Tsuchimoto, Y.
Osaka J. Math.
42 (2005), 435–452

ENDOMORPHISMS OF WEYL ALGEBRA AND *p*-CURVATURES

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(Received October 9, 2003)

Abstract

We first show that for each Weyl algebra over a positive characteristic field, we may obtain an affine space with a projectively flat connection on it. We give a set of differential equations which controls the behavior of the connection under endomorphism of the Weyl algebra. The key is the theory of p-curvatures.

Next we introduce a field $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ of characteristic zero as a limit of fields of positive characteristics. We need to fix an ultrafilter on the set of prime numbers to do this. The field is actually isomorphic to the field $\mathbb{C}$ of complex numbers.

Then we show that we may associate with a Weyl algebra over the field $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ an affine space with a symplectic form in a functorial way. That means, the association is done in such a way that an endomorphism of the Weyl algebra induces a symplectic map of the affine space.

As a result, we show that a solution of the Jacobian conjecture is sufficient for an affirmative answer to the Dixmier conjecture.

1. Introduction

A Weyl algebra $A_n(k)$ over a field k is a non commutative associative algebra with $2n$ generators which satisfy canonical commutation relations.

$$A_n(k) = k\langle \xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots, \eta_n\rangle/(\eta_j\xi_i - \xi_i\eta_j - \delta_{ij}\,;\ 1 \le i, j \le n),$$

where δ_{ij} is the Kronecker's delta. It is common to relate such a "quantum" object to a "semi-classical Hamiltonian" (symplectic) object. Namely, we may associate the Weyl algebra with an affine space

$$\mathbb{A}^{2n} = \operatorname{Spec}(k[T_1, T_2, \dots, T_n, U_1, U_2, \dots, U_n])$$

with symplectic structure given by a symplectic form

$$\Omega = \sum_{i=1}^{n} dT_i \wedge dU_i.$$

2000 *Mathematics Subject Classification* : 14R15, 14A22.

Partly supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science.

In this paper we study its functoriality from an arithmetic point of view. Namely, for any non-principal ultrafilter $\mathcal{U}$ of the set $\mathrm{Spm}(\mathbb{Z})$ of maximal ideals of $\mathbb{Z}$, we first define a field $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ as a ultraproduct of fields $\overline{\mathbb{F}}_p$ of finite characteristics (Definition 6.1). The field is large enough to contain arbitrary field K which is finitely generated over $\mathbb{Q}$. In fact, it is (non canonically) isomorphic to the field $\mathbb{C}$ of complex numbers. Then we see that when the base field k is equal to our field $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ the associated symplectic space $(\mathbb{A}^{2n}, \Omega)$ behaves functorially with endomorphisms of the Weyl algebra (Proposition 7.1).

To this end, we first pass to a case where the characteristic $\mathrm{char}(k)$ of k is non zero. Then the Weyl algebra $A_n(k)$ has a fairly large center $Z_n(k)$. The "p-th root" $S_n(k)$ of $Z_n(k)$ may be regarded as a "ring of eigenvalues of $A_n(k)$." We define a symplectic form on an affine space $\mathbb{A}^{2n} = \mathrm{Spec}(S_n(k))$ by essentially inverting the process known as "geometric quantization." (The author proposes a name "geometric antiquantization" for this procedure.) Elements of the Weyl algebra are related to differential operators on a vector bundle on the affine space $\mathbb{A}^{2n}$ via a connection ∇. We will give a set of differential equations which describes how much the connection varies under an algebra endomorphism of the Weyl algebra (Proposition 3.2). The equation is quite interesting by itself and is derived from an arguments in p-curvature. (Section 8 is devoted for nontrivial examples of the equation.)

Then we go back to the characteristics zero case by taking "limit," namely by considering an ultrafilter (Proposition 7.1).

As an upshot, we show that a solution of the Jacobian conjecture would imply the (generalized) Dixmier conjecture (Corollary 7.3).

2. Preliminaries

In this section we summarize some definitions and facts on Weyl algebras needed for the rest of our arguments. Proofs are fairly easy and are found in [8].

Definition 2.1. A Weyl algebra $A_n(k)$ over a field k is a non commutative associative algebra defined as follows:

$$A_n(k) = k\langle \xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots, \eta_n \rangle / (\eta_j \xi_i - \xi_i \eta_j - \delta_{ij}\,;\, 1 \le i, j \le n),$$

(where δ_{ij} is the Kronecker's delta).

When the characteristic of k is 0, a study of k-algebra endomorphism of $A_n(k)$ is deeply related to Jacobian conjecture [3].

It is also important to study k-algebra endomorphisms of $A_n(k)$ when characteristic p of k is non zero. In that case, the Weyl algebra $A_n(k)$ has the following properties [8].

Proposition 2.2 ([8, Lemma 3]). *Let k be a field of characteristic $p > 0$.*
(1) *The center $Z_n(k)$ of the Weyl algebra $A_n(k)$ is a polynomial ring generated by $\xi_1^p, \xi_2^p, \dots, \xi_n^p, \eta_1^p, \eta_2^p, \dots, \eta_n^p$.*
(2) *A k-algebra endomorphism of $A_n(k)$ induces a k-algebra endomorphism of the center $Z_n(k)$.*

The following easy lemma plays an important role in our study.

Lemma 2.3 ([8, Lemma 1]). *Let k be a field of characteristic p. Then a k-algebra $\mathfrak{M}$ which is generated by $\mu_1, \mu_2, \dots, \mu_n, \nu_1, \nu_2, \dots, \nu_n$ with the relations*

$$[\nu_j, \mu_i] = \delta_{ji}, \quad \nu_j^p = 0, \quad \mu_i^p = 0 \qquad (i, j = 1, 2, \dots, n)$$

(*where δ_{ij} is the Kronecker's delta*) *is isomorphic to the full matrix algebra $M_{p^n}(k)$.*

Proof. We have a representation of the algebra on $k[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots, x_n^p)$ where μ_i is identified with multiplication by x_i, ν_i is identified with $\partial/\partial x_i$. $\square$

From now on, we fix such an isomorphism all the way and regard the elements $\mu_1, \mu_2, \dots, \mu_n, \nu_1, \nu_2 \dots, \nu_n$ as matrices. The next lemma tells us that each of the standard generators $\xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots \eta_n$ is expressed as a sum of its (unique) eigen value and a constant matrix.

We first add some notation. We assume k is a perfect field of characteristic $p > 0$.

DEFINITION 2.4. We put $S_n(k) = Z_n(k)^{1/p} = k[T_1, T_2, \dots, T_n, U_1, U_2, \dots, U_n]$ ($T_i = (\xi_i^p)^{1/p}, U_i = (\eta_i^p)^{1/p}$). (Where $\bullet^{1/p}$ denotes 'p-th root in the sense of commutative algebra.')

We may regard T_i, U_i respectively as the "eigen value" of operators ξ_i, η_i.

Lemma 2.5 ([8, Lemma 5]). *Let us put $M = S_n[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p \dots, x_n^p)$. Then there is a representation of $A_n(k)$ on M defined in the following way.*

$$\Phi(\xi_i) = x_i + T_i, \qquad \Phi(\eta_i) = \frac{\partial}{\partial x_i} + U_i.$$

In other words, there exists a k-algebra homomorphism $\Phi\colon A_n(k) \to M_{p^n}(S_n(k))$ defined by the above representation.

The k-algebra homomorphism Φ may be extended to the following isomorphism.

$$\Phi\colon A_n(k) \otimes_{Z_n(k)} S_n(k) \cong M_{p^n}(S_n(k)).$$

In terms of the "trivialization" of the Weyl algebra $A_n(k)$ as above, endomorphisms of $A_n(k)$ is expressed as follows:

Lemma 2.6 ([8, Lemma 7]). *Let k be an algebraically closed field with characteristic $p > 0$. Let $\phi\colon A_n(k) \to A_n(k)$ be a k-homomorphism, $\psi\colon Z_n(k) \to Z_n(k)$ its restriction to the center. Then we have the following.*
(1) *ψ extends uniquely to a k-algebra homomorphism*

$$\hat{\psi}\colon S_n(k) \to S_n(k)$$

(2) *ϕ extends uniquely to a k-algebra homomorphism*

$$\hat{\phi}\colon A_n(k) \otimes_{Z_n(k)} S_n(k) \to A_n(k) \otimes_{Z_n(k)} S_n(k)$$

(3) *Under the isomorphism Φ of* Lemma 2.5, *$\hat{\phi}$ may be identified with a map*

$$M_p(S_n(k)) \ni m(T,U) \mapsto G(T,U)m(f(T,U))G(T,U)^{-1}$$

where $G(T,U)$ is an element of $GL_p(S_n(k))$ and $f = {}^a\hat{\psi}\colon \mathbb{A}^{2n} \to \mathbb{A}^{2n}$ is a polynomial map associated to the algebra homomorphism $\hat{\psi}$. In other words, we have

$$\Phi(\phi(x)) = Gf^*(\Phi(x))G^{-1}.$$

In short, we may decompose the algebra endomorphism ϕ of the Weyl algebra into two pieces. One is an endomorphism $\hat{\psi}$ of a polynomial algebra, and the other is a conjugation by a matrix valued function G. It is naturally related to the theory of fiber bundles and connections, and that is what we do in the next section.

Corollary 2.7. *ϕ is invertible if and only if ψ is invertible.*

3. Symplectic structure defined by a connection

In this section we fix a field k of characteristic $p > 0$ and write S_n for $S_n(k)$, A_n for $A_n(k)$, and so on.

Now we introduce a symplectic form on the space $\mathrm{Spec} S_n$ of "eigen values of generators $\xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots, \eta_n$ for A_n." This is done by inverting the process known as "geometric quantization." (See [9] for a description of geometric quantization.) Namely, we create a connection on a vector bundle on $\mathrm{Spec} S_n$ by using the non-commutative algebra $B_n = A_n \otimes_{Z_n} S_n$ over S_n and then consider its curvature form. (We may call this procedure "geometric anti-quantization.") Note that B_n is isomorphic to the full matrix algebra $M_{p^n}(S_n)$ over S_n (Lemma 2.5).

Lemma 3.1. *Let $\widetilde{B}_n$ be the sheaf of modules on* $\mathrm{Spec}(S_n)$ *associated to the S_n-module B_n $(= A_n \otimes_{Z_n} S_n)$. Then there exists an unique connection $\nabla^{(0)}$ on the sheaf $\widetilde{B}_n$ such that the following conditions hold.*

(1) $\nabla^{(0)}$ *is flat* (*that means, it has zero curvature*).
(2) *Every section* s *of* $\widetilde{B}_n$ *which corresponds to an element of* A_n *is parallel with respect to* $\nabla^{(0)}$.
The $\nabla^{(0)}$ *above further satisfies the following Leibniz rule.*

$$\nabla^{(0)}(xy) = (\nabla^{(0)}x)y + x(\nabla^{(0)}y) \qquad (\forall x, \forall y \in B_n)$$

Proof. The uniqueness of such a connection is a direct consequence of the fact that B_n is generated by A_n as a S_n-module.

To see the existence, we may identify, as we have already seen in Lemma 2.5, the algebra B_n with $M_{p^n}(S_n)$, elements $\xi_1, \ldots, \xi_n$, $\eta_1, \ldots, \eta_n$ corresponding respectively to $T_1 + \mu_1, \ldots, T_n + \mu_n, U_1 + \nu_1, \ldots, U_n + \nu_n$. Then the connection $\nabla^{(0)}$ may be defined by the following formulae.

$$\left.\begin{aligned} \nabla^{(0)}_{\partial/\partial T_i} &= \frac{\partial}{\partial T_i} + \mathrm{ad}(\nu_i), \\ \nabla^{(0)}_{\partial/\partial U_i} &= \frac{\partial}{\partial U_i} - \mathrm{ad}(\mu_i). \end{aligned}\right\} \qquad (i = 1, 2, \ldots, n) \tag{3.1}$$

We see immediately that the connection $\nabla^{(0)}$ surely satisfies the required conditions. □

We would like to get rid of "Ad" in the equation (3.1). In other words, we "lift" the connection $\nabla^{(0)}$ and introduce a connection ∇ on the trivial sheaf $\widetilde{M}$ (associated to the free S_n-module $M = \bigoplus_{i=1}^{p^n} S_n$) on $\mathrm{Spec} S_n$ by the following formula.

$$\nabla = d + dF, \quad F = \sum_{i=1}^{n} T_i \nu_i - U_i \mu_i.$$

It is not as canonical as the connection $\nabla^{(0)}$. Our task is therefore to find how it varies under an algebra endomorphism of A_n.

The connection ∇ satisfies the following compatibility condition with $\nabla^{(0)}$.

$$\nabla(av) = (\nabla^{(0)}a)v + a(\nabla v) \qquad (a \in B_n,\ v \in M). \tag{3.2}$$

Let $\phi\colon A_n \to A_n$ be a k-algebra endomorphism. Let $f\colon \mathrm{Spec} S_n \to \mathrm{Spec} S_n$ be the associated morphism.

Note that any connection on M may be written as a sum of the exterior derivative d and a matrix valued 1-form φ. We may define pull back of the connection with respect to f as follows:

$$f^*(d + \varphi) = d + f^*\varphi.$$

We would like to know how our connection ∇ varies when pulled back with respect to f. The answer is given by the following proposition.

Proposition 3.2. *Let k be a field of characteristic $p > 0$. Let $\phi\colon A_n \to A_n$ be a k-algebra endomorphism. Let $f\colon \operatorname{Spec} S_n \to \operatorname{Spec} S_n$ be the associated morphism. Let G be a matrix valued function on $\operatorname{Spec} S_n$ which satisfies the property* (3) *of* Lemma 2.6. *Then we have the following identity.*

$$G(f^*\nabla)G^{-1} = \nabla + \omega \tag{3.3}$$

where $\omega = \sum_{i=1}^n (\omega_{T_i} dT_i + \omega_{U_i} dU_i)$ is a unique solution of the following set of differential equations.

$$\left.\begin{aligned} \omega_{T_i}^p + \left(\frac{\partial}{\partial T_i}\right)^{p-1}\left(\omega_{T_i} - \sum_{j=1}^n \overline{T}_j \frac{\partial \overline{U}_j}{\partial T_i}\right) = 0 \\ \omega_{U_i}^p + \left(\frac{\partial}{\partial U_i}\right)^{p-1}\left(\omega_{U_i} - \sum_{j=1}^n \overline{T}_j \frac{\partial \overline{U}_j}{\partial U_i}\right) = 0 \end{aligned}\right\} \qquad (i = 1, 2, \dots, n) \tag{3.4}$$

where $\overline{T}_i = \hat{\psi}(T_i) = f^*(T_i)$, $\overline{U}_i = \hat{\psi}(U_i) = f^*(U_i)$ (see Lemma 2.6 *for notation).*

Note that above proposition immediately implies the following.

Corollary 3.3. *If the total degree of ϕ is less than $p/2$, (that is, elements $\phi(\xi_1), \phi(\xi_2), \dots, \phi(\xi_n), \phi(\eta_1), \phi(\eta_2), \dots, \phi(\eta_n)$ are all expressed as a (non commutative) polynomial of the standard generators $\{\xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots, \eta_n\}$ of degree less than $p/2$) then the connection ∇ is "invariant." To be more precise, we have*

$$G(f^*\nabla)G^{-1} = \nabla$$

in that case. In particular, the associated map f preserves the curvature (symplectic form) $\Omega = \sum_{i=1}^n dT_i \wedge dU_i$. Thus the Jacobian of f is equal to 1.

To prove Proposition 3.2, we use the theory of p-curvature ([4], [6]).
In particular, the following lemma is needed for our argument.

Lemma 3.4 ([6, §10.6.3]). *Let D be a derivation on a commutative algebra C. Assume that there exists a noncommutative algebra A which contains C as a subalgebra and an element $\xi \in A$ such that*

$$[\xi, f](= \xi f - f\xi) = D(f)$$

holds for all $f \in C$. Then for any element f of C we have

$$(\xi + f)^p = \xi^p + D^{p-1}(f) + f^p.$$

Proof of Proposition 3.2. Since the equation (3.4) is additive in ω, the uniqueness of its solution is equivalent to saying that there is no non trivial solution to the following equation.

$$\left.\begin{aligned} \omega_{T_i}^p + \left(\frac{\partial}{\partial T_i}\right)^{p-1}(\omega_{T_i}) = 0 \\ \omega_{U_i}^p + \left(\frac{\partial}{\partial U_i}\right)^{p-1}(\omega_{U_i}) = 0 \end{aligned}\right\} \qquad (i = 1, 2, \dots, n) \tag{3.5}$$

By considering the highest degree of both hand sides, we see that the above equations surely have no nontrivial solution.

Let us now proceed to obtain the differential equations. Unfortunately, p-curvature is only significant for a flat connection, which our connection ∇ fails to be. So we modify the connection ∇ and introduce another connection $\nabla^{(1)}$ on the sheaf $\widetilde{M}$ by the following formula.

$$\nabla^{(1)} = \nabla - \sum_{i=1}^n T_i\, dU_i \left(= d + dF - \sum_{i=1}^n T_i\, dU_i \right) \tag{3.6}$$

In other words,

$$\nabla^{(1)}_{\partial/\partial T_i} = \frac{\partial}{\partial T_i} + \nu_i, \qquad \nabla^{(1)}_{\partial/\partial U_i} = \frac{\partial}{\partial U_i} - \mu_i - T_i.$$

From the above formula we may check immediately that the connection $\nabla^{(1)}$ certainly is a flat connection. Let us compute its p-curvature $\mathrm{curv}_p\, \nabla^{(1)}$. For any vector field D on $\mathrm{Spec} S_n$, the following relation holds.

$$(\mathrm{curv}_p\, \nabla^{(1)})(D) = (\nabla^{(1)}_D)^p - \nabla^{(1)}_{D^p} = -\left\langle \sum_{i=1}^n T_i\, dU_i, D \right\rangle^p. \tag{3.7}$$

(The symbol $\langle \varphi, D \rangle$ denotes the pairing of a 1-form φ with a derivation (= tangent vector field) D.) Indeed, the above formula is clearly valid for $D = \partial/\partial T_i$ or $D = \partial/\partial U_i$. Using p-semi-linearity (the terminology is borrowed from [5])

$$(\mathrm{curv}_p\, \nabla^{(1)})(f_1 D_1 + f_2 D_2) = f_1^p \cdot (\mathrm{curv}_p\, \nabla^{(1)})(D_1) + f_2^p \cdot (\mathrm{curv}_p\, \nabla^{(1)})(D_2)$$

of the p-curvature, we see that the formula is valid in general.

Now, suppose we have an endomorphism of A_n. We have an associated map $f \colon \mathrm{Spec} S_n \to \mathrm{Spec} S_n$. Both the connection $\nabla^{(1)}$ and the conjugate $G(f^*\nabla^{(1)})G^{-1}$ of the pull back are connections on M which satisfy the same compatibility condition as ∇ in equation (3.2).

Thus we see that there exists an scalar valued 1-form $\omega^{(1)}$ such that the following equation holds.

$$G(f^*\nabla^{(1)})G^{-1} = \nabla^{(1)} + \omega^{(1)}$$

If D is a derivation on A_n which satisfies $D^p = 0$, then we use Lemma 3.4 and obtain

$$G((f^*\nabla^{(1)})_D)^p G^{-1} = (\nabla_D^{(1)})^p + D^{p-1}\langle\omega^{(1)}, D\rangle + \langle\omega^{(1)}, D\rangle^p.$$

Applying equation (3.7) to this equation, we obtain the following equation.

$$-\left\langle \sum_{i=1}^n \overline{T}_i\, d\overline{U}_i, D \right\rangle^p = -\left\langle \sum_{i=1}^n T_i\, dU_i, D \right\rangle^p + D^{p-1}\langle\omega^{(1)}, D\rangle + \langle\omega^{(1)}, D\rangle^p$$

Putting $D = \partial/\partial T_i$ or $D = \partial/\partial U_i$, we obtain differential equations for $\omega^{(1)}$. It is now easy to go back and derive the differential equations (3.4) for $\omega = G(f^*\nabla)G^{-1} - \nabla$ using the equation (3.6). □

4. Estimate for degree of inverse

We cite here the following proposition.

Proposition 4.1 ([1]). *Let k be a field with any characteristic. Let $F: k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_n]$ be an algebra automorphism of polynomials over a field k. Then we have*

$$\deg(F^{-1}) \le \deg(F)^{n-1}.$$

We have the similar bound for Dixmier case.

Proposition 4.2. *Let k be a field of characteristic $p > 0$. Let ϕ: $A_n(k) \to A_n(k)$ be a k-algebra automorphism of a Weyl algebra over a field k. Then we have*

$$\deg(\phi^{-1}) \le \deg(\phi)^{2n-1}.$$

Proof. ϕ induces a k-algebra automorphism ψ of the center $Z_n(k)$ of $A_n(k)$. Then we notice that the algebra $Z_n(k)$ is isomorphic to polynomial algebra of $2n$-variables and that the degree of ψ is the same as that of ϕ. □

Corollary 4.3. *ϕ is surjective if and only if its restriction to a finite dimensional vector space*

$$A_n(k)_{\le 2n-1} = \{x \in A_n(k);\ \textit{total degree of } x \textit{ is less than or equal to } 2n-1\}$$

is surjective on standard generators.

Thus the question of surjectivity reduces to a question on a linear map between finite dimensional vector spaces. Thus we see that the following lemma holds.

Lemma 4.4 (A stronger form of [8, Lemma 2]). *Let $\mathfrak{K}$ be an algebraic number field with the ring of integers $\mathfrak{O}$. Assume we are given a $\mathfrak{K}$-algebra endomorphism ϕ of the Weyl algebra $A_n(\mathfrak{K})$. Then*
(1) *For almost all (that is, all except finite number of) prime ideals $\mathfrak{p}$ of $\mathfrak{O}$, ϕ induces an algebra endomorphism $\phi_{\mathfrak{p}}$ of an algebra $A_n(k(\mathfrak{p}))$ over $k(\mathfrak{p})$ where $k(\mathfrak{p}) = \mathfrak{O}/\mathfrak{p}$ is a field of a positive characteristic p.*
(2) *ϕ is invertible if and only if homomorphisms $\phi_{\mathfrak{p}}$ are invertible for infinitely many primes $\mathfrak{p} \in \mathrm{Spec}(\mathfrak{O})$.*

5. Limit using ultrafilter

In this section we use ultrafilter-limit to show how we obtain the results for fields of characteristics 0 from the results for fields of positive characteristics. To avoid unnecessary confusion, we recall some elementary facts in the following. Those readers who are not familiar with the arguments are invited to read for example [7] or the book of Bourbaki [2].

Lemma 5.1. *Let $\mathcal{U}$ be a filter on a set X. The following statements are equivalent.*
(1) *$\mathcal{U}$ is an ultrafilter. That means, a maximal filter.*
(2) *for any subset $S \subset X$, we have either $S \in \mathcal{U}$ or $\complement S \in \mathcal{U}$*

DEFINITION 5.2. A principal filter on a set X is an ultra filter of the form $\mathcal{F}_a = \{S \subset X \mid a \in S\}$ where a is an element of X. A ultrafilter which is not principal filter is called non-principal.

Lemma 5.3. *For any ultrafilter $\mathcal{U}$, the following statements are equivalent.*
(1) *$\mathcal{U}$ is principal.*
(2) *$\mathcal{U}$ is not free. That means, $\bigcap_{U \in \mathcal{U}} U \neq \emptyset$.*
(3) *There exists a member E of $\mathcal{U}$ which is a finite set ($\#E < \infty$).*
(4) *There exists a co-finite subset Y of X (that means, $\#(X \setminus Y) < \infty$,) such that $Y \notin \mathcal{U}$.*
In particular, if $\mathcal{U}$ is a non-principal ultrafilter on a set X, then any co-finite subset Y of X of is a member of $\mathcal{U}$.

An ultrafilter $\mathcal{U}$ on a set X may be identified with a point of Stone-Čech compactification of (X with discrete topology). A non principal ultrafilter is identified with a boundary point. In particular, any infinite sets have non principal ultrafilters.

6. A field $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$

In this section we define a field $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ as a ultraproduct of fields of positive characteristics. It is isomorphic to the field $\mathbb{C}$ of complex numbers and therefore plays a role of 'universal field' in the sense of Weil.

We denote by $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ a finite field with p-elements, and by $\overline{\mathbb{F}}_p = \operatorname{inj\,lim}_{n\to\infty} \mathbb{F}_{p^n}$ its algebraic closure.

DEFINITION 6.1. Let $\mathcal{U}$ be an ultrafilter on $\operatorname{Spm}(\mathbb{Z})$. We put

$$\mathbb{Q}_{\mathcal{U}}^{(\infty)} = \prod_{p\in\operatorname{Spm}(\mathbb{Z})} \overline{\mathbb{F}}_p/\mathfrak{J}_{\mathcal{U}}$$

where $\mathfrak{J}_{\mathcal{U}}$ is an ideal of $\prod \overline{\mathbb{F}}_p$ defined as follows

$$\mathfrak{J}_{\mathcal{U}} = \left\{ (f_p)_{p\in\operatorname{Spm}(\mathbb{Z})} \in \prod_{p\in\operatorname{Spm}(\mathbb{Z})} \overline{\mathbb{F}}_p \;\middle|\; \exists U \in \mathcal{U} \text{ such that } f_p = 0 \text{ for } \forall p \in U \right\}.$$

We also define the following.

$$\mathbb{Q}_{\mathcal{U}} = \prod_{p\in\operatorname{Spm}(\mathbb{Z})} \mathbb{F}_p \Bigg/ \left(\left(\prod_{p\in\operatorname{Spm}(\mathbb{Z})} \mathbb{F}_p \right) \cap J_{\mathcal{U}} \right)$$

We denote by $\pi_{\mathcal{U}} \colon \prod_p \overline{\mathbb{F}}_p \to \mathbb{Q}_{\mathcal{U}}^{(\infty)}$ the canonical projection.

Lemma 6.2. $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ *is an algebraically closed field of characteristic* 0 *which contains* $\mathbb{Q}_{\mathcal{U}}$ *as a subfield.*

Proof. Let us first show that $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ is a field. Let $f = \pi_{\mathcal{U}}((f_{\mathfrak{p}}))$ be a non zero element in $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$. Let $E_1 = \{\mathfrak{p} \in \operatorname{Spm}(\mathfrak{O});\ f_{\mathfrak{p}} \neq 0\}$. Then for any $E \in \mathcal{U}$, intersection $E \cap E_1$ is non empty. Maximality of $\mathcal{U}$ now implies that E_1 itself is a member of $\mathcal{U}$. The inverse $g = (g_{\mathfrak{p}})$ of f in $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ is given by the following formula.

$$g_{\mathfrak{p}} = \begin{cases} f^{-1} & \text{if } \mathfrak{p} \in E_1 \\ 0 & \text{otherwise} \end{cases}$$

Next let us prove that the characteristic of $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ is zero. If $n = 0$ in $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ for a positive integer n, then there exists $E_0 \in \mathcal{U}$ such that $n \in \cap_{\mathfrak{p}\in E_0}\mathfrak{p}$. On the other hand, as we have mentioned in Lemma 5.3 above, being a member of a non-principal filter $\mathcal{U}$, E_0 cannot be a finite set. Thus the characteristic of $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ is zero.

That $\mathbb{Q}_{\mathcal{U}}$ is a field is proved in a similar way. $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ clearly contains $\mathbb{Q}_{\mathcal{U}}$ as a subfield.

Finally let us prove $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ is algebraically closed. Let

$$a(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$$

be a monic element of the polynomial algebra $\mathbb{Q}_{\mathcal{U}}^{(\infty)}[X]$. Then each coefficient a_i may be written as a limit

$$a_i = \pi_{\mathcal{U}}((a_{i,p})) \qquad a_{i,p} \in \overline{\mathbb{F}}_p.$$

Since $\overline{\mathbb{F}}_p$ is algebraically closed, we may find an solution r_p of the polynomial

$$a_{,p}(X) = X^n + a_{n-1,p}X^{n-1} + \cdots + a_{1,p}X + a_{0,p}.$$

Then we may see easily that

$$r = \pi_{\mathcal{U}}((r_p))$$

is a root of a in $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$. □

Lemma 6.3. *The cardinality of $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ is equal to that of $\mathbb{R}$. Hence $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ has infinite transcendence degree over $\mathbb{Q}$.*

Proof. Since $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ is a quotient space of countable product of countable sets $\overline{\mathbb{F}}_p$, we have $\#\mathbb{Q}_{\mathcal{U}}^{(\infty)} \leq \#\mathbb{R}$. To prove the converse inequality, we would like to define a surjective map $\mathbb{Q}_{\mathcal{U}} \to S^1 = \mathbb{R}/\mathbb{Z}$. First we define $\iota_p \colon \mathbb{F}_p \to S^1$ by

$$\iota_p \colon \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \ni (n \bmod \mathbb{Z}) \to (n/p \bmod \mathbb{Z}) \in \mathbb{R}/\mathbb{Z}.$$

It is an well-defined map. Then we define π as follows.

$$\pi(\pi_{\mathcal{U}}((a_p))) = \lim_{\substack{p \to \infty \\ \mathcal{U}}} \iota_p(a_p)$$

Where the limit in the right hand side is the one in the usual sense of filter. It is easy to verify that the limit always converges and π is well-defined. (See the Lemma below) Furthermore, for any $\alpha \in S^1$ and for any prime p, we may choose $a_p \in \mathbb{F}_p$ such that $\iota_p(a_p)$ is the nearest to α. Then the distance $\mathrm{dist}(\iota_p(a_p), \alpha)$ between $\iota_p(a_p)$ and α should be smaller than $1/n$. We see immediately from this that $\pi((a_p)) = \alpha$. Thus our map π is surjective. □

For those of readers who is not used to the arguments in ultrafilter, we record here the proof of the fact used in the proof above.

Lemma 6.4. *π defined in the proof above is well-defined.*

Proof. For $U \in \mathcal{U}$ we put

$$J_U = \{\iota_p(a_p);\ p \in U\}.$$

Then a collection

$$\mathcal{F} = \{J_U\,;\ U \in \mathcal{U}\}$$

is a filter. We need to show that $\mathcal{F}$ is Cauchy. That means, for any $\epsilon > 0$, there exists a member $U \in \mathcal{U}$ such that diameter of J_U is less than ϵ. To do this we divide S^1 into several pieces. Let N be an integer such that $N > 1/\epsilon$. We put

$$S^1_{i,N} = \left[\frac{i}{N}, i + \frac{1}{N}\right) \bmod \mathbb{Z}$$

and set

$$T_{i,N} = \{p \in \mathrm{Spm}(\mathbb{Z});\ \iota_p(a_p) \in S^1_{i,N}\}.$$

Then we clearly have

$$\mathrm{Spm}(\mathbb{Z}) = T_{0,N} \cup T_{1,N} \cup T_{2,N} \cup \cdots \cup T_{N-1,N} \qquad \text{(disjoint union).}$$

Since $\mathcal{U}$ is an ultrafilter, using Lemma 5.1 (2) several times, we notice that we have one and only one i such that $T_{i,N}$ is a member of $\mathcal{U}$. $J_{T_{i,U}}$ is a member of $\mathcal{F}$ and is contained in $S^1_{i,N}$. It follows that the diameter of $J_{T_i,N}$ is less than ϵ. $\square$

7. Main proposition

Proposition 7.1. *For each non-principal ultrafilter $\mathcal{U}$ on* $\mathrm{Spm}(\mathbb{Z})$*, we have the following statements.*

(1) *$\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ is isomorphic to $\mathbb{C}$ as a field.*

(2) *There exists a monoid homomorphism*

$$L_{\mathcal{U}} : \mathrm{End}_{\mathbb{Q}_{\mathcal{U}}^{(\infty)}\text{-}alg}(A_n(\mathbb{Q}_{\mathcal{U}}^{(\infty)})) \to \mathrm{End}_{\mathbb{Q}_{\mathcal{U}}^{(\infty)}\text{-}alg}(\mathbb{Q}_{\mathcal{U}}^{(\infty)}[T_1, T_2, \ldots, T_n, U_1, U_2, \ldots, U_n])$$

such that the following properties hold.

(3) *$L_{\mathcal{U}}(\phi)$ preserves the symplectic form*

$$\Omega = dT_1 \wedge dU_1 + dT_2 \wedge dU_2 + \cdots + dT_n \wedge dU_n.$$

(4) *The Jacobian of $L_{\mathcal{U}}(\phi)$ is constant and is equal to* 1.

(5) *$L_{\mathcal{U}}(\phi)$ is invertible if and only if ϕ is invertible.*

Proof. (1) is already proved in Lemmas 6.2 and 6.3. For the sake of simplicity, we denote

$$\gamma_1, \gamma_2, \ldots, \gamma_{2n} = \xi_1, \xi_2, \ldots, \xi_{2n}, \eta_1, \eta_2, \ldots, \eta_{2n},$$

and for any multi-index $I = \{i_1, i_2, \ldots, i_{2n}\}$, we put

$$\gamma^I = \gamma_1^{i_1} \ldots \gamma_n^{i_n} \gamma_{n+1}^{i_{n+1}} \ldots \gamma_{2n}^{i_{2n}}.$$

Then for any field k, any element of $A_n(k)$ is uniquely written as a k-linear combination of γ^I 's. Now, suppose we have a $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$-algebra endomorphism $\phi\colon A_n(\mathbb{Q}_{\mathcal{U}}^{(\infty)}) \to A_n(\mathbb{Q}_{\mathcal{U}}^{(\infty)})$. In terms of the standard generator γ_i's, we may write

$$\phi(\gamma_i) = \sum_{i,I} a_{i,I} \gamma^I.$$

Regarded as elements of $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$, $a_{i,I}$ may be written as a limit

$$a_{i,I} = \pi_{\mathcal{U}}((a_{i,I,p})) \qquad (a_{i,I,p} \in \overline{\mathbb{F}}_p).$$

Since condition of ϕ being algebra endomorphism is given by a finite set of equations in terms of coefficients $\{a_{i,I}\}$ of $\phi_{\mathcal{U}}$, we see immediately that there exists $U \in \mathcal{U}$ such that for each $p \in U$, we may define an $\overline{\mathbb{F}}_p$-algebra homomorphism

$$\phi_p\colon A_n(\overline{\mathbb{F}}_p) \to A_n(\overline{\mathbb{F}}_p)$$

by the formula

$$\phi_p(\gamma_i) = \sum_{i,I} a_{i,I,p} \gamma^I.$$

The total degree of ϕ_p is less than or equal to the total degree d of ϕ. Applying Corollary 3.3 to ϕ_p for $p > 2d$, we see that the map

$$\psi_p\colon \mathbb{A}^{2n}(\overline{\mathbb{F}}_p) \to \mathbb{A}^{2n}(\overline{\mathbb{F}}_p)$$

associated to ϕ_p ($p \in U, p > 2d$) induces a polynomial map which preserves the symplectic form. So we put

$$\psi = \pi_{\mathcal{U}}((\psi_p)).$$

It is an well-defined polynomial map which preserves the symplectic form. This proves (2) and (3). (4) easily from (3). Let us prove (5). “If” part is clear. Suppose $L_{\mathcal{U}}(\phi)$ is invertible. Let $\overline{\psi} = \{\overline{\psi}_p\}$ be the inverse. There exists an element $E \in \mathcal{U}$ such that $\overline{\psi}_p \circ \psi_p = id$ holds for any element $p \in E$. Then for such p, we see from Corollary 2.7

that $\phi_p\colon A_n(\overline{\mathbb{F}}_p) \to A_n(\overline{\mathbb{F}}_p)$ is invertible. Let $\overline{\phi}_p$ be the inverse. There is a bound for the total degrees of $\overline{\phi}_p$ by virtue of Proposition 4.2. Thus we may gather these $\overline{\phi}_p$ together and define an $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$-algebra homomorphism

$$\overline{\psi} = \pi_{\mathcal{U}}((\overline{\phi}_p))\colon A_n(\mathbb{Q}_{\mathcal{U}}^{(\infty)}) \to A_n(\mathbb{Q}_{\mathcal{U}}^{(\infty)})$$

$\overline{\psi}$ clearly gives an inverse of ϕ. $\square$

Let K be a field of characteristic zero. Then the following conjectures are known.

Conjecture. (generalized) Dixmier conjecture (Dixmier (n)) Any endomorphism of $A_n(K)$ is invertible.

Conjecture. Jacobian conjecture (Jacobian (n)) Any morphism $\mathbb{A}_n(K) \to \mathbb{A}_n(K)$ with the constant Jacobian is invertible.

The following two facts are known.

Proposition 7.2. (1) *Jacobian conjecture may be reduced to the case where* $K = \mathbb{Q}$.
(2) *Dixmier* (n) *implies Jacobian* (n).

(See [3] for details of these two facts.)
The Proposition 7.1 immediately yields the following.

Corollary 7.3. *Jacobian* $(2n)$ *implies Dixmier* (n).

Proof. Suppose we have a K-algebra endomorphism $\phi\colon A_n(K) \to A_n(K)$ for a field K. In terms of the standard generator γ_i's, we may write

$$\phi(\gamma_i) = \sum_{i,I} a_{i,I}\gamma^I.$$

We may concentrate ourselves to the subfield of K generated by all the coefficients $\{a_{i,I}\}$ of ϕ and assume that K is finitely generated over $\mathbb{Q}$. Then we may embed K into our field $\Omega_{\mathcal{U}}$ and apply Proposition 7.1. By the assumption Jacobian $(2n)$, the map $L_{\mathcal{U}}(\phi)$ is invertible. This implies ϕ is invertible. $\square$

So if we leave the number n of variables out of count, we have shown that the Jacobian conjecture and the Dixmier conjecture are equivalent.

8. Examples

In this section we give some examples of endomorphism ϕ of the Weyl algebra $A_n(k)$ over a field k of characteristic $p > 0$.

EXAMPLE 8.1. For any non-negative integer r, s, t such that $(s + t + 1) = p^r$, we define an algebra endomorphism $\phi^{(r,s)}$ of $A_1(k)$ as follows.

$$\phi^{(r,s)}(\xi_1) = (\xi_1\eta_1)^s\xi_1, \qquad \phi^{(r,s)}(\eta_1) = \eta_1(\xi_1\eta_1)^t.$$

It induces the endomorphism $\hat{\psi}^{(r,s)}$ of the polynomial algebra $S_1(k) = k[T_1, U_1]$ given by the following formula.

$$\hat{\psi}^{(r,s)}(T_1) = T_1^{s+1}U_1^s, \qquad \hat{\psi}^{(r,s)}(U_1) = T_1^tU_1^{t+1}$$

The differential equation in Proposition 3.2 tells us that the 1-form ω in the equation (3.3) is given in the following formula.

$$\omega = -\sum_{l=0}^{r-1} tU_1^{p^l}T_1^{p^l-1}dT_1 + \sum_{l=0}^{r-1} sT_1^{p^l}U_1^{p^l-1}dU_1$$

The Jacobian of $\hat{\psi}$ is 0 unless $s = t = r = 0$. Note also that the set of all the endomorphisms given in this examples are closed under composition of endomorphisms.

$$\phi^{(r_2,s_2)} \circ \phi^{(r_1,s_1)} = \phi^{(r_1+r_2, s_1+p^{r_1}s_2)}.$$

EXAMPLE 8.2. Let a be a polynomial in two variables. Then we may consider the following endomorphism ϕ of $A_1(k)$.

$$\phi(\xi_1) = \xi_1 + a(\xi_1^p, \eta_1), \qquad \phi(\eta_1) = \eta_1.$$

It induces an endomorphism $\hat{\psi}$ of the center

$$\hat{\psi}(T_1) = T_1 + a(T_1^p, U_1) + b(T_1, U_1), \qquad \hat{\psi}(U_1) = U_1$$

where $b(T_1, U_1)$ is a polynomial determined by the following equation.

$$b(T_1, U_1)^p = \left(\frac{d}{dU_1}\right)^{p-1} a(T_1^p, U_1).$$

We may easily compute the 1-form ω and the Jacobian $\mathrm{Jac}(\hat{\psi})$ of $\hat{\psi}$.

$$\omega = b(T_1, U_1)\, dU_1,$$

$$\mathrm{Jac}(\hat{\psi}) = 1 + \left(\frac{d}{dT_1}\right) b(T_1, U_1).$$

Let us describe them in terms of coefficients in a,

$$a(T_1, U_1) = \sum_{i,j=0}^{\infty} a_{i,j} T_1^i U_1^j.$$

The functions $b, \mathrm{Jac}(\hat{\psi})$ are written in the following manner.

$$b(T_1, U_1) = -\sum_{i\geq 0,\, l>1} (a_{i,lp-1})^{1/p} T_1^i U_1^{l-1}$$

$$\mathrm{Jac}(\hat{\psi}) = 1 - \sum_{i,l=1}^{\infty} (a_{i,lp-1})^{1/p} i T_1^{i-1} U_1^{l-1}$$

Note that $\mathrm{Jac}(\hat{\psi}) = 1$ if the total degree of a is smaller than p.

A little more computation shows the following. If the total degree of a is less than $p-1$, then G is given by the following formula.

$$G = \mathrm{ex}\left(-\left[\int_{s=0}^{t} a(s + U_1) - a(U_1)\, ds\right]_{t=\nu_1}\right).$$

Where ex is a polynomial obtained by cutting exponential function off the tail after p ([8, §2.2]).

$$\mathrm{ex}(L) = \sum_{i=0}^{p-1} \frac{1}{i!} L^i.$$

EXAMPLE 8.3. Let $a_1, a_2, \ldots, a_n$ be n polynomials in n-variables $x_1, x_2, \ldots, x_n$ with the Jacobian 1.

$$\mathrm{Jac}(a_1, a_2, \ldots, a_n; x_1, x_2, \ldots, x_n) = 1.$$

Then we may obtain an algebra endomorphism ϕ of $A_n(k)$ as follows.

$$\left.\begin{aligned} \phi(\xi_i) &= a_i(\xi_1, \xi_2, \ldots, \xi_n), \\ \phi(\eta_i) &= \sum_{j=1}^{n} b_{ij}(\xi_1, \xi_2, \ldots, \xi_n)\eta_j. \end{aligned}\right\} \quad (i = 1, 2, \ldots, n)$$

Where b is the matrix elements of inverse of the Jacobian matrix of a.

$$\sum_{j=1}^{n} b_{lj} \frac{\partial a_i}{\partial \xi_j} = \delta_{li} \qquad (i, l = 1, 2, \ldots, n).$$

The endomorphism $\hat{\psi}$ of the polynomial algebra $S_n(k) = k[T_1, T_2, \ldots, T_n, U_1, U_2, \ldots, U_n]$ induced by ϕ is given by the following formula.

$$\left.\begin{aligned} \hat{\psi}(T_i) &= a_i(T_1, T_2, \ldots, T_n), \\ \hat{\psi}(U_i) &= \sum_{j=1}^{n} b_{ij}(T_1, T_2, \ldots, T_n)U_j. \end{aligned}\right\} \qquad (i = 1, 2, \ldots, n)$$

Indeed, the formula for $\hat{\psi}(T_i)$ is clear. The formula for $\hat{\psi}(U_i)$ is obtained in the following steps.

STEP 1. The degrees of $\hat{\psi}$ in U-variables should be one. In other words. There exists polynomials c_{ij}, d_i such that

$$\phi(U_i)^p = \sum c_{ij}(T)U_j^p + d_i(T)$$

holds.

STEP 2. We consider their principal symbols of $\phi(U_i)$ and determine the functions c_{ij}.

STEP 3. We consider the left action of $\phi(\eta_i)$ on

$$k[\xi_1, \xi_2, \ldots, \xi_n] = A_n(k)/(A_n(k)\eta_1 + A_n(k)\eta_2 + \cdots A_n(k)\eta_{n-1} + A_n(k)\eta_n)$$

and conclude that the terms $d_i(T)$ are all zero.

We may easily see that the 1-form ω is equal to zero and that the Jacobian $\mathrm{Jac}(\hat{\psi})$ is equal to 1 in this case.

Note that Example 8.2 and Example 8.3 give rise to examples of calculation for $L_{\mathcal{U}}$. The choice of ultrafilter $\mathcal{U}$ is irrelevant for these examples.

ACKNOWLEDGMENT. Deep appreciation goes to Professor Fumiharu Kato for his having guided the author to the theory of p-curvatures. Deep appreciation also goes to Professor Akira Ishii for his good advice. The author is grateful to the colleagues in Kochi University. The author also expresses his gratitude to Takuro Mochizuki for constant encouragement. Last but not least, the author expresses his gratitude to the referee of the paper for the careful proof-reading and comments to improve the paper.

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