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$(\mathbb{Z}_2)^k$-ACTIONS WITH TRIVIAL NORMAL BUNDLE OF FIXED POINT SET

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1. Introduction

Let k be a positive integer. Let $\phi\colon (\mathbb{Z}_2)^k \times M^n \to M^n$ be a smooth action of the group $(\mathbb{Z}_2)^k = \{t_1, \dots, t_k \mid t_i^2 = 1 \text{ and } t_i t_j = t_j t_i\}$ on a closed smooth n-dimensional manifold M^n. It is well known that the fixed point set F of the action ϕ on M^n, i.e.,

$$F = \{m \in M^n \mid \phi(t, m) = m \text{ for all } t \in (\mathbb{Z}_2)^k\}$$

is a disjoint union $\bigsqcup_h F^{n-h}$ of closed submanifolds of M^n.

The purpose of this paper is to study $(\mathbb{Z}_2)^k$-actions having the property that each component of fixed point set has trivial normal bundle. When $k = 1$, Conner and Floyd gave the complete analysis of such actions (see [1, Theorem 25.1]). When $k > 1$, as far as the author knows, some works in this respect are only on the case in which the fixed point set consists of isolated fixed points. For example, see [1], [2], [3], [4], and [5].

In [5], a linear independence condition for the fixed point set of $(\mathbb{Z}_2)^k$-actions on closed manifolds was introduced, and then using the condition, one analyzed the property of fixed point set for $(\mathbb{Z}_2)^k$-actions having only isolated points. Following this idea, we first consider a more general case, i.e., $(\mathbb{Z}_2)^k$-actions with constant dimensional fixed point set satisfying that each component of fixed point set has trivial normal bundle. The result is stated as follows.

Theorem 1.1. *Suppose that (ϕ, M^n) is an $(\mathbb{Z}_2)^k$-action on a closed manifold M^n with constant l-dimensional fixed point set F^l for which each component of F^l has trivial normal bundle. Then either (ϕ, M^n) bounds equivariantly or the fixed point set has the following property*:
(1) *for $l < n$, F^l must possess the linear dependence property*;
(2) *for $l = n$, F^n is bordant to M^n.*

REMARK. In Theorem 1.1, linear dependence forces the fixed points of (ϕ, M^n) to have not only a normal representation, so F^l has at least two connected components

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if (ϕ, M^n) is nonbounding and $l < n$, i.e., F^l is disconnected. This means that there cannot be $(\mathbb{Z}_2)^k$-actions with just an isolated point (see also [1, Theorem 31.3]).

In addition, we also consider the case in which the fixed point set has variable codimensions. However, the general argument is still difficult since the key point for $(\mathbb{Z}_2)^k$-actions ($k > 1$) is the variation in the normal action. For example, see [4], [6] and [7]. With the help of a linear independence condition, when we add a restriction that each part of the fixed point set possesses the linear independence property, this naturally eliminate the existence of isolated points. Then the following result is obtained.

Theorem 1.2. *Suppose that (ϕ, M^n) is an $(\mathbb{Z}_2)^k$-action on a closed manifold for which the fixed point set $F = \bigsqcup_{h>0} F^h$ satisfies that for $h < n$, all Stiefel-Whitney classes of the normal bundle to each part F^h vanish in positive dimension, and each F^h possesses the linear independence property. Then for $h < n$, each connected component of F^h bounds, and M^n is bordant to F^n.*

REMARK. It is well-known that any involution is equivariantly bordant to an involution with fixed point set having the connectedness property, so each part of the fixed point set of any involution always possesses the linear independence property. Thus, when $k = 1$, Theorem 1.2 is just the Theorem 25.1 in [1], so Theorem 1.2 is directly the generalization of [1, Theorem 25.1].

In Section 2, we introduce some notations, such as the linear dependence property, and review a formula given by Kosniowski and Stong. Theorems 1.1 and 1.2 will be proved in Section 3. Throughout this paper, all manifolds and $(\mathbb{Z}_2)^k$-actions are to be smooth. Let $[N]$ denote the fundamental homology class of the closed manifold N. $S_\omega(x_1, \ldots, x_n) = \sum x_1^{i_1} \cdots x_r^{i_r}$ denotes the usual smallest symmetric polynomial containing the given monomial where $\omega = (i_1, \ldots, i_r)$ is a partition of $|\omega| = i_1 + \cdots + i_r$.

2. Notations and a formula

Let $\phi\colon (\mathbb{Z}_2)^k \times M^n \to M^n$ be an action of the group $(\mathbb{Z}_2)^k$ being generated by the elements $t_1, \ldots, t_k$ having the relations $t_i^2 = 1$ and $t_i t_j = t_j t_i$, and let $F = \bigsqcup_d F^{n-d}$ be its fixed point set. Let $\mathrm{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ be the set of homomorphisms $\rho\colon (\mathbb{Z}_2)^k \to \mathbb{Z}_2 = \{+1, -1\}$, which consists of 2^k distinct homomorphisms labeled by ρ_i, $i = 1, \ldots, 2^k$. One agrees to let $\rho_1 = \underline{1}$, i.e., $\rho_1(t_i) = 1$ for all i. Every irreducible real representation of $(\mathbb{Z}_2)^k$ is one-dimensional and has the form $\lambda_\rho\colon (\mathbb{Z}_2)^k \times \mathbb{R} \to \mathbb{R}$ with $\lambda_\rho(t, r) = \rho(t) \cdot r$ for some ρ. $\lambda_{\underline{1}}$ is the trivial representation corresponding to $\rho_1 = \underline{1}$.

For each part F^{n-d} of F, the restriction to each connected component F_i^{n-d} of F^{n-d} of the tangent bundle of M^n decomposes into subbundles under the action of $(\mathbb{Z}_2)^k$

$$\tau_M|_{F_i^{n-d}} \cong \tau_{F_i^{n-d}} \oplus \bigoplus_{\rho \neq \underline{1}} \nu_{\rho,d}$$

where $\nu_{\rho,d}$ is the subbundle on which $(\mathbb{Z}_2)^k$ acts via λ_ρ, and the subbundle on which $(\mathbb{Z}_2)^k$ acts trivially is identified with the tangent bundle of F_i^{n-d}. Let $q_{\rho,d} = \dim \nu_{\rho,d}$, so that $d = \sum_{\rho \neq \underline{1}} q_{\rho,d}$. If F^{n-d} is not connected, then the sequence $(q_{\rho_2,d}, \ldots, q_{\rho_{2^k},d})$ (called the *normal dimensional sequence*) may vary for different components, although $\sum_{\rho \neq \underline{1}} q_{\rho,d}$ is always equal to d. Without loss of generality one may assume that the part of F^{n-d} with a given normal dimensional sequence is connected. This is because one may form a connected sum of the components if $n - d > 0$, and cancel pairs of components if $n - d = 0$, so that up to equivariant bordism, the $(\mathbb{Z}_2)^k$-action is unchanged. Then, one may write $F^{n-d} = \bigsqcup_{i=1}^{u_d} F_i^{n-d}$, $u_d \geq 1$, where F_i^{n-d} is the part of F^{n-d} with a given normal dimensional sequence $(q^i_{\rho_2,d}, \ldots, q^i_{\rho_{2^k},d})$, and the collection

$$\mathfrak{C} = \left\{ (q^i_{\rho_2,d}, \ldots, q^i_{\rho_{2^k},d}) \text{ with } \sum_{\rho \neq \underline{1}} q^i_{\rho,d} = d \;\middle|\; i = 1, \ldots, u_d \right\}$$

of such sequences occuring in F^{n-d} will be called the *normal dimensional sequence set* of F^{n-d}.

Now, assuming $a_1, \ldots, a_k$ to be formal variables, let

$$\alpha_\rho = \sum \{a_i \mid \lambda_\rho(t_i, r) = -r\}$$

for λ_ρ an irreducible representation of $(\mathbb{Z}_2)^k$. Obviously, $\alpha_{\underline{1}} = 0$. With the above understood, we state the definition of the linear independence for the fixed point set.

DEFINITION. We say that the $(n - d)$-dimensional part F^{n-d} of F possesses the *linear independence property* if its normal dimensional sequence set

$$\mathfrak{C} = \{(q^i_{\rho_2,d}, \cdots, q^i_{\rho_{2^k},d}) \mid i = 1, \ldots, u_d\}$$

has the following property:

$$\frac{1}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^1_{\rho,d}}}, \ldots, \frac{1}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^{u_d}_{\rho,d}}}$$

are linearly independent in the quotient field of $\mathbb{Z}_2[a_1, \ldots, a_k]$.

Next, let us review the formula given by Kosniowski and Stong. Let $f(a_1, \ldots, a_k, x_1, \ldots, x_n)$ be a polynomial over $\mathbb{Z}_2$ which is symmetric in the set of variables

$x_1, \dots, x_n$. If we use the j-th Stiefel-Whitney class of M^n to replace the j-th elementary symmetric function $\sigma_j(x) = \sum x_1 \cdots x_j$, then the resulting cohomology class evaluated on the fundamental homology class of M^n is a characteristic number

$$f(a_1, \dots, a_k, x_1, \dots, x_n)[M^n]. \tag{2.1}$$

On the other hand, consider each connected component F_i^{n-d} of F^{n-d} with the normal bundle $\bigoplus_{\rho \neq \underline{1}} \nu^i_{\rho,d}$. In the polynomial, $x_1, \dots, x_n$ is replaced by $z_1, \dots, z_{n-d}$ and, for all $\rho \neq 1$, variables $\alpha_\rho + y^l_\rho$, $1 \leq l \leq q^i_{\rho,d}$. If we let $W_j(F_i^{n-d})$ (resp. $W_j(\nu^i_{\rho,d})$) replace the j-th elementary symmetric function in $\{z_1, \dots, z_{n-d}\}$ $\left(\text{resp. } \left\{y^1_\rho, \dots, y^{q^i_{\rho,d}}_\rho\right\}\right)$, then

$$\frac{f\left(a_1, \dots, a_k, z_1, \dots, z_{n-d}, \dots, \alpha_\rho + y^1_\rho, \dots, \alpha_\rho + y^{q^i_{\rho,d}}_\rho, \dots\right)}{\prod_{\rho \neq \underline{1}} \prod_{l=1}^{q^i_{\rho,d}} (\alpha_\rho + y^l_\rho)}$$

is a class in the cohomology of F_i^{n-d} which may be evaluated on the fundamental homology class of F_i^{n-d}, thus obtaining a characteristic number

$$\frac{f\left(a_1, \dots, a_k, z_1, \dots, z_{n-d}, \dots, \alpha_\rho + y^1_\rho, \dots, \alpha_\rho + y^{q^i_{\rho,d}}_\rho, \dots\right)}{\prod_{\rho \neq \underline{1}} \prod_{l=1}^{q^i_{\rho,d}} (\alpha_\rho + y^l_\rho)}[F_i^{n-d}] \tag{2.2}$$

which can be considered as an element in the quotient field K of $\mathbb{Z}_2[a_1, \dots, a_k]$.

Kosniowski and Stong [4] indicated the relation between (2.1) and (2.2).

Theorem 2.1. *If $f(a_1, \dots, a_k, x_1, \dots, x_n)$ is of degree less than or equal to n, then*

$$f(a_1, \dots, a_k, x_1, \dots, x_n)[M^n]$$
$$= \sum_d \left\{ \sum_{i=1}^{u_d} \frac{f\left(a_1, \dots, a_k, z_1, \dots, z_{n-d}, \dots, \alpha_\rho + y^1_\rho, \dots, \alpha_\rho + y^{q^i_{\rho,d}}_\rho, \dots\right)}{\prod_{\rho \neq \underline{1}} \prod_{l=1}^{q^i_{\rho,d}} (\alpha_\rho + y^l_\rho)}[F_i^{n-d}] \right\}$$

in K.

3. Proofs of Theorems 1.1 and 1.2

This section is devoted to prove Theorems 1.1 and 1.2.

First, let us begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. Write $F^l = \bigsqcup_{i=1}^{u} F_i^l$ with $u \geq 1$, where each F_i^l is connected, and let

$$\nu \longrightarrow F^l = \bigsqcup_{i=1}^{u} \bigoplus_{\rho \neq \underline{1}} \nu^i_{\rho, n-l} \longrightarrow F_i^l$$

be the normal bundle to F^l in M^n such that the total class $W(\nu) = 1$, and

$$\mathfrak{C} = \{(q^i_{\rho_2, n-l}, \ldots, q^i_{\rho_{2^k}, n-l}) \mid i = 1, \ldots, u\}$$

the normal dimensional sequence set of F^l. Recall that for $(\mathbb{Z}_2)^k$-actions the equivariant bordism class is determined by the fixed point data, which consists of the fixed point set and subbundles of the normal bundle on which $(\mathbb{Z}_2)^k$ has given representations (see [8]). Thus, if each component of F^l bounds, then (ϕ, M^n) bounds equivariantly. In other words, the equivariant bordism class of an $(\mathbb{Z}_2)^k$-action having the property that each component of the fixed point set has trivial normal bundle is determined by the bordism class of the fixed point set.

If there exists at least a nonbounding component in F^l, then (ϕ, M^n) must be nonbounding. In this case, without loss of generality, one may assume that all components $F_1^l, \ldots, F_u^l$ of F^l are nonbounding. In fact, if there exist bounding components in F^l, then one can cancel those bounding components in F^l. This doesn't change the $(\mathbb{Z}_2)^k$-action up to equivariant bordism. For $l < n$, consider the symmetric polynomial

$$f_\omega(x) = \sum x_1^{i_1} \cdots x_r^{i_r}$$

where $\omega = (i_1, \ldots, i_r)$ is a non-dyadic partition of l. Then one has that

$$\begin{aligned} & f_\omega(z_1, \ldots, z_l, \underbrace{\alpha_{\rho_2}, \ldots, \alpha_{\rho_2}}_{q_{\rho_2, n-l}}, \ldots, \underbrace{\alpha_{\rho_{2^k}}, \ldots, \alpha_{\rho_{2^k}}}_{q_{\rho_{2^k}, n-l}}) \\ &= \sum z_1^{i_1} \cdots z_r^{i_r} + \text{terms of lower degree in } z\text{'s} \\ &= S_\omega(z) + \text{terms of lower degree in } z\text{'s}. \end{aligned}$$

Since $\deg f_\omega = l < n$, by Theorem 2.1 one has that

$$0 = \frac{S_\omega(z)[F_1^l]}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^1_{\rho, n-l}}} + \cdots + \frac{S_\omega(z)[F_u^l]}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^u_{\rho, n-l}}}.$$

Since each component F_i^l is nonbounding, one can choose a suitable ω such that for some i_0, the characteristic number $S_\omega(z)[F_{i_0}^l]$ is nonzero, so

$$\frac{1}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^1_{\rho, n-l}}}, \ldots, \frac{1}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^u_{\rho, n-l}}}$$

are linearly dependent in the quotient field of $\mathbb{Z}_2[a_1, \ldots, a_k]$. Thus, for $l < n$, F^l possesses the linear dependence property.

For $l = n$, the dimension of the normal bundle of F^n in M^n is zero, and so in this case, each element of the normal dimensional sequence set is $(0, \ldots, 0)$. Thus, for any symmetric polynomial $S_\omega(x) = \sum x_1^{i_1} \cdots x_r^{i_r}$ where $\omega = (i_1, \ldots, i_r)$ with $|\omega| = n$, by Theorem 2.1, one has that

$$S_\omega(x)[M^n] = S_\omega(z)[F_1^n] + \cdots + S_\omega(z)[F_u^n]$$

so F^n is bordant to M^n. This completes the proof. $\square$

NOTE. We see that for $l < n$, if (ϕ, M^n) is nonbounding (i.e., no all components of F^l are bounding), then the linear dependence makes sure that F^l is disconnected. In this case, we claim that all nonbounding components of F^l must be linearly dependent in MO_*, where MO_* is the polynomial algebra over $\mathbb{Z}_2$ formed by all unoriented closed manifolds. In fact, if all nonbounding components of F^l is linearly independent in MO_*, then there exists some ω such that for some component $F^l_{i_0}$, the characteristic number $S_\omega(z)[F^l_{i_0}] \neq 0$, but such the numbers for other components of F^l are zero. Further, we have from the proof of Theorem 1.1 that

$$\frac{1}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^{i_0}_{\rho, n-l}}} = 0.$$

This is impossible. Thus, if (ϕ, M^n) is nonbounding, then all nonbounding components of F^l with $l < n$ must be linearly dependent in MO_*. This means that the conclusion (1) of Theorem 1.1 is equivalent to the statement that all nonbounding components of F^l with $l < n$ is linearly dependent in MO_*.

Next, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Being given a

$$F^{h_0} = \bigsqcup_{i=1}^{u_{n-h_0}} F_i^{h_0}, \quad u_{n-h_0} \geq 1,$$

with each $F_i^{h_0}$ connected and $h_0 < n$, and let

$$\mathfrak{C} = \{(q^i_{\rho_2, n-h_0}, \ldots, q^i_{\rho_{2^k}, n-h_0}) \mid i = 1, \ldots, u_{n-h_0}\}$$

the normal dimensional sequence set of F^{h_0}. We suppose inductively that each component of F^h bounds if $h < h_0$. Thus, up to equivariant bordism, (ϕ, M^n) is equivariantly bordant to an $(\mathbb{Z}_2)^k$-action with fixed point set $F = \bigsqcup_{h \geq h_0} F^h$. Choose

$$f_\omega(x) = \sum x_1^{i_1} \cdots x_r^{i_r}$$

where $\omega = (i_1, \dots, i_r)$ is a non-dyadic partition of h_0, then we have that $\deg f_\omega = h_0$, and for any $h > 0$,

$$\begin{aligned} &f_\omega(z_1, \dots, z_h, \underbrace{\alpha_{\rho_2}, \dots, \alpha_{\rho_2}}_{q_{\rho_2, n-h}}, \dots, \underbrace{\alpha_{\rho_{2^k}}, \dots, \alpha_{\rho_{2^k}}}_{q_{\rho_{2^k}, n-h}}) \\ &= \sum z_1^{i_1} \cdots z_r^{i_r} + \text{terms of lower degree in } z\text{'s} \\ &= S_\omega(z) + \text{terms of lower degree in } z\text{'s}. \end{aligned}$$

Since the highest degree in z's of $f_\omega(z_1, \dots, z_h, \underbrace{\alpha_{\rho_2}, \dots, \alpha_{\rho_2}}_{q_{\rho_2, n-h}}, \dots, \underbrace{\alpha_{\rho_{2^k}}, \dots, \alpha_{\rho_{2^k}}}_{q_{\rho_{2^k}, n-h}})$ is h_0, we have that for any part $F^h = \bigsqcup_{i=1}^{u_{n-h}} F_i^h$ with $h > h_0$,

$$\sum_{i=1}^{u_{n-h}} \frac{f_\omega(z_1, \dots, z_h, \overbrace{\alpha_{\rho_2}, \dots, \alpha_{\rho_2}}^{q^i_{\rho_2, n-h}}, \dots, \overbrace{\alpha_{\rho_{2^k}}, \dots, \alpha_{\rho_{2^k}}}^{q^i_{\rho_{2^k}, n-h}})}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^i_{\rho, n-h}}} [F_i^h] = 0.$$

Furthermore, by Theorem 2.1, we have that

$$0 = \frac{S_\omega(z)[F_1^{h_0}]}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^1_{\rho, n-h_0}}} + \cdots + \frac{S_\omega(z)\left[F^{h_0}_{u_{n-h_0}}\right]}{\prod_{\rho \neq \underline{1}} \alpha_\rho^{q^{u_{n-h_0}}_{\rho, n-h_0}}}.$$

Since F^{h_0} possesses the linear independence property, we have that for each i, the characteristic number $S_\omega(z)[F_i^{h_0}] = 0$, and so $F_i^{h_0}$ bounds. This completes the induction, and thus, for $h < n$, each component of F^h bounds. For the n-dimensional part F^n of F, similarly to the argument of Theorem 1.1, we may obtain that M^n is bordant to F^n. $\square$

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