Kambayashi, T.
Osaka J. Math.
40 (2003), 621–638

SOME BASIC RESULTS ON PRO-AFFINE ALGEBRAS AND IND-AFFINE SCHEMES

TATSUJI KAMBAYASHI

(Received February 18, 2002)

Introduction

The theory of ind-affine varieties was first introduced by Shafarevich, who then employed it to elucidate the structure of the automorphism group of the affine space. (see [3], [4].) More recently we made certain revisions on the theory and applied it to study the Jacobian Problem on the endomorphisms of the complex affine space. (see [2].)

Since these pieces of work appeared, there has not been much progress made. This state may be due, in part, to the fact that the basic theory of these ind-affine or pro-affine objects as presented by us was still *ad hoc* and was rather rudimentary. So, we have embarked on building a theory of pro-affine algebras and ind-affine schemes anew and from the ground up. The outcome of our effort is the contents of the present paper. As we worked on the material we encountered a number of subtle examples, as shown in the main text below. It would seem that these examples perhaps suggest richness and mystery that this theory holds.

We mention a piece of specific result we have of our theory: The set of all morphisms of an affine variety over a field K to another may be identified with the K-rational point set of an appropriately constructed ind-affine scheme over K. This was proven using the theory of Gröbner bases over K, and is expected to be published in the near future along with certain related results about automorphisms of the affine space.

1. Pro-affine algebras

1.1. Definitions and Notations. Throughout we work over a ground field K of any characteristic. A commutative topological K-algebra A is said to be a *pro-affine algebra* if

1. A is *complete* and *separated*.
2. A base of open neighborhoods of 0 is given by a family of *countably many* ideals $\subseteq A$.

Let $\{\mathfrak{a}_i : i \in \mathbb{N}\}$ be a countable base referred to just above. Here, as elsewhere throughout the present paper, $\mathbb{N}$ represents the set of all *nonnegative* integers. We may, and shall always, assume that $\mathfrak{a}_i \supseteq \mathfrak{a}_j$ whenever $i \leq j$. The condition 1 then implies that

$$(1) \qquad \bigcap_{i \in \mathbb{N}} \mathfrak{a}_i = \{0\} \quad \text{and} \quad A \simeq \varprojlim_{i \in \mathbb{N}} (A_i),$$

where, for each $i \in \mathbb{N}$, $A_i := A/\mathfrak{a}_i$ is a *discrete* K-algebra, with all maps $\mu_i \colon A_i \to A_{i-1}$ being *surjective*. Conversely, a K-algebra given as the limit of a *countable*, *surjective* inverse system of *discrete* K-algebras in the form of (1) is evidently pro-affine in our sense.

One recognizes then that a pro-affine K-algebra as above is the same thing as a "filtered commutative K-algebra which is complete and separated" in the sense of Northcott [5, Chap. 9].

Proposition 1.1.1. *Let A and B be pro-affine algebras. Then, the product $A \times B$ and the complete tensor product $A \mathbin{\hat{\otimes}}_K B$ are both pro-affine K-algebras.*

Proof seems hardly necessary. If $\{\mathfrak{a}_i : i \in \mathbb{N}\}$ and $\{\mathfrak{b}_j : j \in \mathbb{N}\}$ are bases of open neighborhoods of 0 for A and B, respectively, then one adopts for $A \times B$ the ideals $\{\mathfrak{a}_k \times \mathfrak{b}_k : k \in \mathbb{N}\}$ as a base of open neighborhoods of 0. As for $A \mathbin{\hat{\otimes}}_K B$, take the ideals $\{\mathfrak{a}_k \otimes B + A \otimes \mathfrak{b}_k : k \in \mathbb{N}\}$ as a base of open neighborhoods of $A \otimes_K B$, and then take its completion. $\square$

A pro-affine algebra A is said to be *algebraic over K*, or *K-algebraic*, if A can be represented as in (1) where *all* $A/\mathfrak{a}_i$ are *finitely generated over K*.

Let A, B be pro-affine K-algebras. A *morphism of* A to B is defined to be a *continuous* K-algebra map $\phi \colon A \to B$. Suppose that A and B are represented as $A = \varprojlim(A/\mathfrak{a}_i)$, $B = \varprojlim(B/\mathfrak{b}_j)$, respectively. Then, the morphism $\phi \colon A \to B$ gives rise to a commutative diagram

$$(2) \qquad \begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle \pi_i^A}\downarrow & & \downarrow{\scriptstyle \pi_j^B} \\ A/\mathfrak{a}_i & \xrightarrow[\phi_{ji}]{} & B/\mathfrak{b}_j \end{array}$$

standing valid for each given $j \in \mathbb{N}$ and for some corresponding $i = i(j) \in \mathbb{N}$ for which $\phi(\mathfrak{a}_i) \subseteq \mathfrak{b}_j$. Here, π_i^A and π_j^B denote the canonical residue-class maps, and $\phi_{ji}(x + \mathfrak{a}_i) \overset{\text{def.}}{=} \phi(x) + \mathfrak{b}_j$ for all $x \in A$.

NOTATIONS. Let us fix some notations we shall be using throughout this paper:

(**a**) Let $A = \lim_{\leftarrow}(A_i)$ be a pro-affine algebra, where we have put $A_i := A/\mathfrak{a}_i$ as before. The canonical surjective maps $A \to A_i$ and $A_j \to A_i$ for $i \le j$ shall be denoted as follows:

$$\pi_i \colon A \longrightarrow A_i\ ; \quad \mu_{ij} \colon A_j \longrightarrow A_i, \tag{3}$$

with $\mathrm{Ker}(\pi_i) = \mathfrak{a}_i$, and $\mu_{ii} = \mathrm{Id}_{A_i}$. We abbreviate $\mu_{i-1,i}$ as μ_i.

(**b**) As a rule, for any *subset* $E \subseteq A$ or any *element* $a \in A$, we denote $\pi_i(E)$ by ${}_iE$ and $\pi_i(a)$ by ${}_ia$. (A notable exception is $\pi_i(A) = A/\mathfrak{a}_i$ which we denote by A_i and *not* by ${}_iA$.) When no fear of confusion is present, we often skip the left suffix and simply write a for ${}_ia$, so that $a = (\cdots \leftarrow {}_{i-1}a \leftarrow {}_ia \leftarrow \cdots)$ is expressed as $(\cdots \leftarrow a \leftarrow a \leftarrow \cdots)$. A sequence $\sigma := (\cdots \leftarrow s_{i-1} \leftarrow s_i \leftarrow \cdots)$ with $s_j \in A_j$ for all $j \in \mathbb{N}$ represents an element of A and thus $\sigma \in A$ if and only if $\mu_j(s_j) = s_{j-1}$ for all j, in which case we shall say σ is *coherent*.

In the notations above, it is then clear that the *closure* $\overline{E}$ of E may be identified with $\lim_{\leftarrow}({}_iE)$. Thus, $E \subseteq A$ is *closed* if and only if every coherent sequence $\epsilon = (\cdots \leftarrow e_i \leftarrow \cdots)$ belongs to E as soon as all $e_i \in {}_iE$ for $i \in \mathbb{N}$.

Proposition 1.1.2. *The group of units $U(A)$ of a pro-affine algebra A is closed.*

Proof. Let $u = (\cdots \leftarrow u_{i-1} \leftarrow u_i \leftarrow \cdots) \in \overline{U(A)}$. For each i there is a unique $v_i \in A_i$ with $u_i \cdot v_i = 1_{A_i}$. Then, $v := (\cdots \leftarrow v_{i-1} \leftarrow v_i \leftarrow \cdots)$ is clearly coherent and satisfies $u \cdot v = 1$ so that $u \in U(A)$. $\square$

EXAMPLE 1.1-A (cf. [2, (1.1), p. 482]). For each $n \in \mathbb{N}$, let $K^{[n]} := K[X_1, \ldots, X_n]$ if $n > 0$, and $K^{[0]} := K$. Define $\mu_n \colon K^{[n]} \to K^{[n-1]}$ by setting $\mu_n(X_i) := X_i$ for all $1 \le i \le n-1$, and $\mu_n(X_n) := 0$. Denote

$$K^{[\infty]} := \lim_{\leftarrow}(K^{[n]})$$

and call it *the pro-affine polynomial algebra* (*over* K). This algebra may be characterized as the set of those formal power-series on $X_1, \ldots, X_m, \ldots$ which become polynomials when reduced *modulo all but finitely many* X_i*'s*.

1.2. The ideals in a pro-affine algebra.

Proposition 1.2.1. *Let $\mathfrak{h}$ be a closed ideal in $A = \lim_{\leftarrow}(A_i)$. Then,*

$$A/\mathfrak{h} \simeq \lim_{\leftarrow}(A_i)/\lim_{\leftarrow}({}_i\mathfrak{h}) \simeq \lim_{\leftarrow}(A_i/{}_i\mathfrak{h}).$$

[This implies that $A/\mathfrak{h}$ is a pro-affine algebra for any closed ideal $\mathfrak{h}$.]

Proof. Since $\mathfrak{h}$ is closed, $\mathfrak{h} \simeq \lim_{\leftarrow}({}_i\mathfrak{h})$ and all maps ${}_i\mathfrak{h} \to {}_{i-1}\mathfrak{h}$ are surjective. So, in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_i\mathfrak{h} & \longrightarrow & A_i & \longrightarrow & A_i/{}_i\mathfrak{h} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}_{i-1}\mathfrak{h} & \longrightarrow & A_{i-1} & \longrightarrow & A_{i-1}/{}_{i-1}\mathfrak{h} & \longrightarrow & 0 \end{array}$$

all vertical maps are surjective. One now applies the functor $\lim_{\leftarrow}$ to this diagram, remembering the Mittag-Leffler condition which holds here. $\square$

EXAMPLE 1.2-B. In the same notations as in Ex. 1.1-A, define an ideal $J_n \subset K^{[n]}$ by $J_n := \langle X_i X_j \mid 1 \le i < j \le n \rangle$, so geometrically the locus of J_n is the union of all coordinate axes in the affine n-space $\mathbb{A}^n$ over K. Let $B_n := K^{[n]}/J_n = K[x_1, \ldots, x_n]$. Consider the exact sequence

$$0 \longrightarrow J_n \longrightarrow K^{[n]} \longrightarrow B_n \longrightarrow 0$$

and take the $\lim_{\leftarrow}$ of this sequence on all $n \in \mathbb{N}$. Since, for all n, $\mu_n \colon K^{[n]} \to K^{[n-1]}$ causes a *surjection* of J_n to J_{n-1}, there results a surjective K-map $K^{[\infty]} \to B := \lim_{\leftarrow} B_n$, and its kernel $J := \lim_{\leftarrow} J_n$ gives an example of a *closed ideal* in $K^{[\infty]}$. [In the subsequent B will be viewed as the coordinate algebra $\mathcal{O}(Y)$ of the closed subscheme Y of all coordinate axes in the ind-affine space $\mathbb{A}^\infty$.]

EXAMPLE 1.2-C. In Example 1.2-B replace each J_n by $J'_n := \langle X_1 \cdots X_n \rangle$, whose locus in $\mathbb{A}^n$ is then the union of all coordinate hyperplanes in $\mathbb{A}^n$. Since the surjections $\mu_n \colon K^{[n]} \to K^{[n-1]}$ all cause *zero* maps of J'_n into J'_{n-1}, the Mittag-Leffler condtion is trivially satisfied, and $J' := \lim_{\leftarrow,n} J_n = \{0\}$ (which is a closed ideal in $K^{[\infty]}$). It follows that $K^{[\infty]} \simeq \lim_{\leftarrow,n}(K^{[n]}/J'_n)$. [So, the union of all coordinate hyperplanes in $\mathbb{A}^n$, as $n \to \infty$, is isomorphic to the whole ind-affine space $\mathbb{A}^\infty$.]

Proposition 1.2.2. *For any maximal ideal* $\mathfrak{m} \subset A$, *the following conditions are equivalent to one another*:
(i) $\mathfrak{m}$ *is closed*;
(ii) *For some* i, $\pi_i(\mathfrak{m}) = {}_i\mathfrak{m} \subsetneqq A_i$;
(iii) *For some* i, $\mathfrak{a}_i \subseteq \mathfrak{m}$;
(iv) *For some* i, $\mathfrak{m} = \pi_i^{-1}$(*some maximal ideal in* A_i);
(v) $\mathfrak{m}$ *is open.*

Proof. (i) $\Rightarrow$ (ii) : If ${}_i\mathfrak{m} = A_i$ for all i, then $(1 \leftarrow \cdots \leftarrow 1 \leftarrow \cdots) \in \overline{\mathfrak{m}} = \mathfrak{m}$, so that $\mathfrak{m} = A$.

(ii) $\Rightarrow$ (iii) : Let ${}_i\mathfrak{m} \subset A_i$ for a particular i. Then, ${}_i\mathfrak{m}$ must be a maximal ideal in

A_i, and $\pi_i^{-1}({}_i\mathfrak{m}) = \mathfrak{m} + \mathfrak{a}_i = \mathfrak{m}$, so $\mathfrak{a}_i \subseteq \mathfrak{m}$.

The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) are obvious. $\square$

The same argument as used in (i) $\Rightarrow$ (ii) above shows the following:

Corollary 1.2.3. *Every closed proper ideal in a pro-affine algebra A is contained in a closed maximal ideal.*

Proposition 1.2.4. *For any prime ideal $\mathfrak{p} \subset A$, the following conditions are equivalent to one another*:
1. $\mathfrak{p}$ *is open*;
2. *For some i, $\mathfrak{p} = \pi_i^{-1}({}_i\mathfrak{p})$*;
3. *For some j and a prime ideal $\mathfrak{q}_j \subset A_j$, $\mathfrak{p} = \pi_j^{-1}(\mathfrak{q}_j)$.*

The proof of this obvious proposition is omitted.

Note that, in view of the two preceding propositions, the *open prime* (resp. *open maximal*) ideals of a pro-affine algebra A are precisely the inverse images of the *prime* (resp. *maximal*) ideals of the A_i's for any $i \in \mathbb{N}$.

Proposition 1.2.5. *Let $\mathfrak{a}$ be a finitely generated proper ideal in a pro-affine algebra A. Then, there exists an open maximal ideal $\mathfrak{m}$ such that $\mathfrak{a} \subseteq \overline{\mathfrak{a}} \subseteq \mathfrak{m}$.*

We first prove the following key lemma due to N. Mohan Kumar:

Lemma 1.2.6 (N. Mohan Kumar). *Let $\mathfrak{a} = \langle x_1, \dots, x_n \rangle$ be a finitely-generated ideal, and let $\overline{\mathfrak{a}}$ be its closure. For any $z \in A$, if $z \in \overline{\mathfrak{a}}$ then $z^{2^n} \in \mathfrak{a}$.*

Proof. The proof goes by induction on the number of generators n. First, take any $x \in A$ and let $z \in \overline{\langle x \rangle} = \lim_{\leftarrow}(A_i \cdot {}_i x)$. Write

$$z = (a_0 x \leftarrow a_1 x \leftarrow \cdots \leftarrow a_i x \leftarrow \cdots), a_j \in A_j \quad \text{for all} \quad j \in \mathbb{N},$$

where the coherence condition

$$\mu_i(a_i x) - a_{i-1} x = \mu_i(a_i \cdot {}_i x) - a_{i-1} \cdot {}_{i-1} x = (\mu_i(a_i) - a_{i-1}) \cdot {}_{i-1} x = 0 \tag{4}$$

is satisfied. Then, $\eta \overset{\text{def.}}{=} (a_0^2 x \leftarrow a_1^2 x \leftarrow \cdots \leftarrow a_i^2 x \leftarrow \cdots)$ is coherent, as one sees from (4) that

$$\begin{aligned} \mu_i(a_i^2 \cdot {}_i x) - a_{i-1}^2 \cdot {}_{i-1} x &= (\mu_i(a_i)^2 - a_{i-1}^2)\ {}_{i-1} x \\ &= (\mu_i(a_i) + a_{i-1})(\mu_i(a_i) - a_{i-1})\ {}_{i-1} x = 0. \end{aligned}$$

So $\eta \in A$. It follows that $z^2 = x\eta \in \langle x \rangle \subseteq A$.

Turning now to the next induction step, we let $z \in \overline{\langle x_1, \dots, x_n \rangle}$. Set $A' \overset{\text{def.}}{=} A/\overline{\langle x_1 \rangle}$, and consider its ideal $\overline{\langle x_2', \dots, x_n' \rangle}$, where $x_2', \dots, x_n'$ denote the canonical images of $x_2, \dots, x_n$, respectively, in A'. Let $z' := z\text{-mod}\overline{\langle x_1 \rangle} \in \overline{\langle x_2', \dots, x_n' \rangle}$. By induction hypothesis, $z'^{2^{n-1}} \in \langle x_2', \dots, x_n' \rangle$. This implies that one can write $z^{2^{n-1}} = z_1 + z_2$, where

$$z_1 \in \overline{\langle x_1 \rangle} \text{ and } z_2 \in \langle x_2, \dots, x_n \rangle.$$

But we saw just above that $z_1 \in \overline{\langle x_1 \rangle}$ gives $z_1^2 \in \langle x_1 \rangle$. Therefore,

$$z^{2^n} = (z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2 \in \langle x_1 \rangle + \langle x_2, \dots, x_n \rangle,$$

and we find $z^{2^n} \in \langle x_1, x_2, \dots, x_n \rangle$, as desired. □

Proof of Proposition 1.2.5. now follows immediately from this lemma. Indeed, if a finitely-generated ideal $\mathfrak{a}$ is such that $\overline{\mathfrak{a}} = A$, then $1 \in \overline{\mathfrak{a}}$, which implies $1 = 1^{2^n} \in \mathfrak{a}$ for some n. So, if $\mathfrak{a}$ is proper, then $\overline{\mathfrak{a}}$ is proper; and one now applies Cor. 1.2.3. □

REMARK. Proposition 1.2.5 fails to hold for ideals *not* finitely generated, as will be shown in §3 below (see Ex. 3-G). Also note that a finitely generated ideal need not be closed. In fact, even a principal ideal can be non-closed, as the following example shows:

EXAMPLE 1.2-D (N. Mohan Kumar). Let $K^{[2]} := K[X, Y]$ be a polynomial ring in X and Y, and for each $i \in \mathbb{N}$ let $A_i := K^{[2]}/\langle XY^{i+1} \rangle = K[x, y]$, with x, y standing for the canonical images of X, Y, respectively, in A_i. Let our pro-affine algebra A be $\lim_{\leftarrow} A_i$. Consider

$$\zeta := (x \leftarrow x(1 + y) \leftarrow x(1 + y + y^2) \leftarrow x(1 + y + y^2 + y^3) \leftarrow \cdots).$$

Clearly, $\zeta \in \overline{\langle x \rangle}$. However, $\zeta \notin \langle x \rangle$. To see this, assume $\zeta \in \langle x \rangle$, and write $\zeta = x\eta$ for some $\eta \in A$. Then, η has to equal

$$(1 + y\, p_1(y) \leftarrow 1 + y + y^2 p_2(y) \leftarrow 1 + y + y^2 + y^3 p_3(y) \leftarrow \cdots),$$

where $p_1(y)$, $p_2(y)$, $p_3(y), \dots$ are polynomials in y only. Now let, for each $i \in \mathbb{N}$, $f_i \colon A_i \to A_i/\langle x \rangle \simeq K[y]$ be the canonical mod-x map. Then,

$$f := \lim_{\leftarrow} f_i \colon \lim_{\leftarrow} A_i = A \longrightarrow K[y]$$

should map η to a polynomial in $K[y]$ of a certain degree, say of degree d. Since $f(\eta) = f_{d+1}(1 + y + \cdots + y^{d+1} + y^{d+2} p_{d+2}(y)) = 1 + y + \cdots + y^{d+1} + y^{d+2} p_{d+2}(y) \in K[y]$ is of degree at least $d + 1$, there results a contradiction.

1.3. The radicals and Nullstellensatz. The *radical* $\mathcal{R}(A)$ and the *nilradical* $\mathcal{N}(A)$ of a pro-affine algebra A are defined as follows:

$$\mathcal{N}(A) = \bigcap_{\forall \mathfrak{p}} \mathfrak{p} \quad \text{and} \quad \mathcal{R}(A) = \bigcap_{\forall \mathfrak{m}} \mathfrak{m}, \tag{5}$$

where the $\mathfrak{p}$'s and the $\mathfrak{m}$'s range over *all open prime* and *all open maximal* ideals, respectively.

Given an ideal $\mathfrak{a} \subseteq A$, the *radical* of $\mathfrak{a}$ is defined as

$$\mathcal{N}(\mathfrak{a}) \overset{\text{def.}}{=} \bigcap_{\forall \mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} \tag{6}$$

with $\mathfrak{p}$ again ranging over *all open prime* ideals containing $\mathfrak{a}$.

As done in [2], for a pro-affine algebra $A = \lim_{\leftarrow} A_i$ we define two kinds of its reductions relative to the radicals:

$$A_{\text{red}} \overset{\text{def.}}{=} A/\mathcal{N}(A) \quad \text{and} \quad A_{\text{RED}} \overset{\text{def.}}{=} \underset{\leftarrow}{\lim}((A_i)_{\text{RED}}) = \underset{\leftarrow}{\lim}((A_i)_{\text{red}}), \tag{7}$$

where $(A_i)_{\text{RED}} := A_i/\mathcal{N}(A_i) = (A_i)_{\text{red}}$ is the usual residue-class ring *modulo* the nilradical of A_i. A is said to be *reduced* or *strongly reduced*, respectively, if $A = A_{\text{red}}$ or $A = A_{\text{RED}}$. One may define likewise two more radicals using the Jacobson radicals $\mathcal{R}(A)$'s and $\mathcal{R}(A_i)$'s, and these were actually what we dealt with in [2, (1.2), (1.3), pp. 483–484]. Just the same, the following counterpart of [2, Prop. (1.2), *loc. cit.*] stands valid, and we state it without proof:

Theorem 1.3.1. *For the canonical map* $\rho\colon A = \lim_{\leftarrow}(A_i) \longrightarrow A_{\text{RED}}$, *we have*

(a) $\text{Ker}(\rho) = \mathcal{N}(A)$;

(b) *The sequence* $0 \longrightarrow \mathcal{N}(A) \longrightarrow A \longrightarrow A_{\text{RED}}$ *is exact with* $\text{Im}(\rho)$ *dense in* A_{RED};

(c) $\mathcal{N}(A) = \{f \in A : \lim_{N\to\infty} f^N = 0\}$ = *topologically nilpotent elements of* A.

REMARKS. **1**. We note that, even in the special context of the theorem above, the exactness of the sequence in (b) at the right-most end fails in general, or ρ is not surjective as a rule. Counter-examples are offered in Section 3 below (see Examples 3-E and 3-F). This point bears critically on the Jacobian Problem (cf. [2, (5.3), (5.4), pp. 497–498]).

2. Since $\mathcal{N}(A)$ is a closed ideal $\subset A$, we deduce from Prop. 1.2.1 that, whereas $\rho\colon A \to \lim_{\leftarrow}(A_i/\mathcal{N}(A_i))$ may not be surjective, the map $A \to \lim_{\leftarrow}(A_i/_i\mathcal{N}(A))$ *is* surjective.

Theorem 1.3.1 and the Jacobson-radical version of it [2, (1.2), p. 483] coincide with each other in the K-algebraic case as seen just below:

Theorem 1.3.2 (Nullstellensatz). *If a pro-affine K-algebra A is algebraic over K, then $\mathcal{R}(A) = \mathcal{N}(A)$.*

Proof. In view of Props. 1.2.2 & 1.2.4, the remarks following these two and the algebraicity, we have

$$\mathcal{R}(A) = \bigcap_{i\in\mathbb{N}} \pi_i^{-1}(\mathcal{R}(A_i)) = \bigcap_{i\in\mathbb{N}} \pi_i^{-1}(\mathcal{N}(A_i)) = \mathcal{N}(A),$$

where the traditional Nullstellensatz $\mathcal{R}(A_i) = \mathcal{N}(A_i)$ has been applied. □

2. Ind-affine schemes and ind-affine varieties

2.1. The spectra of pro-affine algebras and their topology. For any pro-affine algebra A, define its *prime spectrum* $\mathfrak{Sp}(A)$ and *maximal spectrum* $\mathfrak{Spm}(A)$, respectively, as

$$\text{(8)}\qquad \begin{cases} \mathfrak{Sp}(A) & = \text{the set of all } \textit{open, prime} \text{ ideals} \subset A, \text{ and} \\ \mathfrak{Spm}(A) & = \text{the set of all } \textit{open, maximal} \text{ ideals} \subset A. \end{cases}$$

Then, in view of Prop. 1.2.2, $\mathfrak{Spm}(A)$ is the same as the set of all *closed* maximal ideals. Let us now introduce topology on $\mathfrak{Sp}(A)$ and $\mathfrak{Spm}(A)$ by extending Zariski topology: The *closed* sets $\subseteq \mathfrak{Sp}(A)$ are, by definition, those subsets of $\mathfrak{Sp}(A)$ in the form of

$$V(E) \overset{\text{def.}}{=} \{\mathfrak{p} \in \mathfrak{Sp}(A) : \mathfrak{p} \supseteq E\} \text{ for some set } E \subseteq A.$$

Likewise, the closed sets $\subseteq \mathfrak{Spm}(A)$ are defined to be precisely the $V_o(E)$'s where $V_o(E) \overset{\text{def.}}{=} V(E) \cap \mathfrak{Spm}(A)$.

The following proposition which should require no proofs shows that the preceding definition of the topologies on $\mathfrak{Sp}(A)$ and on $\mathfrak{Spm}(A)$ is valid:

Proposition 2.1.1. (i) *Let $\mathfrak{a} := \langle E\rangle$, the ideal generated by E, and let $\mathcal{N}(\mathfrak{a})$ be the radical of $\mathfrak{a}$. Then,*

$$V(\mathfrak{a}) = V(E) = V(\mathcal{N}(\mathfrak{a})).$$

(ii) $V(0) = \mathfrak{Sp}(A)$, $V(1) = \emptyset$.
(iii) *Given a family $\{E_i : i \in I\}$ of subsets of A, we have*

$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i).$$

(iv) *For ideals $\mathfrak{b}$ and $\mathfrak{c}$, $V(\mathfrak{b} \cap \mathfrak{c}) = V(\mathfrak{b}\mathfrak{c}) = V(\mathfrak{b}) \cup V(\mathfrak{c})$.*

Next we define, for each $f \in A$, the *basic open set* $D(f) \subseteq \mathfrak{Sp}(A)$:

$$D(f) \overset{\text{def.}}{=} V(f)^c = \{\, \mathfrak{p} \in \mathfrak{Sp}(A) : f \notin \mathfrak{p} \,\}.$$

Proposition 2.1.2. *Let f, g, f_α $(\alpha \in I)$ be elements of A. Then,*
(i) $D(f) \cap D(g) = D(f \cdot g)$.
(ii) $\bigcup_{\alpha \in I} D(f_\alpha) = V(\langle f_\alpha : \alpha \in I \rangle)^c$.
(iii) $D(f) = \emptyset \iff f \in \mathcal{N}(A) \iff f$ *is topologically nilpotent.*
(iv) $D(f) = \mathfrak{Sp}(A) \iff f$ *is a unit.*
(v) $D(g) \subseteq D(f) \iff g \in \mathcal{N}(\langle f \rangle)$.

Proof. Parts (i), (ii), (iii) immediately follow from relevant definitions. As for (iv), if $f \notin$ any open prime, then by Prop. 1.2.5 $\langle f \rangle$ must equal the unit ideal $\langle 1 \rangle$. Therefore, f must be a unit.

As for part (v), $D(g) \subseteq D(f) \Leftrightarrow \forall \mathfrak{p} \in \mathfrak{Sp}(A)$ $[f \in \mathfrak{p} \Rightarrow g \in \mathfrak{p}]$, clearly, and this last condition is equivalent to $\mathcal{N}(\langle g \rangle) \subseteq \mathcal{N}(\langle f \rangle)$, or $g \in \mathcal{N}(\langle f \rangle)$. $\square$

REMARK. Proposition 2.1.2 goes to show that the $D(f)$'s for all $f \in A$ form a base of open sets in our topology on $\mathfrak{Sp}(A)$, just as in the more traditional theory of affine schemes. Note, however, that in our theory here the open sets $D(f)$'s are *not quasi-compact* in general. This is due to the existence of infinitely-generated proper ideals whose closures are the unit ideal $\langle 1 \rangle$. See Ex. 3-G in §3 below.

2.2. Localization in pro-affine algebras and structure sheaves of ind-affine schemes. Let S be a *multiplicatively closed set* in a pro-affine algebra A. It will be assumed always that $1 \in S$ and $0 \notin \overline{S} = \lim_{\leftarrow}({}_i S)$ for such an S. The localization $S^{-1}A$ can be defined in the standard manner, and this K-algebra naturally inherits its uniform topology from A. We shall adopt the completion of $S^{-1}A$ as our definition of A_S. Namely,

DEFINITION. For A and S as above, the localization A_S of A by S is defined to be

$$A_S \overset{\text{def.}}{=} \lim_{\leftarrow}({}_i S^{-1} A_i).$$

Clearly, $A_S \simeq A_{\overline{S}}$, so one may assume from the beginning that S is closed. For useful examples of S one may mention $(f) \overset{\text{def.}}{=} \{\, f^n \mid n \in \mathbb{N} \,\}$ where f is not topologically nilpotent, and the complement $A - \mathfrak{p}$ of an open prime ideal $\mathfrak{p}$. In these instances, we shall denote $A_{(f)}$, $A_{A-\mathfrak{p}}$ by A_f, $A_{\mathfrak{p}}$, respectively.

Proposition 2.2.1. *Let f, $g \in A$, $U := D(f)$, $V := D(g)$, and let A_f, A_g be as just above. Let $A(U) := A_f$ and $A(V) := A_g$. Then,*

(i) *If* $U = V$, *then* $A(U) \simeq A(V)$. (*Thus* $A(U)$ *depends only on* U, *not on* f.)
(ii) *If* $V \subseteq U$, *then there is a canonical homomorphism of pro-affine* K*-algebras* $\rho_V^U : A(U) \longrightarrow A(V)$, *which depends only on* U *and* V. (*The* ρ_V^U *will be called the restriction homomorphism from* U *to* V.)
(iii) *Let* U, V *be as above and* $W = D(h)$ *for* $h \in A$. *If* $U \supseteq V \supseteq W$, *we have*

$$\rho_U^U = \mathrm{Id}_{A(U)}, \ \ \rho_W^V \circ \rho_V^U = \rho_W^U .$$

Proof. (ii) Assume $V \subseteq U$, or $D(g) \subseteq D(f)$. So, by Props. 1.2.2 & 1.2.4, $g \in \mathcal{N}(\langle f \rangle) = \bigcap_{i \in \mathbb{N}} \pi_i^{-1}$(the radical of $\langle {}_i f \rangle$ in A_i). This means that, for every $i \in \mathbb{N}$, there is an n_i such that ${}_i g^{n_i} \in \langle {}_i f \rangle \subseteq A_i$. So, for each i there is an element $u_i \in A_i$ such that

$$({}_i g)^{n_i} = u_i \cdot {}_i f. \tag{9}$$

Now let $s \in A(U) = A_f = \lim_{\leftarrow, i} (({}_i f)^{-1} A_i)$. Write s as a coherent sequence $s = (\cdots \leftarrow a_{i-1}/({}_{i-1} f)^{m_{i-1}} \leftarrow a_i/({}_i f)^{m_i} \leftarrow \cdots)$. Define $\rho_V^U(s)$ to be equal to $(\cdots \leftarrow s_i' \leftarrow \cdots)$, where

$$s_i' \overset{\text{def.}}{=} a_i \cdot u_i^{m_i}/({}_i g)^{n_i m_i}. \tag{10}$$

If another pair (n_i', u_i') is chosen to make (9) stand, as $({}_i g)^{n_i'} = u_i' \cdot {}_i f$, then s_i' in (10) will have to be replaced by $s_i'' = a_i \cdot u_i'^{m_i}/({}_i g)^{n_i' m_i}$. But one can check out easily that $s_i' = s_i''$ inside $({}_i g)^{-1} A_i$. So, $\rho_V^U(s)$ is well-defined *provided* that $s' := (\cdots \leftarrow s_{i-1}' \leftarrow s_i' \leftarrow \cdots)$ given by (10) just above is coherent.

Let us now check the coherence of s'. Since s is given coherent, one knows

$$[({}_{i-1} f)^{m_i} a_{i-1} - ({}_{i-1} f)^{m_{i-1}} \mu_i(a_i)] \cdot ({}_{i-1} f)^{\text{some power}} = 0, \tag{11}$$

and one need to verify

$$[a_{i-1}(u_{i-1})^{m_{i-1}}({}_{i-1} g)^{m_i n_i} - ({}_{i-1} g)^{m_{i-1} n_{i-1}} \mu_i(a_i) \mu_i(u_i)^{m_i}] \cdot ({}_{i-1} g)^{\text{some power}} = 0. \tag{12}$$

Applying μ_i to both sides of (9) and then raising them to the m_i-th power, one obtains ${}_{i-1} g^{m_i n_i} = \mu_i(u_i)^{m_i} ({}_{i-1} f)^{m_i}$; also, (9) for $i := i-1$ gives $({}_{i-1} g)^{n_{i-1}} = u_{i-1} \cdot {}_{i-1} f$. Substituting the right-hand sides of these two equalities for the appropriate terms inside the "[ ]" of (12), we find the said contents of [ ] to be

$$\begin{aligned} &a_{i-1} u_{i-1}^{m_{i-1}} \mu_i(u_i)^{m_i} ({}_{i-1} f)^{m_i} - u_{i-1}^{m_{i-1}} ({}_{i-1} f)^{m_{i-1}} \mu_i(a_i) \mu_i(u_i)^{m_i} \\ &\qquad = u_{i-1}^{m_{i-1}} \mu_i(u_i)^{m_i} [a_{i-1} ({}_{i-1} f)^{m_i} - ({}_{i-1} f)^{m_{i-1}} \mu_i(a_i)]. \end{aligned} \tag{13}$$

The expression inside the "[ ]" of (13) equals that of (11) and, consequently, gets killed by some power of ${}_{i-1} f$. It follows that either side of (13) will be killed by some power of ${}_{i-1}g$ because $({}_{i-1}g)^{n_{i-1}} = u_{i-1} \cdot {}_{i-1}f$. The proof of (ii) will be complete after (iii) and then (i) are established below.

(iii) That $\rho^U_U = \mathrm{Id}_{A(U)}$ is clear in view of the preceding reasoning. As for the transitivity, we have

$$\forall i \in \mathbb{N}\ \exists n_i \exists l_i \in \mathbb{N} : ({}_ig)^{n_i} = u_i \cdot {}_if \text{ and } ({}_ih)^{l_i} = v_i \cdot {}_ig, \text{with } u_i, v_i \in A_i.$$

It follows that, for each i, $({}_ih)^{l_i n_i} = v_i^{n_i} u_i \cdot {}_if$ holds, which implies that the composition $\rho^V_W \circ \rho^U_V$ maps $s = (\cdots \leftarrow a_i/({}_if)^{m_i} \leftarrow \cdots) \in A_f$ to

$$\rho^V_W \circ \rho^U_V(s) = (\cdots \leftarrow a_i u_i^{m_i} v_i^{m_i n_i}/({}_ih)^{m_i n_i l_i} \leftarrow \cdots).$$

On the other hand, the relations $({}_ih)^{k_i} = w_i \cdot {}_if$ for all $i \in \mathbb{N}$ corresponding to $W \subseteq U$ indicates $\rho^U_W(s) = (\cdots \leftarrow a_i w_i^{m_i}/({}_ih)^{m_i k_i} \leftarrow \cdots)$. We already saw above that such coherent sequences are the same in A_h. Therefore, $\rho^V_W \circ \rho^U_V = \rho^U_W$.

(i) If $U = V$ or $D(f) = D(g)$, we have maps $\rho^U_V : A(U) \to A(V)$ and $\rho^V_U : A(V) \to A(U)$. As we just saw, $\rho^V_U \circ \rho^U_V = \rho^U_U = \mathrm{Id}_{A(U)}$, and likewise for $\rho^U_V \circ \rho^V_U$. Hence $A(U) \simeq A(V)$. With (i) proven now, the proof of (ii) is complete. $\square$

It follows from Prop. 2.2.1 that the assignments $U = D(f) \mapsto A(U) = A_f$ and $[V = D(g) \hookrightarrow U = D(f)] \mapsto \rho^U_V$ produce a *presheaf* $\mathcal{A}$ of pro-affine K-algebras on the base $\mathcal{B} = \{D(f) : f \in A\}$ of open sets of the topological space $\mathfrak{Sp}(A)$. (see [1, Chap. 0, §3.2, p. 25ff.] .)

Proposition 2.2.2. *Let A be a pro-affine algebra, and let $\mathcal{A}$ be the presheaf over the base $\mathcal{B}$ of open sets on $\mathfrak{Sp}(A)$ introduced just above. Let $U = D(g) \in \mathcal{B}$ be any basic open set, and let $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ be a covering of U with each $U_\lambda = D(f_\lambda)$, $f_\lambda \in A_g$. Suppose given for each $\kappa \in \Lambda$ an element $s_\kappa \in \mathcal{A}(U_\kappa)$ such that $\rho^{U_\lambda}_{U_{\lambda\nu}}(s_\lambda) = \rho^{U_\nu}_{U_{\lambda\nu}}(s_\nu)$ for any λ, $\nu \in \Lambda$, where $U_{\lambda\nu}$ denotes $U_\lambda \cap U_\nu$. Then, there is one and only one $s \in \mathcal{A}(U)$ such that $\rho^U_{U_\kappa}(s) = s_\kappa$ for all $\kappa \in \Lambda$.*

Proof. The proof is based on the well-established fact that the proposition holds true in case of the affine schemes. (cf. [1, Th. (1.3.7), p. 86].)

It is clearly enough to prove the proposition in case $U = \mathfrak{Sp}(A)$ and $\mathcal{A}(U) = A$. Assume so and write $A = \lim_{\leftarrow} A_i$, $X_i = \mathrm{Spec}(A_i)$. For each $\lambda \in \Lambda$ and each $i \in \mathbb{N}$, write $f_\lambda = (\cdots \leftarrow {}_if_\lambda \leftarrow \cdots)$ and let

$$U_{\lambda,i} := \{\pi_i^{-1}(P) : P \in X_i \text{ and } {}_if_\lambda \notin P\} = \pi_i^{-1}(D({}_if_\lambda)) \tag{14}$$

where $D({}_if_\lambda)$ is the basic open set in $X_i = \mathrm{Spec}\, A_i$. We then have two types of open coverings for each λ and each i, i. e.:

$$U_\lambda = \bigcup_{i\in\mathbb{N}} U_{\lambda,i} \quad \text{and} \quad X_i = \bigcup_{\lambda\in\Lambda} D({}_if_\lambda). \tag{15}$$

[Uniqueness] Let s', $s'' \in A = \mathcal{A}(U)$ be such that $\rho^U_{U_\kappa}(s') = s_\kappa$, $\rho^U_{U_\kappa}(s'') = s_\kappa$ for all $\kappa \in \Lambda$. So, one may write $s' = (\cdots \leftarrow {}_is' \leftarrow \cdots)$ and $s'' = (\cdots \leftarrow {}_is'' \leftarrow \cdots)$, with ${}_is' \in A_i$, ${}_is'' \in A_i$ for each i. Now, since $\rho^U_{U_\kappa}(s') = \rho^U_{U_\kappa}(s'')$ for all κ, these agree on $U_{\kappa,i}$ for all i in the first covering of (15), or ${}_i(\rho^U_{U_\kappa}(s')) = {}_i(\rho^U_{U_\kappa}(s''))$. This means that ${}_is'$ and ${}_is''$ agree on each piece $D({}_if_\kappa)$ of the second covering of (15) for each κ. It follows that ${}_is' = {}_is''$ on X_i for each i, because of the fact pointed out at the beginning of the proof. Therefore, we have $s' = s''$.
[Existence] We are locally given s_κ on U_κ for all κ such that s_λ and s_ν agree on $U_\lambda \cap U_\nu$ whenever the intersection is nonempty. The data will then induce, at each finite level i, the data of $\{{}_i(s_\kappa) : \kappa \in \Lambda\}$ locally on each open piece $D({}_if_\kappa)$ of the covering $X_i = \bigcup_{\lambda\in\Lambda} D({}_if_\lambda)$. We can patch up the local data of ${}_i(s_\kappa)$'s on the affine scheme X_i so as to obtain $s_i \in A_i$. What remains to be checked out is that $(\cdots \leftarrow s_i \leftarrow s_{i+1} \leftarrow \cdots)$ is coherently defined. So, let $s_i' := \mu_{i+1}(s_{i+1})$, and we will show that $s_i = s_i'$. Now, denote the restriction map of X_i to $D({}_if_\kappa)$ by $\rho_{i,\kappa}$. We have thus $\rho_{i,\kappa} \colon A_i \longrightarrow (A_i)_{f_\kappa}$. By construction, $\rho_{i,\kappa}(s_i) = {}_i(s_\kappa)$ and $\rho_{i+1,\kappa}(s_{i+1}) = {}_{i+1}(s_\kappa)$. It follows that

$$\rho_{i,\kappa}(s_i') = \rho_{i,\kappa}(\mu_{i+1}(s_{i+1})) = \mu'_{i+1}(\rho_{i+1,\kappa}(s_{i+1})) = \mu'_{i+1}({}_{i+1}(s_\kappa)) = {}_i(s_\kappa), \tag{16}$$

with $\mu'_{i+1} \colon (A_{f_\kappa})_{i+1} \rightarrow (A_{f_\kappa})_i$ standing for the map induced by $\mu_{i+1} \colon A_{i+1} \rightarrow A_i$. It is now shown that $\rho_{i,\kappa}(s_i) = \rho_{i,\kappa}(s_i')$ for all $\kappa \in \Lambda$. Once again one draws upon the uniqueness in the affine-scheme case to conclude that $s_i = s_i'$. $\square$

We now extend the presheaf $\mathcal{A}$ to a presheaf over the topological space $\mathfrak{Sp}(A)$ by defining, for any open set $U \subseteq \mathfrak{Sp}(A)$, $\mathcal{A}(U) \overset{\text{def.}}{=} \lim_{\leftarrow} \mathcal{A}(V)$ where the $\lim_{\leftarrow}$ is taken over all *basic* V's for which $V = D(g) \subseteq U$ [1, chap. 0-3.2, pp. 25ff]. The extended presheaf will be denoted by $\mathcal{A}$, too. The next theorem follows immediately from Prop. 2.2.2. (cf. [1, *loc. cit.*].)

Theorem 2.2.3. *The presheaf $\mathcal{A}$ is a sheaf.*

From here on, the topological space $\mathfrak{Sp}(A)$ endowed with the sheaf $\mathcal{A}$ as above will be referred to as the *ind-affine scheme associated with A* and will be denoted by $\mathcal{X}_A$. $\mathcal{A}$ is then, by definition, the *structure sheaf of* $\mathcal{X}_A$. In conformity with standard practice in scheme theory we shall also write $\mathcal{A} = \mathcal{O}(A)$. Similarly, the topological space $\mathfrak{Spm}(A)$ together with the sheaf induced on it from $\mathcal{A}$ is called the *ind-affine variety associated with A*, and this variety will be denoted by $\mathcal{V}_A$.

We next address the issue of stalks of the sheaf $\mathcal{A}$. Let $\mathcal{X}_A$ be an ind-affine scheme, with $A = \lim_{\leftarrow} A_i$. Let $\mathfrak{p}$ be a point on $\mathcal{X}_A$, and let $\Lambda :=$ the filter of all basic open sets containing the point $\mathfrak{p}$, so $\Lambda = \{D(g_\alpha) : \mathfrak{p} \in D(g_\alpha)\}$. Let us write $A_{i,\alpha} := (A_i)_{{}_ig_\alpha}$ for all $i \in \mathbb{N}$ and all g_α for which $D(g_\alpha) \in \Lambda$. We then have the following commutative diagram in which all horizontal arrows represent surjections and vertical ones are restrictions occurring whenever $D(g_\alpha) \supseteq D(g_\beta)$, each column thus forming a direct system:

(17)

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & & & {} \arrow[d] \\
\cdots & A_{i-1,\alpha} \arrow[l] \arrow[d] & A_{i,\alpha} \arrow[l] \arrow[d] & \cdots \arrow[l] & \cdots & \lim_{\leftarrow,n} A_{n,\alpha} \arrow[l] \arrow[d] \\
\cdots & A_{i-1,\beta} \arrow[l] \arrow[d] & A_{i,\beta} \arrow[l] \arrow[d] & \cdots \arrow[l] & \cdots & \lim_{\leftarrow,n} A_{n,\beta} \arrow[l] \arrow[d] \\
 & \cdot \arrow[d] & \cdot \arrow[d] & & & \cdot \arrow[d] \\
 & \cdot \arrow[d] & \cdot \arrow[d] & & & \lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha}) \arrow[d, "\Phi"] \\
\cdots & \lim_{\rightarrow,\gamma} A_{i-1,\gamma} \arrow[l] & \lim_{\rightarrow,\gamma} A_{i,\gamma} \arrow[l] & \cdots \arrow[l] & \cdots & \lim_{\leftarrow,m}(\lim_{\rightarrow,\gamma} A_{m,\gamma}) \arrow[l]
\end{tikzcd}$$

In the diagram (17) one should recognize that $\lim_{\leftarrow,n} A_{n,\alpha} = A_{g_\alpha} = \mathcal{A}(D(g_\alpha))$, and $\lim_{\rightarrow,\gamma} A_{m,\gamma} = (A_m)_{{}_m\mathfrak{p}}$. So, the map Φ on the lower right corner of (17) amounts to $\lim_{\rightarrow,\alpha}(\mathcal{A}(D(g_\alpha)) = \lim_{\rightarrow,\alpha} A_{g_\alpha} \xrightarrow{\Phi} \lim_{\leftarrow,m}((A_m)_{{}_m\mathfrak{p}})$, and Φ gets induced as follows: (i) First, for each α there is a map $A_{g_\alpha} \to \lim_{\rightarrow,\gamma} A_{i,\gamma}$ for all i with appropriate commutativity of arrow paths; (ii) as a consequence there is a map $A_{g_\alpha} \to \lim_{\leftarrow,j}(\lim_{\rightarrow,\lambda} A_{j,\lambda}) = \lim_{\leftarrow,j}((A_j)_{{}_j\mathfrak{p}})$; and finally (iii) the desired map $\lim_{\rightarrow,\gamma} A_{g_\gamma} \to \lim_{\leftarrow,j}((A_j)_{{}_j\mathfrak{p}})$ again because of the appropriate commutativity.

We now come to study the map Φ. In order to describe its kernel, we need to introduce the notion of *elements infinitely near* 0 in the ring $\lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha})$ and, before that, a new *ad hoc* notation: If $a_i \in A_{i,\alpha}$ then $[a_i]$ denotes the equivalence class represented by a_i in the direct limit $\lim_{\rightarrow,\gamma} A_{i,\gamma} = (A_i)_{{}_i\mathfrak{p}}$. Likewise, if $(\cdots \leftarrow a_{q-1,\gamma} \leftarrow a_{q,\gamma} \leftarrow \cdots) \in \lim_{\leftarrow,n} A_{n,\gamma} = \mathcal{A}(D_{g_\gamma})$, then $[\cdots \leftarrow a_{q-1,\gamma} \leftarrow a_{q,\gamma} \leftarrow \cdots]$ is to mean the corresponding equivalence class $\in \lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha}) = \lim_{\rightarrow,\alpha} \mathcal{A}(D(g_\alpha))$. Now, let

$$u := [\cdots \leftarrow u_{n-1,\alpha} \leftarrow u_{n,\alpha} \leftarrow \cdots] \in \lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha}) = \lim_{\rightarrow,\alpha} \mathcal{A}(D(g_\alpha)). \tag{18}$$

We shall say that u is *infinitely near* 0 if $\forall u_{n,\alpha} \exists \beta = \beta(n,\alpha) \geq \alpha$ such that $u_{n,\alpha} \mapsto u_{n,\beta} = 0$ under the restriction map due to the inclusion $D({}_n g_\beta) \subseteq D({}_n g_\alpha)$. The terminology is appropriate because, for such u, $[u_{n,\alpha}] = [0]$ for every n, yet u may not be 0.

It is easy to see that the set of all elements of $R := \lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha}) = \lim_{\rightarrow,\alpha}(\mathcal{A}(D(g_\alpha))$ that are infinitely near 0 form an ideal of the ring R.

Theorem 2.2.4. (a) *Let R be as just above. Then, the kernel of the map $\Phi\colon R \to \lim_{\leftarrow,m}(A_m)_{m\mathfrak{p}}$ is the ideal of all elements infinitely near 0 in R.*
(b) *The image of Φ is everywhere dense in $\lim_{\leftarrow,m}(A_m)_{m\mathfrak{p}}$.*

Proof. (a) If $\Phi(u) = 0$ for u as in (18), that means $(\cdots \leftarrow [u_{n-1,\alpha}] \leftarrow [u_{n,\alpha}] \leftarrow \cdots) = (\cdots \leftarrow 0 \leftarrow 0 \leftarrow \cdots)$ inside $\lim_{\leftarrow,m}((A_m)_{m\mathfrak{p}})$, or $\forall n$, $[u_{n,\alpha}] = 0$. So, u is infinitely near 0. The converse clearly holds also.
(b) Given $\eta = (\cdots \leftarrow [u_{i-1,\alpha_{i-1}}] \leftarrow [u_{i,\alpha_i}] \leftarrow \cdots) \in \lim_{\leftarrow,m}((A_m)_{m\mathfrak{p}})$, write $\eta = (\cdots \leftarrow r_{i-1} \leftarrow r_i \leftarrow \cdots)$ with each $r_i \in (A_i)_{i\mathfrak{p}}$. For an arbitrary high $N > 0$, let $w_N := u_{N,\alpha_N} \in A_{N,\alpha_N}$. Clearly, one can complete w_N to an element

$$w = (\cdots \leftarrow w_{N-1} \leftarrow w_N \leftarrow w_{N+1} \leftarrow \cdots) \in \lim_{\leftarrow,n} A_{n,\alpha_N}$$

such that

$$[w_0] = [u_{0,\alpha_0}],\ \ [w_1] = [u_{1,\alpha_1}], \ldots, [w_{N-1}] = [u_{N-1,\alpha_{N-1}}],\ \ [w_N] = [u_{N,\alpha_N}].$$

So, $[w] := [\cdots \leftarrow w_{N-1} \leftarrow w_N \leftarrow w_{N+1} \leftarrow \cdots] \in R$ is such that $\Phi([w])$ and η agree with each other up to the N-th place from the left. Since N was arbitrary, this shows the density of the image of Φ. $\square$

In view of Th. 2.2.4, we define the *local ring of* a point $\mathfrak{p}$ on an ind-affine scheme $\mathcal{X}_A$, $A = \lim_{\leftarrow} A_m$, to be $\lim_{\leftarrow,m}(A_m)_{m\mathfrak{p}}$. It is a pro-affine K-algebra, and a surjective inverse limit of local rings of the more traditional type.

3. Comments and Examples

(A) The *reduction* A_{red} and the *strong reduction* A_{RED} (see §1.3-(7) above):

In [2] we raised the question as to whether or not $A_{\mathrm{red}} = A_{\mathrm{RED}}$ for the types of pro-affine algebras A of interest to us, and we indicated how this issue bears upon the Jacobian Problem (cf. [2, (1.3), p. 484, and (5.4), p. 498]). As expected, this question is easily answered in the negative, as follows:

EXAMPLE 3-E. For all $i \in \mathbb{N}$ consider the same algebras A_i as occurred in [2, Ex. (1.4), p. 484] but with different connecting maps μ_i. Namely, let

$$A_i := K[T_1, \cdots, T_{i-1}, T_i, T_{i+1}]/\langle T_{i+1}^2\rangle = K[T_1, \ldots, T_{i-1}, T_i, \tau_{i+1}],$$

and define $\mu_i\colon A_i \longrightarrow A_{i-1}$ by stipulating

$$\mu_i(T_j) := T_j \text{ for } j < i\,;\ \ \mu_i(T_i) := \tau_i\,;\ \ \mu_i(\tau_{i+1}) := \tau_i \cdot T_1.$$

Then, in the exact sequence

$$0 \longrightarrow \langle \tau_{i+1} \rangle \longrightarrow A_i \longrightarrow K[T_1, \ldots, T_i] \longrightarrow 0 \tag{19}$$

for all $i > 0$, the Mittag-Leffler condition fails to hold, so that the sequence

$$0 \longrightarrow \mathcal{N}(A) \longrightarrow A \longrightarrow A_{\text{RED}} \longrightarrow 0, \text{ where } A_{\text{RED}} = K^{[[\infty]]} \tag{20}$$

obtained by applying $\lim_{\leftarrow,i}$ to (19), is expected to be nonexact on the right.

We can actually exhibit where the map $A \to A_{\text{RED}}$ fails to be surjective. In fact, let

$$f_i := T_1 + \cdots + T_{i-1} + T_i \quad \text{for all } i \in \mathbb{N},$$

and consider $f := (f_1 \leftarrow \cdots \leftarrow f_{i-1} \leftarrow f_i \leftarrow \cdots) \in A_{\text{RED}}$. Suppose that there existed some $g \in A$ such that $g = (g_1 \leftarrow \cdots \leftarrow g_i \leftarrow \cdots) \mapsto f \in A_{\text{RED}}$. Then, for each $i \in \mathbb{N}$, it must hold that $g_i = f_i + \tau_{i+1} \cdot h_i = f_{i-1} + T_i + \tau_{i+1} \cdot h_i$ for a suitable $h_i \in K[T_1, \ldots, T_{i-1}, T_i]$. On the other hand, $\mu_i(g_i) = g_{i-1}$, or

$$\begin{aligned} f_{i-1} + \tau_i + \tau_i \cdot T_1 \cdot h_i(T_1, \cdots, T_{i-1}, \tau_i) \\ = f_{i-1} + \tau_i + \tau_i \cdot T_1 \cdot h_i(T_1, \cdots, T_{i-1}, 0) \\ = f_{i-1} + \tau_i \cdot h_{i-1}, \end{aligned} \tag{21}$$

which implies that

$$h_{i-1} = 1 + T_1 \cdot h_i(T_1, \cdots, T_{i-1}, 0) \text{ for all } i \in \mathbb{N}. \tag{22}$$

Using this last equation recursively, one would get

$$\begin{aligned} h_1(T_1) &= 1 + T_1 \cdot h_2(T_1, 0) \\ &= 1 + T_1(1 + T_1 \cdot h_3(T_1, 0, 0)) = 1 + T_1 + T_1^2 \cdot h_3(T_1, 0, 0) \\ &= \cdots = 1 + T_1^2 + \cdots + T_1^{k-1} \cdot h_k(T_1, 0, \ldots, 0) = \cdots (ad\ infin.). \end{aligned} \tag{23}$$

This lends an arbitrarily high T_1-degree to the polynomial $h_1(T_1)$, an absurdity.

(B) Closed Embedding and Topology of Ind-affine schemes:

Let A, B be pro-affine algebras, and $X := \mathcal{X}_A$, $Y := \mathcal{X}_B$. A morphism of ind-affine schemes $f : Y \to X$ defined by a continuous K-map $\phi : A \to B$ is said to be a *closed embedding* if ϕ is open and surjective. When that is so, through appropriate representations $A = \lim_{\leftarrow} A_i$, $B = \lim_{\leftarrow} B_i$ of A and B as inverse limits, one may see to it that ϕ is induced by surjections $A_i \to B_i$ for all $i \in \mathbb{N}$. One can then say that the closed embedding $Y \to X$ is the direct limit of the closed embeddings $Y_i \to X_i$

for all i. The converse is inexact. Namely, if $\phi\colon A \to B$ comes as the inverse limit of surjective K-maps $A_i \to B_i$ for all i, ϕ need not be surjective. In other words, if $f\colon Y \to X$ is induced by closed embeddings $Y_i \to X_i$ $(\forall i)$ of finite-dimensional affine K-schemes $X_i = \operatorname{Spec}(A_i)$, $Y_i = \operatorname{Spec}(B_i)$, f need not be a closed embedding of ind-affine schemes. This point is illustrated by the following example:

EXAMPLE 3-F (Burt Totaro). Let $X := \mathbb{A}^\infty = \mathfrak{X}_{K^{[\infty]}}$, so $X = \bigcup_{i=1}^\infty X_i$ with $X_i = \mathbb{A}^i$. Define a subscheme $Y = \bigcup_{i=1}^\infty Y_i$ of X inductively, as follows: (a) $Y_1 := X_1 = \mathbb{A}^1$.
(b) Having built Y_{i-1}, define Y_i to be the union of Y_{i-1} and a finite set of lines through the origin in X_i such that every polynomial function on X_i of degree $\le i$ which vanishes on these lines must be 0 altogether on X_i. [Just take enough number of lines on X_i through the origin and in general position.]

Now consider the morphism $f\colon Y \longrightarrow X$ arising as the dual of the natural map, $\mathcal{O}(X) := \lim_{\leftarrow,i} \mathcal{O}(X_i) \to \mathcal{O}(Y) := \lim_{\leftarrow,i} \mathcal{O}(Y_i)$, where the maps $\mathcal{O}(X_i) \to \mathcal{O}(Y_i)$ are surjections associated with the closed embeddings $Y_i \to X_i$ for all i. This f exhibits some pathological characters, as shall be seen now.
(a) First, let $J_i := \operatorname{Ker}(\mathcal{O}(X_i) \to \mathcal{O}(Y_i))$. Then, J_i is a homogeneous ideal in $K^{[i]}$ whose generators may be taken to be forms of degree $> i$. This shows that the exact sequences $0 \to J_i \to \mathcal{O}(X_i) \to \mathcal{O}(Y_i) \to 0$ taken for all $i \in \mathbb{N}$ do not satisfy the Mittag-Leffler condition, and the non-surjectiveness of $\mathcal{O}(X) \to \mathcal{O}(Y)$ is strongly indicated.
(b) Second, let $\mathfrak{m}_{X_i}$, $\mathfrak{m}_{Y_i}$ be the maximal ideals of the origin (0) on X_i, Y_i in the rings $\mathcal{O}(X_i)$, $\mathcal{O}(Y_i)$, respectively. Then, for every pair of r and i with $0 < r \le i$, the natural surjection

$$\psi_{r,i}\colon \mathcal{O}(X_i)/\mathfrak{m}_{X_i}^r \longrightarrow \mathcal{O}(Y_i)/\mathfrak{m}_{Y_i}^r$$

is also injective because of the make-up of Y_i, so that $\psi_{r,i}$ is an isomorphism. It follows that $\psi_r := \lim_{i\to\infty}(\psi_{r,i})$ gives an isomorphism $\mathcal{O}(X)/\mathfrak{m}_X^{(r)} \simeq \mathcal{O}(Y)/\mathfrak{m}_Y^{(r)}$ for all $r > 0$. Consequently, $\mathfrak{m}_X/\mathfrak{m}_X^{(2)} \simeq \mathfrak{m}_Y/\mathfrak{m}_Y^{(2)}$ and $\mathfrak{m}_X^{(r)}/\mathfrak{m}_X^{(r+1)} \simeq \mathfrak{m}_Y^{(r)}/\mathfrak{m}_Y^{(r+1)}$. Since the point (0) on X satisfies the smoothness condition that $\check{\mathrm{S}}^n(\mathfrak{m}_X/\mathfrak{m}_X^{(2)}) \to \mathfrak{m}_X^{(n)}/\mathfrak{m}_X^{(n+1)}$ be an isomorphism for all $n > 0$ (see [2, p. 488]), so does (0) on Y, or Y is smooth at (0).

We can see that this creates a serious problem for the notion of smoothness of ind-affine varieties, as calling the point (0) a simple point on Y goes against our intuition. It appears that the notion of smoothness (or of simple point) should be reworked (see [2, p. 488], [3, p. 187ff]). We will not, however, go into this issue in this paper. Turning to the more immediate question on Totaro's example at hand, we find it impossible that the K-map $\mathcal{O}(X) \to \mathcal{O}(Y)$ in (a) just above should be surjective, or that the morphism $Y \to X$ should be a closed immersion. For, were this the case, then the embedding theorem [2, (2.6), p. 488] would imply that Y is isomorphic to X as ind-affine scheme. It follows that, for every i, Y_i is a closed subscheme of X_i but $Y \to X$ is not a closed immersion.

(C) Example of a proper ideal whose closure is the unit ideal:
We follow up on Example 1.2-D and Remark that precedes it.

EXAMPLE 3-G. Let

$$\begin{aligned}
w_1 &:= (1 \leftarrow 1 + x_1 \leftarrow 1 + x_1 + x_2 \leftarrow \cdots \leftarrow 1 + x_1 + x_2 + \cdots + x_n \leftarrow \cdots)\\
w_2 &:= (1 \leftarrow 1 \leftarrow 1 + x_2 \leftarrow 1 + x_2 + x_3 \leftarrow \cdots \leftarrow 1 + x_2 + \cdots + x_n \leftarrow \cdots)\\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots\\
w_n &:= (1 \leftarrow 1 \leftarrow \cdots \leftarrow 1 \leftarrow 1 + x_n \leftarrow 1 + x_n + x_{n+1} \leftarrow \cdots)\\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

be a sequence of elements in $K^{[\infty]}$. So, $w_n - w_{n+1} = (0 \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow x_n \leftarrow x_n \leftarrow \cdots)$ and $w_n - 1 = (0 \leftarrow \cdots \leftarrow 0 \leftarrow x_n \leftarrow x_n + x_{n+1} \leftarrow \cdots)$. It follows that $\lim_{n\to\infty} w_n = 1$ and $\overline{\langle w_1, \ldots, w_n, \ldots\rangle} = \langle 1\rangle$. On the other hand, $\langle w_1, \ldots, w_n, \ldots\rangle \subsetneqq \langle 1\rangle$ because no finite linear combination of the w_i's can equal 1. To be more specific, suppose $L = 1$ for an $K^{[\infty]}$-linear combination L of $w_k, w_l, \ldots, w_m$ ($k < l < \cdots < m$), or $\langle w_k, w_l, \ldots, w_m\rangle = \langle 1\rangle$. Then, $\langle w_1, \ldots, w_m\rangle = \langle w_1, w_1 - w_2, \ldots, w_{m-1} - w_m\rangle = \langle 1\rangle$. This implies that an $K^{[\infty]}$-linear combination of

$$\begin{aligned}
w_1 &= (1 \leftarrow 1 + x_1 \leftarrow \cdots \leftarrow 1 + x_1 + \cdots + x_m \leftarrow \cdots)\\
w_1 - w_2 &= (0 \leftarrow x_1 \leftarrow x_1 \leftarrow \cdots \leftarrow x_1 \leftarrow \cdots)\\
w_2 - w_3 &= (0 \leftarrow 0 \leftarrow x_2 \leftarrow \cdots \leftarrow x_2 \leftarrow \cdots)\\
&\ \ \vdots \qquad\qquad\qquad \vdots\\
w_{m-1} - w_m &= (0 \leftarrow 0 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow x_{m-1} \leftarrow \cdots)
\end{aligned}$$

should produce $(1 \leftarrow 1 \leftarrow \cdots \leftarrow 1 \leftarrow \cdots)$. Clearly, this is impossible.

ACKNOWLEDGEMENTS. In conducting research for the present paper, the author has received great benefits from discussions with Masayoshi Miyanishi, M. Pavaman Murthy, N. Mohan Kumar and David Wright. Especially, N. Mohan Kumar contributed some key results as indicated in the main text. E-mail correspondence with Yakov Varshavsky was also valuable, and Burt Totaro kindly permitted us to include his example as in §3 above. The author wishes to express heartfelt thanks to these old and new friends. Also, the hospitality of the University of Chicago's Mathematics Department, where the author has been allowed to spend many summers for research, is very much appreciated.

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Department of Mathematical Sciences
Tokyo Denki University
Saitama 350-0394, JAPAN
e-mail: Kambayashi@gakushikai.jp