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FROM HOMMA'S THEOREM TO PICK'S THEOREM

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Abstract

From Homma's PL Gauss-Bonnet theorem applied on a PL-complex in a plane, many generalizations of Pick's theorem on a lattice PL-figure are obtained in a unified geometric way.

1. Introduction

The purpose of this paper is to provide a new viewpoint for the teaching of mathematics. In about 1976, a teacher at Nakafuji elementary school of Fukui City gave lessons on the area of figures on a plane. Under the influence of the tiling scheme method of Tōyama, she had used a tiled floor of the classroom as the educational tool to demonstrate that there exists no functional relation between the area of a region and the length of the boundary. Last of all, she gave the following homework: *Let's draw various figures of boundary length* 16 *by putting together the square tiles* $\boxed{\mathrm{S}}$ *of area* 1 *by the edges as follows, and calculate the areas of the drawn figures*:

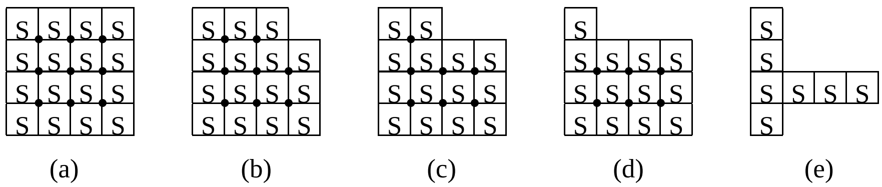

Fig. 1.

Surprisingly, a pupil was aware of the following equation for such figures with the area A and the number I of the intersecting points $\bullet$ in the interior:

$$A - I = 7,$$

which was reported to the teacher on the next day. In the Fig. 1, that is, a: $16 - 9 = 7$, b: $15 - 8 = 7$, c: $14 - 7 = 7$, d: $13 - 6 = 7$, e: $7 - 0 = 7$.

Question. In a meeting held at Fukui University in the last half of 1980's, this teacher asked Kurogi several questions on the above report [15, §2]:
(Q1) Is it true for such all figures? What happens if the boundary length of a figure is changed?
(Q2) It is hard to realize the relation between the area and the number of the intersecting points. How can one consider the relation?
(Q3) Why does the number 7 appear in the right hand side?
(Q4) How can one develop the discovery of this pupil?

Answer. Note that the area of the above each figure are equal to the number of the square tiles. Then one has Corollary 2.2 as well as Pick's theorem (Theorem 2.1). On the other hand, the process from (a) to (e) appears in an elementary method for computing the Euler number of a simple polygonal region [12, §44]. And Pick's formula was proved from Euler's formula of a simple polygonal region [2, 6, 17]. Recently, it was observed that Pick's formula is a corollary of Riemann-Roch theorem for a toric variety [7] or Green's theorem [4]. However, they treated only simple polygonal regions. This paper shows that many generalizations of Pick's theorem [5, 11, 20, 21, 22, 25, 27] are nothing but Homma's PL Gauss-Bonnet theorem on a plane.

Contents. In §2, to state the problem precisely, the notion of a lattice PL-figure in a coordinate plane $\mathbf{R}^2$ is introduced, by which the above formula is related to Pick's formula for the lattice polygonal region. In §3, the area of primitive triangle is proved to be $1/2$ after Gaskell-Klamkin-Watson [9] (cf. [3]), by means of Reeve's anglular-sum formula at vertexes [22] (cf. [27]).

In §4, Homma's alternating angular-sum formula at simplexes on a plane (Theorem 4.1) is proved from Thalēs theorem (cf. [14, §1.3], [23]). It is then obtained that Pick's formula and its generalizations on lattice PL-figures (Theorem 2.5) follow from Homma's PL Gauss-Bonnet formula on a plane (Theorems 4.2, 4.3). In §5, it is observed that Pick's theorem is a consequence of Homma's PL Gauss-Bonnet theorem and Shoenflies theorem for a simple region.

2. Pick's formula and its generalizations

Such figures like the examples in Fig. 1 are called *polyominoes* (cf. [19, §1.9]). However, there are wilder figures of boundary length 16 made by combining square tiles, isolated points or line segments on a plane as follows, where $A - I \neq 7$ for all figures in Fig. 2 except the example (g). In this article, the combination such as the example (f) is not considered, that is, the square tiles should be combined such that each pair of two overlaped edges of the gathered square tiles are glued as one line segment of unit length. In this case, each vertex of the square tiles can be considered as *a lattice point in the coordinate plane* (that is, the point with integer

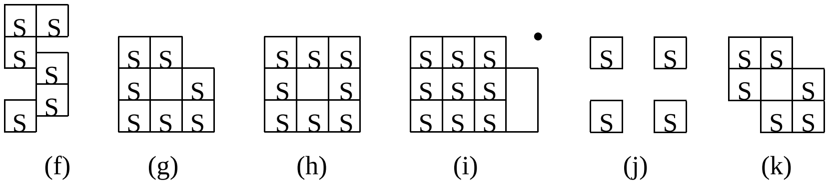

Fig. 2.

xy-coordinates). On the other hand, the figure such as the example (i) is considered in this article, although this is not only a combination of square tiles but also isolated points and line segments (cf. [11]).

In general, a figure such as the examples in Fig. 1 or 2 except the example (f) is called *a lattice PL-figure*, that is precisely defined as follows: By definition, *a PL-figure* is a compact subset P of a coordinate plane $\mathbf{R}^2$ such that the boundary ∂P is a 1-dimensional PL-complex. Let $\mathbf{Z}^2$ be the set of all points in $\mathbf{R}^2$ with integer coordinates, whose each element is called *a lattice point*. By definition, *a lattice line segment* is a line segment joining two lattice points in $\mathbf{R}^2$, and *a primitive line segment* is a lattice line segment such that there exists no lattice point except the two vertexes. Then *a* 1*-dimensional primitive PL-complex* is defined to be a union of a finite set of primitive line segments or lattice points whose intersections are only vertices (those are then lattice points). Note that a 1-dimensional primitive PL-complex is a 1-dimensional PL-complex consisting of their primitive line segments and their vertexes or several lattice points as simplexes, which is not assumed to be connected and may have 0-dimensional connected components. By definition, *a lattice PL-figure* P is a compact subset of $\mathbf{R}^2$ such that the boundary ∂P is a 1-dimensional primitive PL-complex. Note that a lattice PL-figure is not assumed to be connected and may have locally 0-dimensional or 1-dimensional parts.

For a lattice PL-figure P, let $A(P)$ denote the area of P. Put

$$V(P) := B(P) + I(P) = \#(P \cap \mathbf{Z}^2),$$

where $B(P) := \#(\partial P \cap \mathbf{Z}^2)$ is the number of the lattice points on ∂P, and $I(P) := \#(\mathrm{int.}P \cap \mathbf{Z}^2)$ is the number of the lattice points in the interior int. P. In Theorem 3.2, it is stated that a lattice PL-figure P admits a structure of 2-dimensional PL-complex consisting of primitive triangles α_i $(i = 1, \ldots, N(P))$, primitive line seqments L_j $(j = 1, \ldots, e(P))$ and lattice points z_k $(k = 1, \ldots, V(P))$, called *a primitive PL-complex*. Put $e_B(P) := \#\{L_j \mid L_j \subseteq \partial P\}$ and

$$e_B^I(P) := \#\{L_j \mid L_j \subseteq \partial(\mathrm{cl.}(\mathrm{int.}\, P))\}$$

as bounded numbers at most $e(P) = \#\{L_j\}$. By definition, a 2-dimensional compact subset of a coordinate plane is said to be *simple* if the boundary is homeomorphic to a circle, a Jordan curve, whose examples are given in Fig. 1 and 2 (f). And a simple lattice PL-figure is called *a lattice polygonal region*. For example, the polyominoes can be considered as lattice polygonal regions (see Fig. 1), as well as more complicated ones (see [25, Chapter 5, Figure 27]). Then the following formula is well-known after Pick [20], Steinhaus [24, Figure (77), References], Coxeter [1] and Niven-Zuckerman [18] (cf. [10]):

Theorem 2.1 (Pick). *Let P be a lattice polygonal region in $\mathbf{R}^2$. Then*

$$A(P) = \frac{B(P)}{2} + I(P) - 1. \tag{1}$$

Corollary 2.2. *Let P be a polyomino made by combining the square tiles $\boxed{\mathrm{S}}$ of area* 1 *by the edges such that the boundary ∂P is a Jordan curve. Then the following linear equation holds among the area A of P, the length L of ∂P and the number I of the intersecting points $\bullet$ in the interior* int. P:

$$A - I = \frac{L}{2} - 1.$$

Proof. Note that $I(P) = I$, because the vertexes of the unit squares are considered as lattice points. Since the boundary ∂P is homeomorphic to a circle, $B(P) = L$. Combined with the Pick's formula, one has then that $A - I = A(P) - I(P) = B(P)/2 - 1 = L/2 - 1$, as required. □

In particular, when $L = 16$, the right hand side is equal to 7, that is the discovery by the pupil in Nakafuji elementary school. However, for the example (h) in Fig. 2 (cf. [25, Figure 34]), one has that

$$A(P) - I(P) = 8 - 0 \neq 7 = \frac{B(P)}{2} - 1.$$

For such figures, the following formula will be obtained (cf. [20, §4], [5]).

Theorem 2.3. *Let P be a lattice PL-figure such that P is equal to the closure* cl.(int. P) *of the interior* int. P *in the coordinate plane. If the boundary ∂P is homeomorphic to a disjoint union of several circles, then*

$$A(P) = \frac{B(P)}{2} + I(P) - \chi(P) \tag{2}$$

with respect to the Euler number $\chi(P)$ of the figure P.

The formula (2) also holds for a disconnected figure such as the example (j) in Fig. 2. However, the formula (2) does not hold for the examples (g), (k) in Fig. 2. Note that g: $A - I = 7 - 0 = 7$, and that k: $A - I = 6 - 0 = 6 \neq 7$. For such figures, the following formula holds (cf. $\chi(\partial P) = B(P) - e_B(P)$).

Theorem 2.4 (Reeve, Rosenholtz, Varberg). *Let P be a lattice PL-figure such that $P = \mathrm{cl.}(\mathrm{int.}\,P)$ in $\mathbf{R}^2$. Then*

$$A(P) = \frac{B(P)}{2} + I(P) + \frac{\chi(\partial P)}{2} - \chi(P). \tag{3}$$

This paper shows that it is a consequence of Homma's PL Gauss-Bonnet theorem, in contrast to a purely combinatorial proof [21, 27]. However, the formula (3) does not hold for the example (i) in Fig. 2 (cf. [11, Ex. 2, Ex. 3]), and that $A - I = 5 \neq 7$. For all lattice PL-figure containing all examples in Fig. 1 and 2 except the example (f), the following formula is also obtained from Homma's PL Gauss-Bonnet theorem.

Theorem 2.5. *Let P be a lattice PL-figure in $\mathbf{R}^2$. Then*

$$A(P) = V(P) - e_B(P) + \frac{e_B^I(P)}{2} - \chi(P). \tag{4}$$

3. Angular-sum at vertexes

By definition, *a lattice triangle* is a simple lattice PL-figure that is also a triangle with a positive area. And *a primitive triangle* is a lattice triangle P such that $V(P) = 3$. For $r > 0$, put $U_r(v) := \{p \in \mathbf{R}^2 \mid \mathrm{dist.}(v, p) < r\}$ with respect to the Euclidean distance $\mathrm{dist.}(v, p)$ between the two points v and p in $\mathbf{R}^2$. Then one has the following *primitive PL-complex decomposition*:

Definition-Proposition 3.1. *Let P be a lattice PL-figure,* $\mathrm{ext.}\,P$ *the set of all exterior points of P, and* $\mathrm{int.}\,P$ *the set of all interior points of P. Then*:
(i) *The boundary $\partial(\mathrm{int.}\,P)$ of $\mathrm{int.}\,P$ is a subcomplex of ∂P;*
(ii) *For any $v \in \partial(\mathrm{int.}\,P) \cap \mathbf{Z}^2$, there exist two distinct primitive line segments $vw_1, vw_2 \subseteq \partial(\mathrm{int.}\,P)$;*
(iii) *There exist a positive integer k and primitive line segments*

$$L_i = x_i y_i := \{t x_i + (1 - t) y_i \mid 0 \leq t \leq 1\} \quad (i = 1, \ldots, k)$$

such that $L_i \cap L_j = \emptyset, \{x_i\}$ or $\{y_i\}$ for $i \neq j$, and that $\partial(\mathrm{int.}\,P) = L_1 \cup \cdots \cup L_k$;
(iv) *For each $z \in L_i^\circ = x_i y_i^\circ := \{t x_i + (1 - t) y_i \mid 0 < t < 1\}$, there exists $r > 0$ such that $U_r(z) \backslash L_i$ consists of the following two connected components*:

$$U_r(z) \cap \mathrm{ext.}\,P \quad \text{and} \quad U_r(z) \cap \mathrm{int.}\,P.$$

Theorem 3.2. *Any lattice PL-figure P admits a decomposition as a primitive PL-complex consisting of primitive triangles, primitive line segments, and lattice points as* 2*-dimensional,* 1*-dimensional, and* 0*-dimensional simplexes such that the boundary ∂P is a subcomplex.*

By means of the distance between two compact subsets in a Euclidean plane, Definition-Proposition 3.1 and Theorem 3.2 can be proved (cf. [16]).

For a subset P and a point z in a coordinate plane, put

$$\angle(z, P) := \lim_{\epsilon \to +0} \frac{A(B_\epsilon(z) \cap P)}{A(B_\epsilon(z))}, \quad \angle P := \sum_{z \in P \cap \mathbf{Z}^2} \angle(z, P). \tag{5}$$

Note that $0 \leq \angle(z, P) \leq 1$. Moreover, one has the following result.

Theorem 3.3. *For a lattice PL-figure P, let $N(P)$ be the number of primitive triangles in a primitive PL-complex decomposition of primitive triangles, primitive line segments, and lattice points. Then*

$$\angle P = \frac{N(P)}{2}. \tag{6}$$

In particular, the number $N(P)$ is uniquely determined by P for any PL-complex decomposition of primitive triangles, primitive line segments, and lattice points.

Proof. For two lattice PL-figures P_1, P_2 such as $P_1 \cap P_2 \subseteq \partial P_1 \cap \partial P_2$,

$$\angle(P_1 \cup P_2) = \angle P_1 + \angle P_2. \tag{7}$$

By Theorem 3.2, $P = \bigcup_{i=1}^N P_i \cup P_0$ for a certain 1-dimensional primitive PL-complex $P_0 \subset \partial P$ and primitive triangles P_i such that $P_i \cap P_j \subseteq \partial P_i \cap \partial P_j$ $(i, j = 0, 1, \dots, N)$. By Thales theorem, $\angle P_i = 1/2$ $(i = 1, \dots, N)$. And $\angle P_0 = 0$ by $0 = \angle(\partial P) \geq \angle P_0 \geq 0$. Then $\angle P = \sum_{i=1}^N \angle P_i + \angle P_0 = N/2$ by (7). □

REMARK 3.4. The statements of Definition-Proposition 3.1, Theorems 3.2 and 3.3 also hold when the definition of "lattice points" $\mathbf{Z}^2$ is replaced by a subset Z in $\mathbf{R}^2$ such that $D \cap Z$ is a finite set for any bounded subsets D in $\mathbf{R}^2$.

Theorem 3.5. *The area of any primitive triangle is equal to* $1/2$.

Proof. (cf. [9], [3]). (i) *If P is a lattice triangle, then the area $A(P)$ is a half integer, which is non less than* $1/2$: Let (x_i, y_i) be the three vertexes of the lattice triangle $(i = 0, 1, 2)$. For $j = 1, 2$, put $a_j := x_j - x_0$ and $b_j := y_j - y_0$, which are integers. Then the area $|a_1 b_2 - a_2 b_1|/2$ is a half integer.

(ii) Let P be a primitive triangle. Take a square S containing P in $\mathrm{int}.S$ of boundary length $4n$ such that the four vertexes are all lattice points. In general, S consists of n^2-unit squares, and that each unit square consists of two primitive triangles, so that S has a standard structure of PL-complex of such $2n^2$-primitive triangles. Put $Q := \mathrm{cl}.(S \backslash P)$. Then $\partial Q = \partial S \sqcup \partial P$ is a 1-dimensional primitive PL-complex. Hence, Q is a lattice PL-figure. By Theorem 3.2, Q is a PL-complex of primitive triangles, primitive line segments, and lattice points. Hence, S has another PL-complex structure given by the PL-complex structure of Q and the primitive triangle P, in which the number of the primitive triangles is the same number $2n^2$ with the standard one by the last statement of Theorem 3.3. Hence, the number of the triangle in the PL-complex of Q is $2n^2 - 1$. By (i) and the additivity of the area function $A(*)$, one has that

$$n^2 = A(S) = A(Q) + A(P) \geq \frac{2n^2 - 1}{2} + \frac{1}{2} = n^2,$$

so that $A(P) = 1/2$. □

Corollary 3.6 (Reeve). *Let P be a lattice PL-figure. Then*

$$A(P) = \angle P.$$

Proof. It follows from Theorems 3.2, 3.3, 3.5 and the additivity of the area function $A(*)$. □

REMARK 3.7. Note that Theorem 3.5 was also proved from the area of a lattice parallelograms (cf. [12, §5], [1, 18, 2]) or more concrete computations on the area of lattice rectangles (cf. [24, 26, 17, 27, 25]). The above proof was given in [9] and [3] except for the proof of the possibility of primitive triangular decomposition. Since $SL_2(\mathbf{R})$ preserves the area of figures, the statements of Theorem 3.5 and Corollary 3.6 hold also when $\mathbf{Z}^2$ is replaced by $Z := g \cdot \mathbf{Z}^2$ with $g \in SL_2(\mathbf{R})$.

4. Homma's PL Gauss-Bonnet theorem

In general, for a line segment $L = zv$ and a 2-dimensionl simplex α contained in a PL-complex P in a coordinate plane, as simplexes of P, taking some points $u \in L \backslash \{z, v\}$ and $w \in \mathrm{int}.\alpha$, put

$$\begin{aligned}
&\angle(L, P) := \angle(u, P), \quad \angle(\alpha, P) := \angle(w, P);\\
&\angle^E(z, P) := 1 - \angle(z, P), \quad \angle^E(L, P) := 1 - \angle(L, P),\\
&\angle^E(\alpha, P) := 1 - \angle(\alpha, P) = 1 - 1 = 0.
\end{aligned}$$

As well as the equation (7), for PL-complexes P_1, P_2 in a coordinate plane such that $P_1 \cap P_2 \subseteq \partial P_1 \cap \partial P_2$, one has that

$$\angle(L, P_1 \cup P_2) = \angle(L, P_1) + \angle(L, P_2); \tag{8}$$

$$\angle(\alpha, P_1 \cup P_2) = \angle(\alpha, P_1) + \angle(\alpha, P_2). \tag{9}$$

As a generalization of Thalēs theorem, one has the following formula (cf. [13, p.113, Cor. 1], [14, pp.20–24], [23, pp.164–165]).

Theorem 4.1 (Homma). *Let P be a PL-complex in a coordinate plane with the vertexes z_i $(i = 1, \ldots, p)$, the edges L_j $(j = 1, \ldots, q)$, and the triangles α_k $(k = 1, \ldots, r)$. Then*

$$\sum_{i=1}^{p} \angle(z_i, P) - \sum_{j=1}^{q} \angle(L_j, P) + \sum_{k=1}^{r} \angle(\alpha_k, P) = 0. \tag{10}$$

Proof. If P is a triangle $\triangle z_1 z_2 z_3$, then for $L_1 = z_2 z_3, L_2 = z_3 z_1, L_3 = z_1 z_2$,

$$\sum_{i=1}^{3} \angle(z_i, P) - \sum_{j=1}^{3} \angle(L_j, P) + \angle(z_1 z_2 z_3, P) = \frac{1}{2} - \frac{3}{2} + 1 = 0.$$

If int. $P = \emptyset$, then $r = 0$, and $\angle(z_i, P) = \angle(L_j, P) = 0$ for any i, j $(i = 1, \ldots, p;$ $j = 1, \ldots, q)$. By the additivity equations (7), (8) and (9), one has then the required equation for any PL-complex P in a coordinate plane. $\square$

Then Homma's PL Gauss-Bonnet theorem on a plane is obtained as follows (cf. [13, p.113, Thm. 1], [14, p.24, Thm. 1.6], [23, pp.170–172]):

Theorem 4.2 (Homma). *Let P be a PL-complex in a coordinate plane with the vertexes z_i $(i = 1, \ldots, p)$, the edges L_j $(j = 1, \ldots, q)$, and the triangles α_k $(k = 1, \ldots, r)$. Then*

$$\sum_{i=1}^{p} \angle^E(z_i, P) - \sum_{j=1}^{q} \angle^E(L_j, P) + \sum_{k=1}^{r} \angle^E(\alpha_k, P) = \chi(P), \tag{11}$$

where the right hand side $\chi(P) := p - q + r$ is the Euler number of P.

Proof. In the equation (10), substitute $\angle(\ast, P) = 1 - \angle^E(\ast, P)$ for $\ast = z_i, L_j, \alpha_k$. Then $p - q + r -$ (the left hand side) $= 0$, as required. $\square$

Theorem 4.3. *Let P be a primitive PL-complex consisting of lattice points z_i $(i = 1, \ldots, V(P))$, primitive line segments L_j $(j = 1, \ldots, e(P))$ and primitive triangles*

α_k $(k = 1, \ldots, N(P))$. *Then*

$$\angle P = V(P) - e_B(P) + \frac{e_B^I(P)}{2} - \chi(P). \tag{12}$$

Proof. In the equation (11) of Theorem 4.2, one has that

$$\sum_{i=1}^{p} \angle^E(z_i, P) + \frac{e_B^I(P)}{2} - e_B(P) = \chi(P). \tag{13}$$

In fact, $\sum_{k=1}^{N(P)} \angle^E(\alpha_k, P) = 0$, at first. At second, if $L_j \not\subseteq \partial P$, then $L_j^\circ \cap \partial P = \emptyset$, because ∂P is a subcomplex of P. In this case, the midpoint of L_j is contained in int. P, so that $\angle^E(L_j, P) = 0$. At third, for $(e_B(P) - e_B^I(P))$-primitive line segments $L_j \subseteq \partial P$ such as $L_j \not\subseteq \partial(\text{int.}\, P)$, $L_j^\circ \cap \partial(\text{int.}\, P) = \emptyset$ because $\partial(\text{int.}\, P)$ is a subcomplex of ∂P. In this case, the midpoint of L_j is not contained in $\partial(\text{int.}\, P)$ but contained in ∂P, so that $\angle^E(L_j, P) = 1 - \angle(L_j, P) = 1$. At last, for $e_B^I(P)$-primitive line segments $L_j \subseteq \partial(\text{int.}\, P)$, L_j is an edge of just one triangle in the PL-complex P, so that $\angle^E(L_j, P) = 1/2$. Hence,

$$\sum_{j=1}^{q} \angle^E(L_j, P) = (e_B(P) - e_B^I(P)) + \frac{e_B^I(P)}{2} = e_B(P) - \frac{e_B^I(P)}{2}.$$

Summing up the above arguments, (13) is obtained. By virtue of the equations (13), $p = V(P)$, $q = e(P)$ and

$$\sum_{i=1}^{V(P)} \angle^E(z_i, P) = V(P) - \sum_{i=1}^{V(P)} \angle(z_i, P) = V(P) - \angle P,$$

one has then that $V(P) - \angle P + e_B^I(P)/2 - e_B(P) = \chi(P)$, as required. □

REMARK 4.4. The statement of Theorem 4.3 holds also when the definition of "lattice points" $\mathbf{Z}^2$ is replaced by a subset Z in $\mathbf{R}^2$ such as Remark 3.4.

Proof of Theorem 2.5. By Corollary 3.6 and Theorem 4.3, the claim of Theorem 2.5 is proved. □

Proof of Theorem 2.4 and Theorem 2.3. By $P = \text{cl.}(\text{int.}\, P)$, one has $e_B(P) = e_B^I(P)$, so that $-e_B(P) + e_B^I(P)/2 = -e_B(P)/2$. Combined with $V(P) = I(P) + B(P)$ and $\chi(\partial P) = B(P) - e_B(P)$, one has Theorem 2.4 by Theorem 2.5. Assume that ∂P is a disjoint union of several circles. Then $\chi(\partial P) = 0$. Hence, Theorem 2.3 is obtained by Theorem 2.4. □

REMARK 4.5. The statements of Theorems 2.5, 2.4 and 2.3 hold also when $\mathbf{Z}^2$ is replaced by $Z := g \cdot \mathbf{Z}^2$ with $g \in SL_2(\mathbf{R})$, as well as Remarks 3.7 and 4.4.

5. Pick's theorem revised

Let X be a subset of $\mathbf{R}^2$ homeomorphic to a circle. By Jordan curve theorem (cf. [8, p.68, Thm. 5.10]), $\mathbf{R}^2\backslash X$ is a disjoint union of the bounded connected component D_1 and the unbounded connected component D_2 of $\mathbf{R}^2\backslash X$ such that $X = \partial D_1 = \partial D_2$:

$$\mathbf{R}^2\backslash X = D_1 \sqcup D_2.$$

Lemma 5.1. (i) *If Q is a non-empty bounded open subset of $\mathbf{R}^2\backslash X$ such as $\partial Q \subseteq X$, then $Q = D_1$.*
(ii) *If P is a compact subset of $\mathbf{R}^2$ such that ∂P is homeomorphic to a circle, then $P = \text{cl.}(\text{int.}\,P)$.*

Proof. (i-1) *If $Q \cap D_i \neq \emptyset$, then $Q \supseteq D_i$*: Assume that $Q \not\supseteq D_i$. Take $p \in D_i\backslash Q$ and $q \in Q \cap D_i$. Then there exists a continuous curve $\gamma : [0, 1] \to D_i$ such that $\gamma(0) = q$ and $\gamma(1) = p$. Put $t_1 := \sup\{t \in [0, 1] \mid \gamma(t) \in Q\}$. Then $\gamma(t_1) \in \text{cl.}\,Q$. If $\gamma(t_1) \in Q$, then there exists $\varepsilon > 0$ such that $\gamma(t) \in Q$ if $-\varepsilon + t_1 < t < t_1 + \varepsilon$, so that $t_1 = 1$, hence $Q \ni \gamma(t_1) = \gamma(1) = p$, a contradiction. Hence, $\gamma(t_1) \notin Q$, so that $\gamma(t_1) \in \partial Q$. Because of $\partial Q \subseteq X$, $\gamma(t_1) \in X \cap D_i = \emptyset$, that is a contradiction.

(i-2) Because Q is bounded, $Q \not\supseteq D_2$. By (i-1), $Q \cap D_2 = \emptyset$. Note that $D_1 \cup D_2 \supseteq Q \neq \emptyset$, so that $Q \cap D_1 \neq \emptyset$. By (i-1), $Q \supseteq D_1$. Then $Q = D_1$, as required.

(ii) Take $X := \partial P$ and $Q := \text{int.}\,P$ in (i). Then $\text{int.}\,P = D_1$, so that $\text{cl.}(\text{int.}\,P) = \text{cl.}\,D_1 \supseteq X = \partial P$. Then $P = \partial P \cup \text{int.}\,P = \text{cl.}(\text{int.}\,P)$. □

Proof of Theorem 2.1. By Shoenflies theorem (cf. [8, p.75]), P is homeomorphic to a disk, so that $\chi(P) = 1$. Combined with Theorem 2.3 and Lemma 5.1 (ii), one has the result. □

The essential points of the traditional arguments on Pick's formula (cf. [20, 24, 1, 18]) are its additivity and the calculation of the area of primitive triangles, which are related with a nature of Euler number (cf. [11]), so that the difficulty for proving traditional Pick's theorem is essentially in the difficulty of the *direct computation* of Euler number. In this article, it is decomposed into the additivity of anglular-sum at vertexes, the possibility of primitive PL-complex decomposition, Homma's PL Gauss-Bonnet theorem and Shoenflies theorem. On the other hand, the following fact was used in the literature [2, 27, 3]: *the external anglular-sum at the boundary vertexes of a polygonal region P is 2π radian*. In general, the following result is also obtained from the equation (13):

Proposition 5.2. *Let P be a PL-figure in a plane such that $P = \mathrm{cl.(int.}\,P)$ and $\chi(\partial P) = 0$. Then "the external anglular-sum at the boundary vertexes" of P is equal to $2\pi \cdot \chi(P)$ radian.*

Proof. By Remark 3.4 and Theorem 3.2, a primitive PL-complex decomposition of P is given by considering the p-vertexes in ∂P as all "lattice points." Note that $p - e_B(P) = \chi(\partial P) = 0$, and that $e_B(P) = e_B^I(P)$ by $P = \mathrm{cl.(int.}\,P)$. By Remark 4.4, the equation (13) gives $\sum_{i=1}^{p} \angle^E(z_i, P) - p/2 = \chi(P)$, so that $\sum_{i=1}^{p}(1 - \angle(z_i, P)) = \chi(P) + p/2$, i.e.

$$\sum_{i=1}^{p} \left(\frac{1}{2} - \angle(z_i, P) \right) = \chi(P), \tag{14}$$

where the left hand side means the external angular-sum at the boundary vertexes in usual terminology, where the straight angle is measured to be $1/2$. □

REMARK 5.3. A geometric proof of Pick's formula (Theorem 2.1) and its generalization (Theorem 2.3) given in DeTemple-Robertson [2, Theorem 4], Varberg [27] and DeTemple [3] can be considered as a proof based on angular-sum formula at vertexes, Proposition 5.2, Lemma 5.1 (ii) and Schoenflies theorem.

Although Theorems 2.5 and 4.3 are firstly observed by Homma's PL Gauss-Bonnet theorem, another purely combinatorial proof is also obtained as well as a proof of Theorem 2.4 in the literature [21, 27]:

Purely combinatorial proof of Theorem 4.3. By Theorem 3.3 and Corollary 3.6, it is enough to prove the following formula:

$$\frac{N(P)}{2} = V(P) - e_B(P) + \frac{e_B^I(P)}{2} - \chi(P). \tag{15}$$

Note that each primitive triangle has three edges; each primitive line segment not contained in ∂P is the common edge of just two primitive triangles; each primitive line segment of $\partial(\mathrm{int.}\,P)$ is an edge of just one primitive triangle; and that each primitive line segment of $\mathrm{cl.}(\partial P \backslash \partial(\mathrm{int.}\,P))$ is not an edge of any primitive triangle. As well as the literature [2, 6, 9, 17, 21, 27], the total number $e(P)$ of primitive line segments of the primitive PL-complex P then satisfies the following equation:

$$2e(P) = 3N(P) + e_B^I(P) + 2(e_B(P) - e_B^I(P)). \tag{16}$$

Hence, $e(P) = (3N(P) - e_B^I(P) + 2e_B(P))/2$. Then

$$\chi(P) = V(P) - e(P) + N(P)$$

$$= V(P) - \frac{3N(P) - e_B^I(P) + 2e_B(P)}{2} + N(P),$$

as required. □

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