} \dfrac{f}{g}\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{bmatrix}, \tag{2}$$

where $f \in \kappa[T_1/T_0, T_2/T_0]$ and $g \in \kappa[T_1/T_0, T_2/T_0] \setminus (T_1/T_0, T_2/T_0)$.

$M(\mathfrak{p})$, being an injective hull of $\kappa_{\mathfrak{p}}$, is also a module over the completion $\mathcal{O}^{\wedge}_{\mathbb{P}^2,\mathfrak{p}}$ of $\mathcal{O}_{\mathbb{P}^2,\mathfrak{p}}$. This can be seen from the following properties of generalized fractions.

Proposition 2 (Linearity Law).

$$\begin{bmatrix} \left(\dfrac{f_1}{g_1} + \dfrac{f_2}{g_2}\right) d\dfrac{T_2}{T_1}\, d\dfrac{T_0}{T_1} \\ \left(\dfrac{T_2}{T_1}\right)^i \end{bmatrix} = \begin{bmatrix} \dfrac{f_1}{g_1}\, d\dfrac{T_2}{T_1}\, d\dfrac{T_0}{T_1} \\ \left(\dfrac{T_2}{T_1}\right)^i \end{bmatrix} + \begin{bmatrix} \dfrac{f_2}{g_2}\, d\dfrac{T_2}{T_1}\, d\dfrac{T_0}{T_1} \\ \left(\dfrac{T_2}{T_1}\right)^i \end{bmatrix},$$

$$\begin{bmatrix} \left(\dfrac{f_1}{g_1} + \dfrac{f_2}{g_2}\right) d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{bmatrix} = \begin{bmatrix} \dfrac{f_1}{g_1}\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{bmatrix} + \begin{bmatrix} \dfrac{f_2}{g_2}\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{bmatrix}.$$

Proposition 3 (Vanishing Law). *If* $f \in (T_2/T_1)^i$,

$$\begin{bmatrix} \dfrac{f}{g}\, d\dfrac{T_2}{T_1}\, d\dfrac{T_0}{T_1} \\ \left(\dfrac{T_2}{T_1}\right)^i \end{bmatrix} = 0.$$

If f is contained in the ideal generated by $(T_1/T_0)^i$ and $(T_2/T_0)^j$, then

$$\left[\begin{array}{c} \dfrac{f}{g}\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{array}\right] = 0.$$

Denominators of generalized fractions ($(T_2/T_1)^i$ in (1) and $(T_1/T_0)^i, (T_2/T_0)^j$ in (2)) can be any system of parameters of $\mathcal{O}_{\mathbb{P}^2,\mathfrak{p}}$. The relations of generalized fractions in different system of parameters are given by the transformation law, which we refer to [2, Lemma 2.3.ii] or [5, Lemma 7.2.b]. Elements of $M(\mathfrak{p})$ represented by generalized fractions are convenient to handle.

EXAMPLE 4. Elements of $M(\mathbb{P}^0)$ can be written as

$$\left[\begin{array}{c} h\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{array}\right],$$

where $h \in \kappa[T_1/T_0, T_2/T_0]$.

Proof. Write f/g in (2) as $f_0/(1-g_0)$, where $f_0 \in \kappa[T_1/T_0, T_2/T_0]$ and $g_0 \in (T_1/T_0, T_2/T_0)$.

$$\frac{1}{1-g_0} - \left(1 + g_0 + g_0^2 + \cdots + g_0^{i+j-2}\right)$$

is contained in the ideal generated by $(T_1/T_0)^i$ and $(T_2/T_0)^j$. Let

$$h = f_0\left(1 + g_0 + g_0^2 + \cdots + g_0^{i+j-2}\right).$$

By the linearity law and the vanishing law,

$$\left[\begin{array}{c} \dfrac{f}{g}\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{array}\right] = \left[\begin{array}{c} h\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{array}\right] + \left[\begin{array}{c} \left(\dfrac{f_0}{1-g_0} - h\right) d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{array}\right]$$
$$= \left[\begin{array}{c} h\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^i, \left(\dfrac{T_2}{T_0}\right)^j \end{array}\right]. \qquad \square$$

Let $J(\mathfrak{p})$ be the quasi-coherent $\mathcal{O}_{\mathbb{P}^2}$-module which is the constant sheaf $M(\mathfrak{p})$ on $\{\mathfrak{p}\}^-$, and zero elsewhere. We write $J(\mathbb{P}^2)$ (resp. $J(C)$) for $J(\mathfrak{p})$ if $\mathfrak{p}$ is the generic point of $\mathbb{P}^2$ (resp. a curve C). In [3, 4], a residual complex

$$J(\mathbb{P}^2) \to \bigoplus_{\text{curves}} J(C) \to \bigoplus_{\text{closed points}} J(\mathfrak{m}) \to 0 \tag{3}$$

on $\mathbb{P}^2$ is described. (3) is an injective resolution of $\mathcal{O}_{\mathbb{P}^2}(-3)$. Tensoring with $\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}^2}(3)$, we get an injective resolution

$$\mathcal{F} \otimes J(\mathbb{P}^2)(3) \to \mathcal{F} \otimes \left(\bigoplus J(C)\right)(3) \to \mathcal{F} \otimes \left(\bigoplus J(\mathfrak{m})\right)(3) \to 0$$

of $\mathcal{F}$. By definition, the cohomology of the complex

$$\begin{aligned}&\Gamma\left(\mathbb{P}^2, \mathcal{F} \otimes J\left(\mathbb{P}^2\right)(3)\right) \to \Gamma\left(\mathbb{P}^2, \mathcal{F} \otimes \left(\bigoplus J(C)\right)(3)\right)\\ &\to \Gamma\left(\mathbb{P}^2, \mathcal{F} \otimes \left(\bigoplus J(\mathfrak{m})\right)(3)\right) \to 0\end{aligned} \tag{4}$$

is the cohomology of $\mathcal{F}$. It was observed in [4] that a subcomplex $\mathcal{F}^{(\bullet)}$ of (4) is quasi-isomorphic to (4).

DEFINITION 5. Let $\{u_i\}$ (resp. $\{v_i\}$ and $\{w_i\}$) be a minimal generating set for the free module $\mathcal{F}(D_+(T_2))$ (resp. $\mathcal{F}(D_+(T_1))$ and $\mathcal{F}(D_+(T_0))$) over $\mathcal{O}_{\mathbb{P}^2}(D_+(T_2))$ (resp. $\mathcal{O}_{\mathbb{P}^2}(D_+(T_1))$ and $\mathcal{O}_{\mathbb{P}^2}(D_+(T_0))$). We define $\mathcal{F}^{(0)}$ to be the submodule of $\Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^2)(3)) = \Gamma(D_+(T_2), \mathcal{F} \otimes J(\mathbb{P}^2)(3))$ generated by

$$u_i \otimes d\frac{T_0}{T_2}\, d\frac{T_1}{T_2} \otimes T_2^3.$$

We define $\mathcal{F}^{(1)}$ to be the submodule of $\Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^1)(3)) = \Gamma(D_+(T_1), \mathcal{F} \otimes J(\mathbb{P}^1)(3))$ generated by

$$v_i \otimes \begin{bmatrix} d\frac{T_2}{T_1}\, d\frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^j \end{bmatrix} \otimes T_1^3 \quad (j \in \mathbb{N}). \tag{5}$$

We define $\mathcal{F}^{(2)}$ to be the submodule of $\Gamma(\mathbb{P}^2, \mathcal{F} \otimes J(\mathbb{P}^0)(3)) = \Gamma(D_+(T_0), \mathcal{F} \otimes J(\mathbb{P}^0)(3))$ generated by

$$w_i \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^j, \left(\frac{T_2}{T_0}\right)^k \end{bmatrix} \otimes T_0^3 \quad (j, k \in \mathbb{N}).$$

Assume that $\mathcal{F}$ has rank n. Then $\mathcal{F}^{(i)}$ is isomorphic to n copies of $\mathcal{O}^{(i)}_{\mathbb{P}^2}$. As κ-vector spaces, $\mathcal{F}^{(0)}$ has a basis

$$\left\{ u_i \otimes \left(\frac{T_0}{T_2}\right)^j \left(\frac{T_1}{T_2}\right)^k d\frac{T_0}{T_2}\, d\frac{T_1}{T_2} \otimes T_2^3 \;\middle|\; 1 \le i \le n \text{ and } 0 \le j, k \right\}, \tag{6}$$

and $\mathcal{F}^{(1)}$ has a basis

$$\left\{ v_i \otimes \begin{bmatrix} \left(\frac{T_0}{T_1}\right)^j d\frac{T_2}{T_1}\, d\frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^k \end{bmatrix} \otimes T_1^3 \;\middle|\; 1 \le i \le n,\ 0 \le j \text{ and } 0 < k \right\}, \tag{7}$$

and $\mathcal{F}^{(2)}$ has a basis

$$\left\{ w_i \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^j, \left(\frac{T_2}{T_0}\right)^k \end{bmatrix} \otimes T_0^3 \;\middle|\; 1 \le i \le n \text{ and } 0 < j, k \right\}. \tag{8}$$

The coboundary maps of the residual complex (3) are decomposed into

$$\delta_{\mathfrak{p},\mathfrak{q}} : J(\mathfrak{p}) \to J(\mathfrak{q})$$

for $\mathfrak{p}, \mathfrak{q} \in \mathbb{P}^2$. We recall two special cases of $\delta_{\mathfrak{p},\mathfrak{q}}$ needed for defining the coboundary maps of $\mathcal{F}^{(\bullet)}$.

EXAMPLE 6.

- Let $\mathfrak{p}$ be the generic point of $\mathbb{P}^2$ and $\mathfrak{q}$ be the generic point of the line $T_2 = 0$. $\delta_{\mathbb{P}^2,\mathbb{P}^1} := \delta_{\mathfrak{p},\mathfrak{q}}$ is determined by the map $M(\mathbb{P}^2) \to M(\mathbb{P}^1)$ satisfying

$$\frac{f}{g}\, d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \mapsto \begin{bmatrix} f\, d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ g \end{bmatrix}, \tag{9}$$

where $f \in \kappa[T_1/T_0, T_2/T_0]$ and $g \in \kappa[T_1/T_0, T_2/T_0] \setminus (0)$.

- Let $\mathfrak{p}$ be the generic point of the line $T_2 = 0$ and $\mathfrak{q}$ be the closed point $T_2 = T_1 = 0$. $\delta_{\mathbb{P}^1,\mathbb{P}^0} := \delta_{\mathfrak{p},\mathfrak{q}}$ is determined by the map $M(\mathbb{P}^1) \to M(\mathbb{P}^0)$ satisfying

$$\begin{bmatrix} \frac{f}{g}\, d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^i \end{bmatrix} \mapsto \begin{bmatrix} f\, d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ g, \left(\frac{T_2}{T_0}\right)^i \end{bmatrix} \tag{10}$$

where $f \in \kappa[T_1/T_0, T_2/T_0]$ and $g \in \kappa[T_1/T_0, T_2/T_0] \setminus (T_2/T_0)$.

In Example 1, elements of $M(\mathbb{P}^2)$ (resp. $M(\mathbb{P}^1)$) are represented in terms of T_0/T_2 and T_1/T_2 (resp. T_2/T_1 and T_0/T_1). We may use the formula

$$(11)\qquad \begin{gathered} T_2^3 d\frac{T_0}{T_2} d\frac{T_1}{T_2} = T_0^3 d\frac{T_1}{T_0} d\frac{T_2}{T_0}, \\ \begin{bmatrix} d\frac{T_2}{T_1}\, d\frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^3 \end{bmatrix} = \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^3 \end{bmatrix} \end{gathered}$$

to rewrite elements of $M(\mathbb{P}^2)$ and $M(\mathbb{P}^1)$ before applying (9) and (10).

For $i = 0, 1$, the image of $\mathcal{F}^{(1-i)}$ under the map

$$\left(\mathrm{id}_{\mathcal{F}} \otimes \delta_{\mathbb{P}^{i+1},\mathbb{P}^i} \otimes \mathrm{id}_{\mathcal{O}_{\mathbb{P}^2}(3)}\right)\left(\mathbb{P}^2\right) : \Gamma\left(\mathbb{P}^2, \mathcal{F} \otimes J\left(\mathbb{P}^{i+1}\right)(3)\right) \to \Gamma\left(\mathbb{P}^2, \mathcal{F} \otimes J\left(\mathbb{P}^{i}\right)(3)\right)$$

is contained in $\mathcal{F}^{(2-i)}$.

DEFINITION 7. For $i = 0, 1$, let $d^{(1-i)} : \mathcal{F}^{(1-i)} \to \mathcal{F}^{(2-i)}$ be the restriction of $\left(\mathrm{id}_{\mathcal{F}} \otimes \delta_{\mathbb{P}^{i+1},\mathbb{P}^i} \otimes \mathrm{id}_{\mathcal{O}_{\mathbb{P}^2}(3)}\right)(\mathbb{P}^2)$ on $\mathcal{F}^{(1-i)}$.

To make $d^{(1-i)}$ explicit, we consider $\mathrm{id}_{\mathcal{F}} \otimes \delta_{\mathbb{P}^{i+1},\mathbb{P}^i} \otimes \mathrm{id}_{\mathcal{O}_{\mathbb{P}^2}(3)}$ on $D_+(T_0)$. Restricted to $D_+(T_2) \cap D_+(T_0)$,

$$u_i = \sum_j \frac{f_{ij}}{(T_2/T_0)^{n_{ij}}} w_i$$

for some $f_{ij} \in \kappa[T_1/T_0, T_2/T_0]$ and $n_{ij} \geq 0$. In terms of these transition functions,

$$\begin{aligned} d^0\left(u_i \otimes d\frac{T_0}{T_2}\, d\frac{T_1}{T_2} \otimes T_2^3\right) &= d^0\left(\sum_j w_i \otimes \frac{f_{ij}}{(T_2/T_0)^{n_{ij}}}\, d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \otimes T_0^3\right) \\ &= \sum_j w_i \otimes \begin{bmatrix} f_{ij}\, d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^{n_{ij}} \end{bmatrix} \otimes T_0^3. \end{aligned}$$

We may use (11) to write the image of $d^{(0)}$ in terms of the generators (5) of $\mathcal{F}^{(1)}$. Restricted to $D_+(T_1) \cap D_+(T_0)$,

$$v_i = \sum_j \frac{h_{ij}}{(T_1/T_0)^{n_{ij}}} w_i$$

for some $n_{ij} \geq 0$ and $h_{ij} \in \kappa[T_1/T_0, T_2/T_0]$. In terms of these transition functions,

$$d^{(1)}\left(v_i \otimes \begin{bmatrix} d\dfrac{T_2}{T_1}\, d\dfrac{T_0}{T_1} \\ \left(\dfrac{T_2}{T_1}\right)^l \end{bmatrix} \otimes T_1^3\right)$$

$$= d^{(1)}\left(\sum_j w_i \otimes \begin{bmatrix} h_{ij}\left(\dfrac{T_1}{T_0}\right)^{l-n_{ij}} d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_2}{T_0}\right)^l \end{bmatrix} \otimes T_0^3\right)$$

$$= \sum_j w_i \otimes \begin{bmatrix} h_{ij}\, d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^{n_{ij}-l}, \left(\dfrac{T_2}{T_0}\right)^l \end{bmatrix} \otimes T_0^3.$$

The following is our main tool.

Theorem 8 ([4, Theorem 3.2]). *The i-th cohomology of $\mathcal{F}^{(\bullet)}$ is isomorphic to* $\mathrm{H}^i(\mathbb{P}^2, \mathcal{F})$.

3. Vector bundles $\mathcal{F}_{p,q}$

Let S be the graded ring $\kappa[X_0, X_1, X_2, X_3]/(X_0X_1 - X_2X_3)$ over a field κ. Denote by x_i the image of X_i under the canonical map $\kappa[X_0, X_1, X_2, X_3] \to S$. So, as a κ-algebra, S is generated by x_0, x_1, x_2, x_3 with a relation $x_0x_1 = x_2x_3$. Proj(S) is a hypersurface of $\mathbb{P}^3$ covered by three affine open sets:

$$\mathrm{Proj}(S) = D_+(x_3) \cup D_+(x_2) \cup D_+(x_1 - x_0).$$

On $D_+(x_3)$ and $D_+(x_2)$, the regular functions of Proj(S) form polynomial rings $\kappa[x_0/x_3, x_1/x_3]$ and $\kappa[x_0/x_2, x_1/x_2]$, respectively. On $D_+(x_1 - x_0)$, its regular functions are

$$\kappa\left[\frac{x_1}{x_1-x_0}, \frac{x_2}{x_1-x_0}, \frac{x_3}{x_1-x_0}\right] \Big/ \left(\left(\frac{x_1}{x_1-x_0}\right)^2 - \frac{x_1}{x_1-x_0} - \frac{x_2}{x_1-x_0}\frac{x_3}{x_1-x_0}\right).$$

We identify Proj(S) with the fiber product of two projective lines, which can be described using a Cartesian product (that is, the scheme Proj$(\kappa[Y_0, Y_1] \times_\kappa \kappa[Z_0, Z_1])$). The identification is given by the homomorphism of κ-algebras

$$\begin{aligned} \kappa[x_0, x_1, x_2, x_3] &\to \kappa[Y_0, Y_1] \times_\kappa \kappa[Z_0, Z_1], \\ x_0 &\mapsto Y_0Z_0, \\ x_1 &\mapsto Y_1Z_1, \end{aligned}$$

$$x_2 \mapsto Y_1 Z_0,$$
$$x_3 \mapsto Y_0 Z_1.$$

Let π_1 and π_2 be the two projections from Proj(S) to $\mathbb{P}^1$. For $p, q \in \mathbb{Z}$,

$$\mathcal{L}_{p,q} := \pi_1^* \mathcal{O}(p) \otimes \pi_2^* \mathcal{O}(q)$$

is an invertible sheaf on Proj(S), which is the sheaf associated to the graded module

$$\kappa[Y_0, Y_1](p) \times_\kappa \kappa[Z_0, Z_1](q).$$

On $D_+(x_3)$, $\mathcal{L}_{p,q}$ is generated by $Y_0^p Z_1^q$. On $D_+(x_2)$, it is generated by $Y_1^p Z_0^q$.

Proposition 9. *Let* $\epsilon \geq \max\{0, -p, -q\}$. $\mathcal{L}_{p,q}(D_+(x_1 - x_0))$ *is generated by* $Y_0^{\epsilon+p} Z_0^{\epsilon+q}/(x_1 - x_0)^\epsilon$ *and* $Y_1^{\epsilon+p} Z_1^{\epsilon+q}/(x_1 - x_0)^\epsilon$.

Proof. $\mathcal{L}_{p,q}(D_+(x_1 - x_0))$ is generated by $Y_0^i Y_1^j Z_0^k Z_1^l/(x_1 - x_0)^n$, where the indices $i, j, k, l, n \geq 0$ satisfy $i + j = n + p$ and $k + l = n + q$. Restricting to $D_+(x_1) \cap D_+(x_1 - x_0)$,

$$\frac{Y_0^i Y_1^j Z_0^k Z_1^l}{(x_1 - x_0)^n} = \left(\frac{x_1}{x_1 - x_0}\right)^{n-i-k-\epsilon} \left(\frac{x_2}{x_1 - x_0}\right)^k \left(\frac{x_3}{x_1 - x_0}\right)^i \frac{Y_1^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^\epsilon}.$$

Restricting to $D_+(x_0) \cap D_+(x_1 - x_0)$,

$$\frac{Y_0^i Y_1^j Z_0^k Z_1^l}{(x_1 - x_0)^n} = \left(\frac{x_0}{x_1 - x_0}\right)^{n-j-l-\epsilon} \left(\frac{x_2}{x_1 - x_0}\right)^j \left(\frac{x_3}{x_1 - x_0}\right)^l \frac{Y_0^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1 - x_0)^\epsilon}.$$

Since $D_+(x_1 - x_0)$ is covered by the subsets $D_+(x_1) \cap D_+(x_1 - x_0)$ and $D_+(x_0) \cap D_+(x_1 - x_0)$, $\mathcal{L}_{p,q}(D_+(x_1 - x_0))$ is generated by $Y_0^{\epsilon+p} Z_0^{\epsilon+q}/(x_1 - x_0)^\epsilon$ and $Y_1^{\epsilon+p} Z_1^{\epsilon+q}/(x_1 - x_0)^\epsilon$. □

Let $\mathcal{O}$ be the point of $\mathbb{P}^3 \setminus$ Proj(S) with homogeneous coordinate $[1, 1, 0, 0]$. Let

$$\pi : \text{Proj}(S) \to \mathbb{P}^2$$

be the double cover of $\mathbb{P}^2$ defined by the immersion Proj(S) $\to \mathbb{P}^3 \setminus \{\mathcal{O}\}$ followed by the projection from $\mathcal{O}$ to the plane $X_0 = 0$, which is identified with $\mathbb{P}^2 = \text{Proj}(\kappa[T_0, T_1, T_2])$. The morphism π is determined by the graded homomorphism

$$\kappa[T_0, T_1, T_2] \to \kappa[x_0, x_1, x_2, x_3]$$

given by

$$T_0 \mapsto x_1 - x_0,$$

$$T_1 \mapsto x_2,$$
$$T_2 \mapsto x_3.$$

We consider the locally free sheaf of modules

$$\mathcal{F}_{p,q} := \pi_* \mathcal{L}_{p,q}$$

on $\mathbb{P}^2$, which has rank 2. On $D_+(T_2)$, $\mathcal{F}_{p,q}$ is generated by $(x_0/x_3)Y_0^p Z_1^q$ and $Y_0^p Z_1^q$. On $D_+(T_1)$, it is generated by $(x_0/x_2)Y_1^p Z_0^q$ and $Y_1^p Z_0^q$.

Proposition 10. *Let $\epsilon = \max\{0, -p, -q\}$. $\mathcal{F}_{p,q}(D_+(T_0))$ is generated by $Y_0 Z_0/T_0^{\epsilon+1}$ and $Y_1 Z_1/T_0^{\epsilon+1}$ if $p = q \leq 0$, otherwise by $Y_0^{\epsilon+p} Z_0^{\epsilon+q}/T_0^{\epsilon}$ and $Y_1^{\epsilon+p} Z_1^{\epsilon+q}/T_0^{\epsilon}$.*

Proof. $\mathcal{F}_{p,q}(D_+(T_0))$ is generated by $Y_0^i Y_1^j Z_0^k Z_1^l / T_0^n$, where $i, j, k, l, n \geq 0$ satisfy $i + j = n + p$ and $k + l = n + q$. Note that, if j and k are both positive, then

$$\frac{Y_0^i Y_1^j Z_0^k Z_1^l}{T_0^n} = \frac{T_1}{T_0} \frac{Y_0^i Y_1^j Z_0^{k-1} Z_1^{l+1}}{T_0^n} - \frac{T_1}{T_0} \frac{Y_0^{i+1} Y_1^{j-1} Z_0^k Z_1^l}{T_0^n}; \tag{12}$$

if i and l are both positive, then

$$\frac{Y_0^i Y_1^j Z_0^k Z_1^l}{T_0^n} = \frac{T_2}{T_0} \frac{Y_0^{i-1} Y_1^{j+1} Z_0^k Z_1^l}{T_0^n} - \frac{T_2}{T_0} \frac{Y_0^i Y_1^j Z_0^{k+1} Z_1^{l-1}}{T_0^n}. \tag{13}$$

Assume that $n > \epsilon$. Then $n, n + p, n + q > 0$ and

$$\frac{Y_0^i Y_1^j Z_0^k Z_1^l}{T_0^n} = \begin{cases} \dfrac{T_1}{T_0} \dfrac{Y_0^i Y_1^{j-1} Z_0^{k-1} Z_1^l}{T_0^{n-1}}, & \text{if} \quad j, k > 0; \\[2ex] \dfrac{T_2}{T_0} \dfrac{Y_0^{i-1} Y_1^j Z_0^k Z_1^{l-1}}{T_0^{n-1}}, & \text{if} \quad i, l > 0, \end{cases} \tag{14}$$

$$\frac{Y_0^{n+p} Z_0^{n+q}}{T_0^n} = \frac{Y_0^{n+p-1} Y_1 Z_0^{n+q-1} Z_1}{T_0^n} - \frac{Y_0^{n+p-1} Z_0^{n+q-1}}{T_0^{n-1}}, \tag{15}$$

$$\frac{Y_1^{n+p} Z_1^{n+q}}{T_0^n} = \frac{Y_0 Y_1^{n+p-1} Z_0 Z_1^{n+q-1}}{T_0^n} + \frac{Y_1^{n+p-1} Z_1^{n+q-1}}{T_0^{n-1}}. \tag{16}$$

We consider first the case $p \neq q$ or $p = q > 0$, in which either $n + p - 1 > 0$ or $n + q - 1 > 0$. Using (14), (15) and (16), induction on n shows that $\mathcal{F}_{p,q}(D_+(T_0))$ is generated by $Y_0^i Y_1^j Z_0^k Z_1^l / T_0^{\epsilon}$, where $i, j, k, l \geq 0$ satisfy $i + j = \epsilon + p$ and $k + l = \epsilon + q$. Applying (12) and (13) with $n = \epsilon$, we see that $\mathcal{F}_{p,q}(D_+(T_0))$ is generated by $Y_0^{\epsilon+p} Z_0^{\epsilon+q}/T_0^{\epsilon}$ and $Y_1^{\epsilon+p} Z_1^{\epsilon+q}/T_0^{\epsilon}$.

Now we consider the case $p = q \leq 0$. Assume that $n > \epsilon + 1$. In this case, $n + p - 1 = n + q - 1 > 0$. Using (14), (15) and (16), induction on n shows that $\mathcal{F}_{p,q}(D_+(T_0))$

is generated by $1/T_0^\epsilon$ and $Y_0^i Y_1^j Z_0^k Z_1^l / T_0^{\epsilon+1}$, where $i, j, k, l \geq 0$ satisfy $i + j = \epsilon + p + 1$ and $k + l = \epsilon + q + 1$. Applying (12) and (13) with $n = \epsilon + 1$, we see that $\mathcal{F}_{p,q}(D_+(T_0))$ is generated by $Y_0 Z_0 / T_0^{\epsilon+1}$, $Y_1 Z_1 / T_0^{\epsilon+1}$ and $1/T_0^\epsilon$. The proposition follows from the identity

$$\frac{Y_1 Z_1}{T_0^{\epsilon+1}} - \frac{Y_0 Z_0}{T_0^{\epsilon+1}} = \frac{1}{T_0^\epsilon}. \qquad \square$$

4. Complexes $\mathcal{F}_{p,q}^{(\bullet)}$

From now on, we always assume that $\epsilon = \max\{0, -p, -q\}$. First we would like to write down bases of the κ-vector spaces $\mathcal{F}_{p,q}^{(i)}$ explicitly.

DEFINITION 11. For $i, j \geq 0$, we define

$$\mathbf{u}^{ij} := \left(\frac{x_0}{x_3}\right)^i \left(\frac{x_1}{x_3}\right)^j Y_0^p Z_1^q \otimes d\frac{T_0}{T_2}\, d\frac{T_1}{T_2} \otimes T_2^3 \in \Gamma\left(\mathbb{P}^2, \mathcal{F}_{p,q} \otimes J\left(\mathbb{P}^2\right)(3)\right).$$

For $i, j, m, n \in \mathbb{Z}$, we choose $\delta \geq \max\{-i, -j\}$ and define

$$\mathbf{v}_n^{ij} := \left(\frac{x_0}{x_2}\right)^{\delta+i} \left(\frac{x_1}{x_2}\right)^{\delta+j} Y_1^p Z_0^q \otimes \begin{bmatrix} d\dfrac{T_2}{T_1}\, d\dfrac{T_0}{T_1} \\ \left(\dfrac{T_2}{T_1}\right)^{\delta+n} \end{bmatrix} \otimes T_1^3$$

in $\Gamma(\mathbb{P}^2, \mathcal{F}_{p,q} \otimes J(\mathbb{P}^1)(3))$ and

$$\mathbf{w}_{mn}^{ij} := \left(\frac{x_0}{x_1 - x_0}\right)^{\delta+i} \left(\frac{x_1}{x_1 - x_0}\right)^{\delta+j} \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^\epsilon} \otimes \begin{bmatrix} d\dfrac{T_1}{T_0}\, d\dfrac{T_2}{T_0} \\ \left(\dfrac{T_1}{T_0}\right)^{\delta+m}, \left(\dfrac{T_2}{T_0}\right)^{\delta+n} \end{bmatrix} \otimes T_0^3$$

in $\Gamma(\mathbb{P}^2, \mathcal{F}_{p,q} \otimes J(\mathbb{P}^0)(3))$.

The definitions of $\mathbf{v}_n^{ij}$ and $\mathbf{w}_{mn}^{ij}$ are independent of the choice of δ. By Proposition 3, $\mathbf{v}_0^{ij} = 0$ if $i, j \geq 0$ and $\mathbf{w}_{m(n+\epsilon+q)}^{(\epsilon+q)0} = \mathbf{w}_{m(n+\epsilon+p)}^{0(\epsilon+p)} = 0$ if $m \leq 0$ or $n \leq 0$. Sometimes, $\mathbf{w}_{mn}^{ij}$ are treated differently according to the values of p and q. The following notations are handy.

$$\mathbf{w}_{mn}^{\geq} := \begin{cases} \mathbf{w}_{m(n+\epsilon+q)}^{10} & \text{if} \quad p = q \leq 0; \\ \mathbf{w}_{m(n+\epsilon+q)}^{(\epsilon+q)0} & \text{otherwise.} \end{cases}$$

$$\mathbf{w}_{mn}^{\leq} := \begin{cases} \mathbf{w}_{m(n+\epsilon+p)}^{01} & \text{if} \quad p = q \leq 0; \\ \mathbf{w}_{m(n+\epsilon+p)}^{0(\epsilon+p)} & \text{otherwise.} \end{cases}$$

Proposition 12.

- *The elements* $\mathbf{u}^{ij}$*, where* $i, j \geq 0$*, form a basis of* $\mathcal{F}^{(0)}_{p,q}$.
- *The elements* $\mathbf{v}^{ij}_0$*, where* $i < 0$ *or* $j < 0$*, form a basis of* $\mathcal{F}^{(1)}_{p,q}$.
- *The elements* $\mathbf{w}^{\geq}_{mn}$ *and* $\mathbf{w}^{\leq}_{mn}$*, where* $m, n > 0$*, form a basis of* $\mathcal{F}^{(2)}_{p,q}$.

Proof. As an $\mathcal{O}_{\mathbb{P}^2}(D_+(T_2))$-module, $\mathcal{F}_{p,q}(D_+(T_2))$ has a minimal generating set $\{Y_0^p Z_1^q, (x_0/x_3)Y_0^p Z_1^q\}$. Indicated in (6), as a κ-vector space, $\mathcal{F}^{(0)}_{p,q}$ has a basis consisting of

$$Y_0^p Z_1^q \otimes \left(\frac{T_0}{T_2}\right)^i \left(\frac{T_1}{T_2}\right)^j d\frac{T_0}{T_2}\, d\frac{T_1}{T_2} \otimes T_2^3 \quad \text{and} \quad \frac{x_0}{x_3} Y_0^p Z_1^q \otimes \left(\frac{T_0}{T_2}\right)^i \left(\frac{T_1}{T_2}\right)^j d\frac{T_0}{T_2}\, d\frac{T_1}{T_2} \otimes T_2^3,$$

where $i, j \geq 0$. Since $\kappa[x_0/x_3, x_1/x_3]$ is freely generated by 1 and x_0/x_3 as a $\kappa[T_0/T_2, T_1/T_2]$-module, these elements are exactly u_{ij}, where $i, j \geq 0$.

For the second statement of the proposition, we use the fact that

$$\mathbf{v}^{(i+1)(j+1)}_{n+1} = \mathbf{v}^{ij}_n$$

for any i, j and n. Since $\mathcal{F}^{(1)}_{p,q}$ is generated by all $\mathbf{v}^{ij}_n$, it is also generated by those $\mathbf{v}^{00}_n$, $\mathbf{v}^{i0}_n$ and $\mathbf{v}^{0j}_n$ with $i, j > 0$ and $n \in \mathbb{Z}$. Note that $\mathbf{v}^{00}_n = \mathbf{v}^{i0}_n = \mathbf{v}^{0j}_n = 0$ if $i, j > 0$ and $n \leq 0$ by Proposition 3. The generating set $\{\mathbf{v}^{00}_n, \mathbf{v}^{i0}_n, \mathbf{v}^{0j}_n \mid i, j, n > 0\}$ for $\mathcal{F}^{(1)}_{p,q}$ is exactly $\{\mathbf{v}^{ij}_0 \mid i < 0 \text{ or } j < 0\}$. To prove that they are linearly independent, we recall (7) that

$$\left\{ \left(\frac{T_0}{T_1}\right)^j \mathbf{v}^{00}_n,\ \left(\frac{T_0}{T_1}\right)^j \mathbf{v}^{10}_n \;\middle|\; n > 0,\ \ j \geq 0 \right\}$$

is a basis of $\mathcal{F}^{(1)}_{p,q}$. For $i, j, n > 0$,

$$\mathbf{v}^{0j}_n - \left(\frac{T_0}{T_1}\right)^j \mathbf{v}^{00}_n - \left(\frac{T_0}{T_1}\right)^{j-1} \mathbf{v}^{10}_n \quad \text{and} \quad \mathbf{v}^{i0}_n - (-1)^{i-1} \left(\frac{T_0}{T_1}\right)^{i-1} \mathbf{v}^{10}_n$$

are contained in the subspace generated by those $\mathbf{v}^{ij}_m$ with $m < n$ and $i, j \geq 0$. This implies that $\mathbf{v}^{00}_n$, $\mathbf{v}^{i0}_n$ and $\mathbf{v}^{0j}_n$ are linearly independent.

For the last statement of the proposition, there are two cases. If $p = q \leq 0$, the elements

$$\frac{Y_0 Z_0}{T_0^{\epsilon+1}} \otimes \left[\frac{d\frac{T_1}{T_0}\, d\frac{T_2}{T_0}}{\left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^n} \right] \otimes T_0^3 \quad \text{and} \quad \frac{Y_1 Z_1}{T_0^{\epsilon+1}} \otimes \left[\frac{d\frac{T_1}{T_0}\, d\frac{T_2}{T_0}}{\left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^n} \right] \otimes T_0^3,$$

where $m, n > 0$, form a basis of $\mathcal{F}^{(2)}_{p,q}$. These elements are exactly $\mathbf{w}^{10}_{m(n+\epsilon+q)}$ and $\mathbf{w}^{01}_{m(n+\epsilon+p)}$. If $p \neq q$ or $p = q > 0$, the elements

$$\frac{Y_0^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1 - x_0)^{\epsilon}} \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^n \end{bmatrix} \otimes T_0^3$$

and

$$\frac{Y_1^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^{\epsilon}} \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^n \end{bmatrix} \otimes T_0^3,$$

where $m, n > 0$, form a basis of $\mathcal{F}^{(2)}_{p,q}$. These elements are exactly $\mathbf{w}^{(\epsilon+q)0}_{m(n+\epsilon+q)}$ and $\mathbf{w}^{0(\epsilon+p)}_{m(n+\epsilon+p)}$ as seen from the computation:

$$\begin{aligned} &\frac{Y_0^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1 - x_0)^{\epsilon}} \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^n \end{bmatrix} \otimes T_0^3 \\ &= \left(\frac{x_0}{x_1 - x_0}\right)^{\epsilon+q} \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^{\epsilon}} \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^{n+\epsilon+q} \end{bmatrix} \otimes T_0^3 = \mathbf{w}^{(\epsilon+q)0}_{m(n+\epsilon+q)} \end{aligned}$$

and

$$\begin{aligned} &\frac{Y_1^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^{\epsilon}} \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^n \end{bmatrix} \otimes T_0^3 \\ &= \left(\frac{x_1}{x_1 - x_0}\right)^{\epsilon+p} \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1 - x_0)^{\epsilon}} \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^m, \left(\frac{T_2}{T_0}\right)^{n+\epsilon+p} \end{bmatrix} \otimes T_0^3 = \mathbf{w}^{0(\epsilon+p)}_{m(n+\epsilon+p)}. \end{aligned}$$ □

The coboundary maps of $\mathcal{F}^{(\bullet)}_{p,q}$ have easy descriptions.

Proposition 13.

$$
\begin{aligned}
d^{(0)}\mathbf{u}^{ij} &= \mathbf{v}_{i+j}^{(p+i)(q+j)}, \\
d^{(1)}\mathbf{v}_n^{ij} &= \mathbf{w}_{(i+j-n)(n+\epsilon+p+q)}^{(i+q)(j+p)}.
\end{aligned}
$$

Proof. The proposition follows from direct computations:

$$
\begin{aligned}
\mathbf{u}^{ij} &= \left(\frac{x_0}{x_1-x_0}\right)^i \left(\frac{x_1}{x_1-x_0}\right)^j \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1-x_0)^\epsilon} \otimes \left(\frac{T_2}{T_0}\right)^{-i-j-\epsilon} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \otimes T_0^3 \\
&\mapsto \left(\frac{x_0}{x_1-x_0}\right)^i \left(\frac{x_1}{x_1-x_0}\right)^j \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1-x_0)^\epsilon} \otimes \begin{bmatrix} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^{i+j+\epsilon} \end{bmatrix} \otimes T_0^3 \\
&= \left(\frac{x_0}{x_2}\right)^{i+\epsilon+p} \left(\frac{x_1}{x_2}\right)^{j+\epsilon+q} Y_1^p Z_0^q \otimes \begin{bmatrix} d\frac{T_2}{T_1}\, d\frac{T_0}{T_1} \\ \left(\frac{T_2}{T_1}\right)^{i+j+\epsilon} \end{bmatrix} \otimes T_1^3 \\
&= \mathbf{v}_{i+j}^{(p+i)(q+j)},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}_n^{ij} &= \left(\frac{x_0}{x_1-x_0}\right)^{\delta+i} \left(\frac{x_1}{x_1-x_0}\right)^{\delta+j} \frac{Y_1^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1-x_0)^\epsilon} \otimes \begin{bmatrix} \left(\frac{T_1}{T_0}\right)^{n-\delta-i-j-\epsilon} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_2}{T_0}\right)^{\delta+n} \end{bmatrix} \otimes T_0^3 \\
&\mapsto \left(\frac{x_0}{x_1-x_0}\right)^{\delta+i} \left(\frac{x_1}{x_1-x_0}\right)^{\delta+j} \frac{Y_1^{\epsilon+p} Z_0^{\epsilon+q}}{(x_1-x_0)^\epsilon} \otimes \begin{bmatrix} \left(\frac{T_1}{T_0}\right)^{n} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^{\delta+i+j+\epsilon}, \left(\frac{T_2}{T_0}\right)^{\delta+n} \end{bmatrix} \otimes T_0^3 \\
&= \left(\frac{x_0}{x_1-x_0}\right)^{\delta+i+\epsilon+q} \left(\frac{x_1}{x_1-x_0}\right)^{\delta+j+\epsilon+p} \frac{Y_0^{\epsilon+p} Z_1^{\epsilon+q}}{(x_1-x_0)^\epsilon} \\
&\quad \otimes \begin{bmatrix} \left(\frac{T_1}{T_0}\right)^{n} d\frac{T_1}{T_0}\, d\frac{T_2}{T_0} \\ \left(\frac{T_1}{T_0}\right)^{\delta+i+j+\epsilon}, \left(\frac{T_2}{T_0}\right)^{\delta+n+2\epsilon+p+q} \end{bmatrix} \otimes T_0^3 \\
&= \mathbf{w}_{(i+j-n)(n+\epsilon+p+q)}^{(i+q)(j+p)}. \qquad \square
\end{aligned}
$$

5. Module structure of $\mathcal{F}^{(2)}_{p,q}$

We need polynomials f_i and g_i with integer coefficients which are defined inductively:

$$f_1 = g_1 = 0$$

and

$$f_{n+1} = f_n + g_n,$$
$$g_{n+1} = X + Xf_n$$

for $n \geq 1$. Induction on n, it is easy to see that

$$g_n(1 + f_{n+1}) - g_{n+1}(1 + f_n) = (-X)^n. \tag{17}$$

If a and b are elements in a commutative ring satisfying $b^2 = b + a$, then

$$b^n = (1 + f_n(a))b + g_n(a).$$

f_n and g_n are divisible by X. With $f = f_n/X$ and $g = g_n/X$,

$$b^n - b = a(f(a)b + g(a)).$$

This is a special case of the following lemma.

Lemma 14. *Let a and b be elements in a commutative ring satisfying $b^2 = b + a$. Then, for any $n_0, n_1, l > 0$ and $n_2 \geq 0$, there exist $f, g \in \mathbb{Z}[X]$ and $h \in \mathbb{Z}[X, Y]$ such that*

$$b^{n_0} = (1 + af(a))b^{n_1} + ag(a)(1 - b)^{n_2} + a^l h(a, b).$$

Proof. We consider first the case that $n_2 > 0$. Choose $h_{01}, h_{02}, h_{11}, h_{12} \in \mathbb{Z}[X]$ such that

$$\begin{aligned} b^{n_0} - b &= a(h_{01}(a)b + h_{02}(a)),\\ b^{n_1} - b &= a(h_{11}(a)b + h_{12}(a)). \end{aligned} \tag{18}$$

With $h_0 = h_{01} - h_{11} + h_{02} - h_{12}$ and $g = h_{02} - h_{12}$, we have

$$b^{n_0} = b^{n_1} + ag(a)(1 - b) + ah_0(a)b.$$

Note that $1-b$ also satisfies the condition $(1-b)^2 = (1-b)+a$. Choose $h_{21}, h_{22} \in \mathbb{Z}[X]$ such that

$$(1 - b)^{n_2} - (1 - b) = a(h_{21}(a)(1 - b) + h_{22}(a)). \tag{19}$$

With $h_1 = h_0 - agh_{22}$ and $h_2 = -ag(h_{21} + h_{22})$, we have

$$b^{n_0} = b^{n_1} + ag(a)(1-b)^{n_2} + abh_1(a) + a(1-b)h_2(a).$$

Fix n_0, n_1, n_2. Assume that for an $l > 1$, there exist $f, g, h_1, h_2 \in \mathbb{Z}[X]$ such that

$$b^{n_0} = (1 + af(a))b^{n_1} + ag(a)(1-b)^{n_2} + a^l bh_1(a) + a^l(1-b)h_2(a).$$

Choose $h_{11}, h_{12}, h_{21}, h_{22} \in \mathbb{Z}[X]$ such that (18) and (19) hold. Then

$$\begin{aligned} b^{n_0} = {} & (1 + af(a) + a^l h_1(a))b^{n_1} \\ & + a(g(a) + a^{l-1}h_2(a))(1-b)^{n_2} \\ & + a^{l+1}b(-h_1(a)h_{11}(a) - h_1(a)h_{12}(a) - h_2(a)h_{22}(a)) \\ & + a^{l+1}(1-b)(-h_2(a)h_{21}(a) - h_2(a)h_{22}(a) - h_1(a)h_{12}(a)). \end{aligned}$$

This induction process on l proves the lemma for the case $n_2 > 0$.

Now we consider the case that $n_2 = 0$. Choose $f, g, h_{11}, h_{12} \in \mathbb{Z}[X]$ and $h \in \mathbb{Z}[X, Y]$ such that (18) and

$$b^{n_0} = (1 + af(a))b^{n_1} + ag(a)(1-b) + a^l h(a, b)$$

hold. Denote

$$\frac{1}{1 + ah_{11}(a)} := 1 - ah_{11}(a) + (ah_{11}(a))^2 - (ah_{11}(a))^3 + \cdots + (-ah_{11}(a))^{l-1}$$

by abusing the notation. Then

$$\begin{aligned} b^{n_0} = {} & \left(1 + af(a) - \frac{ag(a)}{1 + ah_{11}(a)}\right) b^{n_1} \\ & + ag(a)\left(1 + \frac{ah_{12}(a)}{1 + ah_{11}(a)}\right) + a^l \left(h(a,b) - ag(a)(-h_{11}(a))^l b\right). \end{aligned}$$ □

For the rest of this paper, we consider elements

$$\begin{aligned} a &:= \frac{x_2 x_3}{(x_1 - x_0)^2}, \\ b &:= \frac{x_1}{x_1 - x_0} \end{aligned}$$

in the ring $\Gamma(D_+(x_1 - x_0), \mathrm{Proj}(S))$, which satisfy the condition $b^2 = b + a$. The multiplications of elements in $\mathcal{F}^{(2)}_{p,q}$ by a and b are easy to describe:

$$a\mathbf{w}^{ij}_{mn} = \mathbf{w}^{(i+1)(j+1)}_{mn},$$

$$b\mathbf{w}_{mn}^{ij} = \mathbf{w}_{mn}^{i(j+1)}.$$

The condition $(1-b)^2 = (1-b) + a$ also holds. The multiplication by $1-b$ gives rise to a negative sign:

$$(1-b)^l \mathbf{w}_{mn}^{ij} = (-1)^l \mathbf{w}_{mn}^{(i+l)j}.$$

This is the reason that we include the condition "sum" in the following definition.

DEFINITION 15. An element $\mathbf{w} \in \mathcal{F}_{p,q}^{(2)}$ is approximated by $\mathbf{w}_{mn}^{\leq}$ (resp. $\mathbf{w}_{mn}^{\geq}$), denoted by $\mathbf{w} \approx \mathbf{w}_{mn}^{\leq}$ (resp. $\mathbf{w} \approx \mathbf{w}_{mn}^{\geq}$), if their difference or *sum* $\mathbf{w} \pm \mathbf{w}_{mn}^{\leq}$ (resp. $\mathbf{w} \pm \mathbf{w}_{mn}^{\geq}$) is contained in the κ-vector subspace generated by the elements $\mathbf{w}_{ij}^{\leq}$ *and* $\mathbf{w}_{ij}^{\geq}$ with $i < m$.

Proposition 16. *Let $i, m > 0$ and $n \in \mathbb{Z}$. If $p = q \leq 0$,*

$$\mathbf{w}_{m(n+\epsilon+p)}^{0i} \approx \mathbf{w}_{mn}^{\leq}, \tag{20}$$

$$\mathbf{w}_{m(n+\epsilon+q)}^{i0} \approx \mathbf{w}_{mn}^{\geq}. \tag{21}$$

If $p \neq q$ or $p = q > 0$, the approximation (20) *holds for $\epsilon + p > 0$ and the approximation* (21) *holds for $\epsilon + q > 0$.*

Proof. We prove only (20) and leave (21) to the reader. So we have the assumption $\epsilon + p > 0$ if $p \neq q$ or $p = q > 0$. We choose $f, g \in \mathbb{Z}[X]$ and $h \in \mathbb{Z}[X, Y]$ such that

$$b^i - a^m h(a,b) = \begin{cases} (1 + af(a))b + ag(a)(b-1), & \text{if} \quad p = q \leq 0; \\ (1 + af(a))b^{\epsilon+p} + ag(a)(b-1)^{\epsilon+q}, & \text{otherwise.} \end{cases}$$

Then

$$\mathbf{w}_{m(n+\epsilon+p)}^{0i} - \mathbf{w}_{mn}^{\leq} = b^i \mathbf{w}_{m(n+\epsilon+p)}^{00} - \mathbf{w}_{mn}^{\leq} = af(a)\mathbf{w}_{m(n+p-q)}^{\leq} + ag(a)\mathbf{w}_{mn}^{\geq},$$

from which we get the required approximation (20). □

If $\mathbf{w}'_{mn}, \mathbf{w}''_{mn} \in \mathcal{F}_{p,q}^{(2)}$ satisfy $\mathbf{w}'_{mn} \approx \mathbf{w}_{mn}^{\leq}$ and $\mathbf{w}''_{mn} \approx \mathbf{w}_{mn}^{\geq}$ for all positive m and n, then $\{\mathbf{w}'_{mn}, \mathbf{w}''_{mn}\}_{m,n>0}$ is a basis of $\mathcal{F}_{p,q}^{(2)}$. More generally, if $\mathbf{w}'_{mn} \approx \mathbf{w}_{mn}^{\leq}$ and $\mathbf{w}''_{mn} - \mathbf{w}'''_{mn} \approx \mathbf{w}_{mn}^{\geq}$ for some $\mathbf{w}'''_{mn}$ contained in the subspace generated by $\mathbf{w}_{ij}^{\leq}$ with $i < m + l$ for a fixed l independent of m and n, then $\{\mathbf{w}'_{mn}, \mathbf{w}''_{mn}\}_{m,n>0}$ is still a basis of $\mathcal{F}_{p,q}^{(2)}$. This observation is useful accompanied with the following fact.

Proposition 17. *Let $i, m > 0$ and $n \in \mathbb{Z}$. Assume that $p \neq q$ or $p = q > 0$.*

$$\mathbf{w}_{m(n+\epsilon+p)}^{0i} \pm \mathbf{w}_{m(n+p-q)}^{\geq} \approx \mathbf{w}_{mn}^{\leq}, \quad \textit{if} \quad \epsilon + p = 0.$$

$$\mathbf{w}^{i0}_{m(n+\epsilon+q)} \pm \mathbf{w}^{\leq}_{m(n+q-p)} \approx \mathbf{w}^{\geq}_{mn}, \quad \textit{if} \quad \epsilon + q = 0.$$

Proof. We prove only the first approximation and leave the second to the reader. So we have the conditions $\epsilon + p = 0$ and $\epsilon + q > 0$. We choose $f, g \in \mathbb{Z}[X]$ and $h \in \mathbb{Z}[X, Y]$ such that

$$\begin{aligned} b^i &= (1 + af(a))b + ag(a)(b-1)^{\epsilon+q} + a^m h(a,b) \\ &= (1 + af(a)) + (1 + af(a))(b-1) + ag(a)(b-1)^{\epsilon+q} + a^m h(a,b). \end{aligned}$$

Since $\epsilon + q > 0$, we may also choose $f', g' \in \mathbb{Z}[X]$ and $h' \in \mathbb{Z}[X, Y]$ such that

$$1 - b = (1 + af'(a))(1-b)^{\epsilon+q} + ag'(a) + a^m h'(a,b).$$

Then

$$\begin{aligned} &\mathbf{w}^{0i}_{m(n+\epsilon+p)} \\ &= (1 + af(a))\mathbf{w}^{00}_{m(n+\epsilon+p)} + (1 + af(a))\mathbf{w}^{10}_{m(n+\epsilon+p)} + ag(a)\mathbf{w}^{(\epsilon+q)0}_{m(n+\epsilon+p)} \\ &= (1 + af(a))\mathbf{w}^{00}_{m(n+\epsilon+p)} - ag'(a)(1 + af(a))\mathbf{w}^{00}_{m(n+\epsilon+p)} \\ &\quad - (-1)^{\epsilon+q}(1 + af(a))(1 + af'(a))\mathbf{w}^{(\epsilon+q)0}_{m(n+\epsilon+p)} + ag(a)\mathbf{w}^{(\epsilon+q)0}_{m(n+\epsilon+p)}. \end{aligned}$$

From the equality

$$\begin{aligned} &\mathbf{w}^{0i}_{m(n+\epsilon+p)} + (-1)^{\epsilon+q}\mathbf{w}^{\geq}_{m(n+p-q)} - \mathbf{w}^{\leq}_{mn} \\ &= a(f(a) - g'(a) - af(a)g'(a))\mathbf{w}^{\leq}_{mn} \\ &\quad - (-1)^{\epsilon+q}a(f(a) + f'(a) + af(a)f'(a))\mathbf{w}^{\geq}_{m(n+p-q)} + ag(a)\mathbf{w}^{\geq}_{m(n+p-q)}, \end{aligned}$$

we get the required approximation. □

Corollary 18. *Let $\mathbf{w}'_{mn}, \mathbf{w}''_{mn} \in \mathcal{F}^{(2)}_{p,q}$. Assume that, for each m and n,*

$$\begin{aligned} \mathbf{w}'_{mn} &= \mathbf{w}^{0i}_{m(n+\epsilon+p)}, \\ \mathbf{w}''_{mn} &= \mathbf{w}^{j0}_{m(n+\epsilon+q)} \end{aligned}$$

for some positive i and j. Then $\left\{\mathbf{w}'_{mn}, \mathbf{w}''_{mn}\right\}_{m,n>0}$ is a basis of $\mathcal{F}^{(2)}_{p,q}$.

Proof. If $\epsilon + p$ and $\epsilon + q$ are both zero, then $p = q \leq 0$. Proposition 16 proves the corollary. If $\epsilon + p > 0$ or $\epsilon + q > 0$, the corollary follows from Proposition 16 and Proposition 17. □

In Section 6, we need also the following approximations.

Proposition 19. *Let* $i, m > 0$ *and* $n \in \mathbb{Z}$. *Assume that* $p \neq q$ *or* $p = q > 0$. *There exist* $g_1, g_2 \in \mathbb{Z}[X]$ *such that*

$$\mathbf{w}_{mn}^{0(i-\epsilon-q)} \pm a g_1(a) \mathbf{w}_{(m+\epsilon+q)n}^{\geq} \approx \mathbf{w}_{mn}^{\leq}, \quad \text{if} \quad \epsilon + p = 0;$$
$$\mathbf{w}_{mn}^{(i-\epsilon-p)0} \pm a g_2(a) \mathbf{w}_{(m+\epsilon+p)n}^{\leq} \approx \mathbf{w}_{mn}^{\geq}, \quad \text{if} \quad \epsilon + q = 0.$$

Proof. We prove the second approximation and leave the first to the reader. So we have the conditions $\epsilon + p > 0$ and $\epsilon + q = 0$. Choose $f_2, g_2 \in \mathbb{Z}[X]$ and $h_2 \in \mathbb{Z}[X, Y]$ such that

$$(1-b)^i = (1 + a f_2(a))(1-b)^{\epsilon+p} + a g_2(a) + a^{m+\epsilon+p} h_2(a, b).$$

Then

$$\begin{aligned}\mathbf{w}_{mn}^{(i-\epsilon-p)0} &= \mathbf{w}_{(m+\epsilon+p)(n+\epsilon+p)}^{i(\epsilon+p)} \\ &= (-1)^{i+\epsilon+p}(1 + a f_2(a)) \mathbf{w}_{mn}^{00} + (-1)^i a g_2(a) \mathbf{w}_{(m+\epsilon+p)(n+\epsilon+p)}^{0(\epsilon+p)},\end{aligned}$$

from which we get the required approximation. □

6. Cohomology of $\mathcal{F}_{p,q}$

Proposition 20. *Let* $p, q \geq 0$. $\mathrm{H}^1(\mathbb{P}^2, \mathcal{F}_{p,q}) = \mathrm{H}^2(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$. *The elements* $\mathbf{u}^{ij}$, *where* $0 \leq i \leq q$ *and* $0 \leq j \leq p$, *form a basis of* $\mathrm{H}^0(\mathbb{P}^2, \mathcal{F}_{p,q})$.

Proof. In this proposition, $\epsilon = 0$. $d^{(0)}\mathbf{u}^{ij} = \mathbf{v}_0^{(p-j)(q-i)} = 0$ if and only if $i \leq q$ and $j \leq p$. Those non-zero $d^{(0)}\mathbf{u}^{ij}$ are linearly independent. Therefore the elements $\mathbf{u}^{ij}$, where $0 \leq i \leq q$ and $0 \leq j \leq p$, form a basis of $\mathrm{H}^0(\mathbb{P}^2, \mathcal{F}_{p,q})$.

Now we compute the images of $\mathbf{v}_0^{ij}$.

- For indices $i \leq p$ and $j \leq q$, $\mathbf{v}_0^{ij}$ is the image of $\mathbf{u}^{(q-j)(p-i)}$. Therefore $d^{(1)}\mathbf{v}_0^{ij} = 0$.
- For indices $i < 0$ and $j > q$,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{(j-q)(p-i)}^{0(j-q+p-i)},$$

where the index $j - q + p - i$ is positive.

- For indices $j < 0$ and $i > p$,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{(i-p)(q-j)}^{(i-p+q-j)0},$$

where the index $i - p + q - j$ is also positive.

As noted in Corollary 18, except those $\mathbf{v}_0^{ij}$ being images of $d^{(0)}$, images of other $\mathbf{v}_0^{ij}$ form a basis of $\mathcal{F}_{p,q}^{(2)}$. This implies $\mathrm{H}^1(\mathbb{P}^2, \mathcal{F}_{p,q}) = \mathrm{H}^2(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$. □

Proposition 21. *Let* $q < 0 \leq p$. $\mathrm{H}^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = \mathrm{H}^2(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$. *The elements* $\mathbf{v}_0^{ij}$, *where* $0 \leq i \leq p$ *and* $q < j < 0$, *form a basis of* $\mathrm{H}^1(\mathbb{P}^2, \mathcal{F}_{p,q})$.

Proof. In this proposition $\epsilon = -q$. The condition $\epsilon + p > 0$ holds. The images of $\mathbf{u}^{ij}$ are linearly independent. Therefore $\mathrm{H}^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$. Other assertions of the proposition follows from the computations of the images of $\mathbf{v}_0^{ij}$:

- For indices $i \leq p$ and $j \leq q$, $\mathbf{v}_0^{ij}$ is the image of $\mathbf{u}^{(q-j)(p-i)}$. Therefore $d^{(1)}\mathbf{v}_0^{ij} = 0$.
- For indices $i < 0$ and $j > q$,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{(j-q)(-i+p-q)}^{0(j-q+p-i)} \approx \mathbf{w}_{(j-q)(-i)}^{\leq}$$

by Proposition 16. The latter elements are exactly those $\mathbf{w}_{mn}^{\leq}$ with positive indices m and n.

- For indices $j < 0$ and $i > p$, by Proposition 19, there exists $g_2 \in \mathbb{Z}[X]$ such that

$$d^{(1)}\mathbf{v}_0^{ij} \pm a g_2(a)\mathbf{w}_{(i-q)(-j)}^{\leq} \approx \mathbf{w}_{(i-p)(-j)}^{\geq}.$$

The latter elements are exactly those $\mathbf{w}_{mn}^{\geq}$ with positive indices m and n.

- For indices $0 \leq i \leq p$ and $q < j < 0$, we write $m = p - i$ and $n = j - q$. With the polynomials f_i and g_i defined in the beginning of Section 5,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}_{n(m-q)}^{0(m+n)} = (1 + f_{m+n}(a))\mathbf{w}_{n(m-q)}^{01} + g_{m+n}(a)\mathbf{w}_{n(m-q)}^{00}. \tag{22}$$

As $m - p = -i \leq 0$,

$$\mathbf{w}_{n(m-q)}^{0(p-q)} = \mathbf{w}_{n(m-p)}^{\leq} = 0.$$

Apply the relation

$$\begin{aligned}\mathbf{w}_{n(m-q)}^{01} &= \mathbf{w}_{n(m-q)}^{0(p-q)} - f_{p-q}(a)\mathbf{w}_{n(m-q)}^{01} - g_{p-q}(a)\mathbf{w}_{n(m-q)}^{00}\\ &= -f_{p-q}(a)\mathbf{w}_{n(m-q)}^{01} - g_{p-q}(a)\mathbf{w}_{n(m-q)}^{00}\end{aligned}$$

repeatedly l times to (22), we get

$$\begin{aligned}d^{(1)}\mathbf{v}_0^{ij} = {}& (1 + f_{m+n}(a))(-f_{p-q}(a))^l \mathbf{w}_{n(m-q)}^{01}\\ & - (1 + f_{m+n}(a))(1 - f_{p-q}(a) + (f_{p-q}(a))^2 - \cdots)g_{p-q}(a)\mathbf{w}_{n(m-q)}^{00}\\ & + g_{m+n}(a)\mathbf{w}_{n(m-q)}^{00}.\end{aligned}$$

For $l \geq n$,

$$(f_{p-q}(a))^l \mathbf{w}_{n(m-q)}^{00} = 0 = (f_{p-q}(a))^l \mathbf{w}_{n(m-q)}^{01}.$$

Without ambiguity, we may write

$$d^{(1)}\mathbf{v}_0^{ij} = (1 + f_{m+n}(a)) \left(\frac{g_{m+n}(a)}{1 + f_{m+n}(a)} - \frac{g_{p-q}(a)}{1 + f_{p-q}(a)} \right) \mathbf{w}_{n(m-q)}^{00}$$

$$= (1 + f_{m+n}(a)) \sum_{i=m+n}^{p-q-1} \left(\frac{g_i(a)}{1 + f_i(a)} - \frac{g_{i+1}(a)}{1 + f_{i+1}(a)} \right) \mathbf{w}^{00}_{n(m-q)}.$$

By (17),

$$d^{(1)}\mathbf{v}_0^{ij} = (1 + f_{m+n}(a)) \sum_{i=m+n}^{p-q-1} \frac{(-a)^i}{(1 + f_i(a))(1 + f_{i+1}(a))} \mathbf{w}^{00}_{n(m-q)}.$$

Since $a^{m+n}\mathbf{w}^{00}_{n(m-q)} = 0$, $d^{(1)}\mathbf{v}_0^{ij} = 0$ for $0 \le i \le p$ and $q < j < 0$. □

Similarly, we have the following proposition.

Proposition 22. *Let* $p < 0 \le q$. $\mathrm{H}^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = \mathrm{H}^2(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$. *The elements* $\mathbf{v}_0^{ij}$, *where* $0 \le j \le q$ *and* $p < i < 0$, *form a basis of* $\mathrm{H}^1(\mathbb{P}^2, \mathcal{F}_{p,q})$.

Proposition 23. *Let* $q \le p < 0$. $\mathrm{H}^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = \mathrm{H}^1(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$. *The elements* $\mathbf{w}^{\ge}_{mn}$, *where* $0 < m < -p$ *and* $0 < n < -q-m$, *together with the elements* $\mathbf{w}^{\le}_{mn}$, *where* $m > 0$, $n > 0$ *and* $m + n \le -p$, *form a basis of* $\mathrm{H}^2(\mathbb{P}^2, \mathcal{F}_{p,q})$.

Proof. In this proposition, $\epsilon = -q$. The images of $\mathbf{u}^{ij}$ are exactly those $\mathbf{v}_0^{ij}$ with indices $i \le p$ and $j \le q$. They are linearly independent. Therefore $\mathrm{H}^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$.

Now we compute the images of $\mathbf{v}_0^{ij}$.

- For indices $i \le p$ and $j \le q$, $\mathbf{v}_0^{ij}$ is the image of $\mathbf{u}^{(q-j)(p-i)}$. Therefore $d^{(1)}\mathbf{v}_0^{ij} = 0$.
- For indices $i < 0$ and $j > q$ satisfying $p - i > q - j$,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}^{0(j-q+p-i)}_{(j-q)(-i-q+p)} \approx \mathbf{w}^{\le}_{(j-q)(-i)}$$

by Proposition 16. The latter elements are exactly those $\mathbf{w}^{\le}_{mn}$ with positive indices m and n satisfying $m + n + p > 0$.

- For indices $j < 0$ and $i > p$ satisfying $q - j > p - i$,

$$d^{(1)}\mathbf{v}_0^{ij} = \mathbf{w}^{(i-p+q-j)0}_{(i-p)(-j)} \approx \mathbf{w}^{\ge}_{(i-p)(-j)}, \quad \text{if} \quad p = q,$$
$$d^{(1)}\mathbf{v}_0^{ij} \pm \mathbf{w}^{\le}_{(i-p)(-j+q-p)} \approx \mathbf{w}^{\ge}_{(i-p)(-j)}, \quad \text{f} \quad p > q$$

by Proposition 16 and Proposition 17. The latter elements are exactly those $\mathbf{w}^{\ge}_{mn}$ with positive indices m and n satisfying $m + n + q > 0$.

- For indices $j < 0$ and $i > p$ satisfying $q - j = p - i$,

$$d^{(1)}\mathbf{v}_0^{ij} - \mathbf{w}^{01}_{(i-p)(-j)} = -\mathbf{w}^{10}_{(i-p)(-j)}.$$

If $p = q$,

(23) $$d^{(1)}\mathbf{v}_0^{ij} - \mathbf{w}^{\le}_{(i-p)(-j)} = -\mathbf{w}^{\ge}_{(i-p)(-j)} \approx \mathbf{w}^{\ge}_{(i-p)(-j)}.$$

If $p > q$, by Proposition 16 and Proposition 17, there are approximations

$$\mathbf{w}^{01}_{(i-p)(-j)} \approx \mathbf{w}^{\leq}_{(i-p)(-j+q-p)},$$
$$\mathbf{w}^{10}_{(i-p)(-j)} \pm \mathbf{w}^{\leq}_{(i-p)(-j+q-p)} \approx \mathbf{w}^{\geq}_{(i-p)(-j)},$$

that is, their differences or sums are contained in the subspace generated by the elements $\mathbf{w}^{\leq}_{mn}$ and $\mathbf{w}^{\geq}_{mn}$ with $m < i - p$. For suitable negative signs and an integer l,

$$\begin{aligned} &d^{(1)}\mathbf{v}^{ij}_0 + l\mathbf{w}^{\leq}_{(i-p)(-j+q-p)} \pm \mathbf{w}^{\geq}_{(i-p)(-j)} \\ &= \left(\mathbf{w}^{01}_{(i-p)(-j)} \pm \mathbf{w}^{\leq}_{(i-p)(-j+q-p)}\right) \\ &\quad - \left(\mathbf{w}^{10}_{(i-p)(-j)} \pm \mathbf{w}^{\leq}_{(i-p)(-j+q-p)} \pm \mathbf{w}^{\geq}_{(i-p)(-j)}\right). \end{aligned}$$

Therefore

$$d^{(1)}\mathbf{v}^{ij}_0 + l\mathbf{w}^{\leq}_{(i-p)(-j+q-p)} \approx \mathbf{w}^{\geq}_{(i-p)(-j)}. \tag{24}$$

The latter elements of (23) or (24) are exactly those $\mathbf{w}^{\geq}_{mn}$ with positive indices m and n satisfying $m + n + q = 0$.

- In order to have indices $i \geq 0$ and $j < 0$ satisfying $p - i > q - j$, the condition $p > q$ has to be satisfied. With this condition, by Proposition 19, there exist $g_2 \in \mathbb{Z}[X]$ such that

$$d^{(1)}\mathbf{v}^{ij}_0 \pm a g_2(a)\mathbf{w}^{\leq}_{(i-q)(-j)} \approx \mathbf{w}^{\geq}_{(i-p)(-j)}.$$

The latter elements are exactly those $\mathbf{w}^{\geq}_{mn}$ with indices $m \geq -p$ and $n > 0$ satisfying $m + n + q < 0$.
These computations show that the non-zero images of $\mathbf{v}^{ij}_0$ together with the elements $\mathbf{w}^{\leq}_{mn}$, where $m > 0$, $n > 0$ and $m+n \leq -p$, and the elements $\mathbf{w}^{\geq}_{mn}$, where $0 < m < -p$ and $0 < n < -q - m$, form a basis of $\mathcal{F}^{(2)}_{p,q}$. This concludes the proposition. □

Similarly, we have the following proposition.

Proposition 24. *Let* $p < q < 0$. $\mathrm{H}^0(\mathbb{P}^2, \mathcal{F}_{p,q}) = \mathrm{H}^1(\mathbb{P}^2, \mathcal{F}_{p,q}) = 0$. *The elements* $\mathbf{w}^{\leq}_{mn}$, *where* $0 < m < -q$ *and* $0 < n < -p-m$, *together with the elements* $\mathbf{w}^{\geq}_{mn}$, *where* $m > 0$, $n > 0$ *and* $m + n \leq -q$, *form a basis of* $\mathrm{H}^2(\mathbb{P}^2, \mathcal{F}_{p,q})$.

Counting the cardinality of the bases of $\mathrm{H}^r(\mathbb{P}^2, \mathcal{F}_{p,q})$ given in previous propositions, we recover the following.

Corollary 25 ([6, Proposition 12]).

$$\dim_\kappa \mathrm{H}^r(\mathbb{P}^2, \mathcal{F}_{p,q}) = (-1)^r(p + 1)(q + 1)$$

if $r = 0$ *and* $p, q \geq 0$; *or if* $r = 1$ *and* $p \geq 0$, $q < 0$ *or* $p < 0$, $q \geq 0$; *or if* $r = 2$ *and* $p, q < 0$; *and is zero otherwise*.

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