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CARLESON TYPE CONDITIONS AND WEIGHTED INEQUALITIES FOR HARMONIC FUNCTIONS

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1. Introduction

Let D be the unit disc of the complex plane and dA be the normalized area measure on D. Littlewood and Paley [6] proved the following theorem.

Theorem (Littlewood-Paley). *Let $2 \le p < \infty$. If $f \in L^p(\partial D)$ and if F is the Poisson integral of f, then*

$$\int_D |\nabla F(z)|^p (1-|z|^2)^{p-1}\, dA(z) \le C^p \int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}$$

where C is a constant independent of f and p.

In relation with this theorem, the following problem has been extensively studied (see [3], [4], [5], [6], [7], [10], [11] and references therein): Let Ω be a domain in $\mathbf{R}^n$. Given p, q and a differential monomial ∂^m of order m, find (locally finite) positive Borel measures $d\mu$ and $d\nu$ such that the inequality

$$\left(\int_\Omega |\partial^m f|^q\, d\mu\right)^{1/q} \le C \left(\int_{\partial\Omega} |f|^p\, d\nu\right)^{1/p} \tag{1.1}$$

holds for all f harmonic on $\overline{\Omega}$. In the case where $d\nu$ is given by the Lebesgue measure on the boundary, complete characterizations have been known either on the ball or on the upper half-space. The case $2 \le p = q < \infty$ was solved by Shirokov [10, 11] on the disc and the case $0 < p < q < \infty$ is solved by Luecking [5] on the upper half-space. All those characterizations are given in terms of Carleson type criterion. For other cases where $0 < p = q < 2$ or $0 < q < p < \infty$, characterizations are given in terms of the so-called "tent" spaces ([5]) or "balayées" conditions ([3]) on the upper half-space. In [3] Gu actually studied the case where $d\nu$ is given by an A_p-weight, but only for $m = 0$.

More recently, on the unit ball of $\mathbf{C}^n$ with an A_p-weight given on the boundary, Kang and Koo [7] considered holomorphic functions and their ordinary, normal and

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complex tangential derivatives of all orders. In this paper, we continue investigating the problem in that direction for harmonic functions on the ball. Here, we confine ourselves to the cases $2 \le p = q < \infty$ and $1 < p < q < \infty$.

Fix an integer $n \ge 2$ and let $B = B_n$ be the unit ball of $\mathbf{R}^n$. In this paper we take $\Omega = B$, consider various derivatives of all orders, and characterize locally finite positive Borel measures $d\mu$ which satisfies (1.1) for all harmonic functions, in case $d\nu$ is given by an A_p-weight. To state our results, let us introduce some notations. For $\zeta \in \partial B$ and $\delta > 0$, define balls $S_\delta(\zeta)$ and their "tents" $\widehat{S}_\delta(\zeta)$ by

$$
\begin{aligned}
S_\delta(\zeta) &= \{\eta \in \partial B : |\zeta - \eta| < \delta\},\\
\widehat{S}_\delta(\zeta) &= \{z \in B : |\zeta - z| < \delta\}.
\end{aligned}
\tag{1.2}
$$

Also, let $\mathcal{D}f$ denote the radial derivative of f and let $\mathcal{T}^\alpha f$ denote tangential derivatives of f (see Section 2). For an A_p-weight ω on ∂B (simply $\omega \in A_p$), we write $h^p(\omega)$ for the harmonic Hardy space with weight ω. For simplicity we let

$$\omega(S) = \int_S \omega \, d\sigma$$

for a Borel set $S \subset \partial B$. Here, $d\sigma$ denotes the surface area measure on ∂B.

The following is our main result. As expected, weighted inequalities are characterized by weighted Carleson type conditions of measures under consideration. Here, we use the conventional multi-index notation.

Main Theorem. *Assume $2 \le p = q < \infty$ or $1 < p < q < \infty$. Let $\omega \in A_p$ and α be a multi-index with $|\alpha| = m \ge 1$. Then, for a locally finite positive Borel measure $d\mu$ on B, the following are equivalent.*

(1) $\mu\big[\widehat{S}_\delta(\zeta)\big] \le C\,\omega\big[S_\delta(\zeta)\big]^{q/p}\delta^{mq}$ *for all* $\zeta \in \partial B$ *and* $\delta > 0$.

(2) $||\mathcal{D}^m f||_{L^q(\mu)} \le C||f||_{h^p(\omega)}$ *for all* $f \in h^p(\omega)$.

(3) $\sum_{|\beta|=m} ||\mathcal{T}^\beta f||_{L^q(\mu)} \le C||f||_{h^p(\omega)}$ *for all* $f \in h^p(\omega)$.

(4) $||\partial^\alpha f||_{L^q(\mu)} \le C||f||_{h^p(\omega)}$ *for all* $f \in h^p(\omega)$.

As mentioned above, the case $m = 0$ (on the upper half-space) is contained in [3]. On the other hand, our results extend those of [7] concerning holomorphic functions. Proofs are divided into two cases. See Section 3 for $2 \le p = q < \infty$ and Section 4 for $1 < p < q < \infty$. In Section 5, we prove the "little oh" version of our main theorem.

NOTATION. Throughout the paper we use the same letter C (often with subscripts) for various constants which may depend on given measures and some parameters such as n, p, q and m, but it will always be independent of particular functions, balls or points, etc. Also, we use the abbreviated notation $A \lesssim B$ if there exists an inessential positive constant C such that $A \le CB$. Thus, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

For a given multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with each α_j a nonnegative integer, we use notations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where ∂_j denotes the differentiation with respect to j-th variable.

For a function $f \in C^1(B)$, we let $\mathcal{D}f$ denote the radial derivative of f. More explicitly, we let

$$\mathcal{D}f(x) = \sum_{j=1}^{n} x_j \partial_j f(x) \qquad (x \in B).$$

Note that if f is harmonic, then so is $\mathcal{D}f$.

Since there is no smooth nonvanishing tangential vector field on ∂B for $n > 2$, we define tangential derivatives by means of a family of tangential vector fields generating all the tangent vectors. We define tangential derivatives $\mathcal{T}_{ij}f$ of $f \in C^1(B)$ by

$$\mathcal{T}_{ij}f(x) = (x_i \partial_j - x_j \partial_i) f(x) \qquad (x \in B)$$

for $1 \le i < j \le n$. As in the case of radial derivatives, tangential derivatives of harmonic functions are again harmonic. Given a nontrivial multi-index α, we abuse the notation $\mathcal{T}^\alpha = \mathcal{T}_{i_1 j_1}^{\alpha_1} \cdots \mathcal{T}_{i_n j_n}^{\alpha_n}$ for any choice of $i_1, \dots, i_n$ and $j_1, \dots, j_n$.

By the mean value property of harmonic functions and Cauchy's estimates, we have the following lemma. See [1, Chapter 8] for a proof. Here and in what follows, dV denotes the Lebesgue measure on $\mathbf{R}^n$.

Proposition 2.1. *Let $1 \le p < \infty$ and α be a multi-index. Suppose f is harmonic on a domain Ω in $\mathbf{R}^n$. Then, we have*

$$|\partial^\alpha f(x)|^p \le \frac{C^p}{d(x, \partial\Omega)^{n+p|\alpha|}} \int_\Omega |f|^p \, dV \qquad (x \in \Omega)$$

where $d(x, \partial\Omega)$ denotes the distance from x to ∂D. The constant C depends only on n and α.

Let $1 < p < \infty$ and ω be a weight function on ∂B. We say $\omega \in A_p$ if ω satisfies the A_p condition of Muckenhoupt (see [9]), that is, there exists a constant C such that

$$\omega(S) \left(\int_S \omega^{-1/(p-1)} \, d\sigma \right)^{p-1} < C|S|^p$$

for all $S = S_\delta(\zeta)$. Here, $|S| = \sigma(S)$. Note that A_p-weights are doubling measures by Hölder's inequality. Namely, to each $\omega \in A_p$ there corresponds a "doubling" constant C_ω such that

$$\omega\left[S_{2\delta}(\zeta)\right] \le C_\omega \omega\left[S_\delta(\zeta)\right] \tag{2.1}$$

for any $\delta > 0$ and $\zeta \in \partial B$.

For $\omega \in A_p$, let $L^p(\omega) = L^p(\omega d\sigma)$. The weighted harmonic Hardy space $h^p(\omega)$ is then the space of all harmonic functions f on B for which $\mathcal{N} f \in L^p(\omega)$ and define $||f||_{h^p(\omega)} = ||\mathcal{N} f||_{L^p(\omega)}$. Here, $\mathcal{N} f$ denotes the nontangential maximal function of f defined by

$$\mathcal{N} f(\zeta) = \sup_{x\in\Gamma(\zeta)} |f(x)|, \qquad \zeta \in \partial B$$

where $\Gamma(\zeta)$ is the nontangential approach region

$$\Gamma(\zeta) = \{x \in B : |x - \zeta| < 2(1 - |x|)\}.$$

By the local Fatou theorem every $f \in h^p(\omega)$ has nontangential limit, which we again denote by f, at almost all boundary points. Note that $f \in L^p(\omega)$ for $f \in h^p(\omega)$, because $|f| \le \mathcal{N} f$ on ∂B. It is well known that $||f||_{h^p(\omega)} \approx ||f||_{L^p(\omega)}$ (see Lemma 3.1 below). Also, note that $L^p(\omega) \subset L^1(\sigma)$ for $\omega \in A_p$. Thus, for each $f \in h^p(\omega)$, the Poisson integral of its boundary function is well defined. Moreover, it is not hard to see that each $f \in h^p(\omega)$ is recovered by the Poisson integral of $f \in L^p(\omega)$.

3. The Case $2 \le p = q < \infty$

This section is devoted to the proof of the main theorem for the case $2 \le p = q < \infty$. The proof will be completed in the following order:

$$\begin{aligned} &(1) \implies (4), \quad (1) \implies (2) + (3) \\ &(2) \implies (1), \quad (3) \implies (1), \quad (4) \implies (1). \end{aligned}$$

Our proof of (1) $\implies$ (4) depends on the weighted inequalities for the nontangential operator and the so-called area integral operator. For $x \in B$, put

$$r(x) = 1 - |x|.$$

For a function f harmonic on B, the area integral function $\mathcal{S} f$ is then defined by

$$\mathcal{S} f(\zeta) = \left(\int_{\Gamma(\zeta)} |\nabla f|^2 r^{2-n}\, dV\right)^{1/2}$$

for $\zeta \in \partial B$. For the operators $\mathcal{S}$ and $\mathcal{N}$, the weighted inequalities below with respect to A_p-weights are well known. In fact, the first inequality below is proved on the upper half-space in [12, Theorem 2 of Chapter VI], and one may use a similar argument to obtain the same on the ball. On the other hand, since A_p-weights are precisely those ones with respect to which the standard Hardy-Littlewood maximal operator satisfies weighted inequalities, the second inequality below is a consequence of the fact that the nontangential maximal operator is dominated by the Hardy-Littlewood maximal operator (see [8, Theorem 3] or [1, Theorem 6.23]).

Lemma 3.1. *For $1 < p < \infty$ and $\omega \in A_p$, the inequalities*

$$||\mathcal{S}f||_{L^p(\omega)} \leq C||f||_{L^p(\omega)}, \qquad ||\mathcal{N}f||_{L^p(\omega)} \leq C||f||_{L^p(\omega)}$$

hold for functions $f \in h^p(\omega)$.

We also need relations between various balls. For $x \in B$, let $B(x)$ be the ball centered at x with radius $r(x)/4$. Note $r(x) \approx r(y)$ for $y \in B(x)$ or $x \in B(y)$.

Lemma 3.2. *Let $x \in B$ and $y \in B(x)$. Put $y = |y|\eta$ where $\eta \in \partial B$. Then the following hold.*
(1) $B(x) \subset \widehat{S}_{2r(y)}(\eta)$.
(2) $y \in \Gamma(\zeta)$ for any $\zeta \in S_{r(y)}(\eta)$.

Proof. For $y \in B(x)$, we have

$$|r(x) - r(y)| \leq |x - y| < \frac{r(x)}{4}$$

and thus $r(x) < 2r(y)$. It follows that, for $z \in B(x)$,

$$|\eta - z| \leq |\eta - y| + |y - z| < r(y) + \frac{r(x)}{2} < 2r(y).$$

This shows the first part of the lemma. Next, assume $\zeta \in S_{r(y)}(\eta)$. Then

$$|\zeta - y| \leq |\zeta - \eta| + |\eta - y| = |\zeta - \eta| + (1 - |y|) < 2r(y)$$

and therefore $y \in \Gamma(\zeta)$. □

Proof of (1) $\Longrightarrow$ (4). Assume (1) holds. Let $f \in h^p(\omega)$. First, note that we have by Proposition 2.1

$$|\partial^\alpha f(x)|^p \lesssim r(x)^{-pm+p-n} \int_{B(x)} |\nabla f|^p \, dV \qquad (x \in B). \tag{3.1}$$

Also, for any $y \in B$, we have by assumption and doubling property

$$\mu\left[\widehat{S}_{2r(y)}(\eta)\right] \lesssim \omega\left[S_{2r(y)}(\eta)\right] r(y)^{mp} \lesssim C_\omega\, \omega\left[S_{r(y)}(\eta)\right] r(y)^{mp} \tag{3.2}$$

where $y = |y|\eta$, $\eta \in \partial B$.

Now, integrate both sides of (3.1) against the measure $d\mu$, interchange the order of integrations using Lemma 3.2, and then apply (3.2). What we have is

$$\begin{aligned}\int_B |\partial^\alpha f(x)|^p\, d\mu(x) &\lesssim \int_B r(x)^{-mp+p-n} \int_{B(x)} |\nabla f(y)|^p\, dy\, d\mu(x)\\ &\lesssim \int_B |\nabla f(y)|^p r(y)^{-mp-n+p}\, \mu\left[\widehat{S}_{2r(y)}\left(\frac{y}{|y|}\right)\right]\, dy\\ &\lesssim C_\omega \int_B |\nabla f(y)|^p r(y)^{-n+p}\, \omega\left[S_{r(y)}\left(\frac{y}{|y|}\right)\right] dy.\end{aligned}$$

Here and elsewhere, $dy = dV(y)$. Thus, interchanging the order of integrations once more, we have

$$\begin{aligned}\int_B |\partial^\alpha f(x)|^p\, d\mu(x) &\lesssim \int_B |\nabla f(y)|^p r(y)^{-n+p} \int_{S_{r(y)}(y/|y|)} \omega(\zeta)\, d\sigma(\zeta)\, dy\\ &\lesssim \int_{\partial B}\int_{\Gamma(\zeta)} |\nabla f(y)|^p r(y)^{-n+p}\, dy\, \omega(\zeta)\, d\sigma(\zeta)\end{aligned} \tag{3.3}$$

where the second inequality holds by Lemma 3.2. Note that

$$\begin{aligned}\int_{\Gamma(\zeta)} |\nabla f|^p r^{-n+p}\, dV &\le \left(\sup_{y\in\Gamma(\zeta)} r(y)|\nabla f(y)|\right)^{p-2} \int_{\Gamma(\zeta)} |\nabla f|^2 r^{-n+2}\, dV\\ &\lesssim |\mathcal{N} f(\zeta)|^{p-2} |\mathcal{S} f(\zeta)|^2\end{aligned}$$

for all $\zeta \in \partial B$. The second inequality of the above holds by Proposition 2.1. Inserting the above into (3.3) and then applying Hölder's inequality, we finally have

$$\begin{aligned}\int_B |\partial^\alpha f|^p\, d\mu &\lesssim \int_{\partial B} |\mathcal{N} f|^{p-2} |\mathcal{S} f|^2 \omega\, d\sigma\\ &\lesssim \left(\int_{\partial B} |\mathcal{N} f|^p \omega\, d\sigma\right)^{1-2/p} \left(\int_{\partial B} |\mathcal{S} f|^p \omega\, d\sigma\right)^{2/p}\\ &\lesssim \int_{\partial B} |f|^p \omega\, d\sigma.\end{aligned}$$

Here, the last inequality follows from Lemma 3.1. This completes the proof. □

Proof of (1) $\Longrightarrow$ (2)+(3). Assume (1) holds. Then, for each nonnegative integer $k \le m$, we have

$$\mu\big[\widehat{S}_\delta(\zeta)\big] \lesssim \omega\big[S_\delta(\zeta)\big]\,\delta^{kp}$$

for all $\zeta \in \partial B$ and $\delta > 0$. Thus, since we already have (1) $\Longrightarrow$ (4), the above yields

$$\sum_{|\alpha|\le m} \int_B |\partial^\alpha f|^p \, d\mu \lesssim \int_{\partial B} |f|^p \, \omega \, d\sigma$$

for all $f \in h^p(\omega)$, which trivially implies (2) and (3). The proof is complete. $\square$

Before proceeding to the proofs of other implications, we first introduce some notations. Let φ be the Newtonian potential. i.e.,

$$\varphi(x) = \begin{cases} \log|x| & \text{for} \quad n = 2 \\ |x|^{-n+2} & \text{for} \quad n > 2. \end{cases} \tag{3.4}$$

By induction one may verify that, to each multi-index $\gamma \neq 0$, there corresponds a (harmonic) homogeneous polynomial g_γ of degree $|\gamma|$ such that

$$\partial^\gamma \varphi(x) = g_\gamma(x)|x|^{-(n+2|\gamma|-2)}. \tag{3.5}$$

Now, for $\delta > 0$ and $\zeta,\ \eta \in \partial B$, define $\varphi_{\delta,\zeta,\eta}$ by

$$\varphi_{\delta,\zeta,\eta}(x) = \varphi(x - \zeta - \delta\eta).$$

Note that we have by (3.5) and homogeneity of g_γ

$$|\partial^\gamma \varphi_{\delta,\zeta,\eta}(x)| \le \frac{||g_\gamma||_{L^\infty(\sigma)}}{|x - \zeta - \delta\eta|^{n+|\gamma|-2}}. \tag{3.6}$$

We will use these functions as test functions in the proofs of all other remaining implications. We first prove some properties of those functions. For simplicity we write $S_\delta(\zeta) = S_\delta$ and $\widehat{S}_\delta(\zeta) = \widehat{S}_\delta$. Recall that C_ω is the "doubling" constant of $\omega \in A_p$ introduced in (2.1). In what follows $\zeta \cdot \eta$ denotes the euclidean inner product on $\mathbf{R}^n$.

Lemma 3.3. *Let* $\omega \in A_p$ *and* γ *be a multi-index such that* $C_\omega \le 2^{|\gamma|}$. *Then, there exists a constant* $C_{p,\gamma}$ *such that*

$$\int_{\partial B} |\partial^\gamma \varphi_{\delta,\zeta,\eta}|^p \omega \, d\sigma \le C_{p,\gamma} \frac{\omega(S_\delta)}{(\delta\zeta\cdot\eta)^{p(n+|\gamma|-2)}}$$

for all $0 < \delta < 1$ *and* $\zeta,\ \eta \in \partial B$ *with* $\zeta \cdot \eta > 0$.

Proof. Assume $\delta < 1$ and $\zeta,\ \eta \in \partial B$ with $\zeta \cdot \eta > 0$. Let $N = |\gamma|$. Note $\sqrt{1 + 2\delta\zeta\cdot\eta} < |\zeta + \delta\eta| < 2$. Thus, for $\xi \in \partial B$, we have

$$|\xi - \zeta - \delta\eta| \ge |\zeta + \delta\eta| - 1 \gtrsim \delta\zeta\cdot\eta$$

and thus, by (3.6),

$$|\partial^\gamma \varphi_{\delta,\zeta,\eta}(\xi)| \lesssim \frac{||g_\gamma||_{L^\infty(\sigma)}}{(\delta\zeta\cdot\eta)^{n+N-2}}, \qquad \xi\in\partial\boldsymbol{B}. \tag{3.7}$$

For $\xi\notin S_{2^k\delta}$, $k\ge 1$, we have

$$|\xi-\zeta-\delta\eta|\ge|\xi-\zeta|-\delta\ge 2^k\delta-\delta\gtrsim 2^k\delta$$

and thus, by (3.6),

$$|\partial^\gamma \varphi_{\delta,\zeta,\eta}(\xi)| \lesssim \frac{||g_\gamma||_{L^\infty(\sigma)}}{(2^k\delta)^{n+N-2}}, \qquad \xi\in S_{2^{k+1}\delta}\setminus S_{2^k\delta} \tag{3.8}$$

for all $k\ge 1$. Also, since $C_\omega<2^N$, we have by doubling property

$$\omega(S_{2^{k+1}\delta})\le C_\omega^{k+1}\omega(S_\delta)\le 2^{N(k+1)}\omega(S_\delta)$$

for each $k\ge 0$. Thus, it follows from (3.7) and (3.8) that

$$\begin{aligned}\int_{\partial B}|\partial^\gamma\varphi_{\delta,\zeta,\eta}|^p\omega\,d\sigma &= \int_{S_{2\delta}}|\partial^\gamma\varphi_{\delta,\zeta,\eta}|^p\omega\,d\sigma+\sum_{k=1}^\infty\int_{S_{2^{k+1}\delta}\setminus S_{2^k\delta}}|\partial^\gamma\varphi_{\delta,\zeta,\eta}|^p\omega\,d\sigma\\ &\lesssim ||g_\gamma||_{L^\infty(\sigma)}\frac{2^N\omega(S_\delta)}{(\delta\zeta\cdot\eta)^{p(n+N-2)}}\sum_{k=0}^\infty 2^{-k[p(n+N-2)-N]}\\ &= C_{p,\gamma}\frac{\omega(S_\delta)}{(\delta\zeta\cdot\eta)^{p(n+N-2)}}\end{aligned}$$

as desired. The proof is complete. □

For $\xi\in\partial B$, let D_ξ denote the differentiation in the direction of ξ.

Lemma 3.4. *For each positive integer m there exists a constant $C_m\neq 0$ such that*

$$D_\xi^m\varphi(x)=\frac{C_m}{|x|^{n+m-2}}\left[1+O\left(1-\left|\xi\cdot\frac{x}{|x|}\right|\right)\right]$$

for all $\xi\in\partial B$ and $x\neq 0$ such that $\xi\cdot x<0$. The constant involved in $O\big(1-|\xi\cdot x/|x||\big)$ depends only on n and m.

Proof. Let m be a positive integer. Let $\xi\in\partial B$, $x\neq 0$ and assume $\xi\cdot x<0$. Put $\eta=x/|x|$. A simple calculation yields

$$D_\xi(|x|^{-k})=-k\frac{\xi\cdot x}{|x|^{k+2}}, \qquad D_\xi\left[(\xi\cdot x)^k\right]=k(\xi\cdot x)^{k-1}$$

for integers $k \geq 0$. Thus, by induction, one can show that there are coefficients $c_j = c_j(m)$ such that

$$
\begin{aligned}
D_\xi^m \varphi(x) &= \sum_{0 \leq j \leq m/2} c_j \frac{(\xi \cdot x)^{m-2j}}{|x|^{n-2+2m-2j}} \\
&= (-1)^m |x|^{-(n+m-2)} \sum_{0 \leq j \leq m/2} c_j |\xi \cdot \eta|^{m-2j}.
\end{aligned}
$$

Note $\sum c_j = D_\xi^m \varphi(\xi)$ which is a nonzero (by a direct calculation) constant depending only on n and m. Thus, letting $C_m = \sum c_i \neq 0$, we have

$$
\sum_{0 \leq j \leq m/2} c_j |\xi \cdot \eta|^{m-2j} = C_m \left[1 + O\left(1 - |\xi \cdot \eta|\right)\right]
$$

where the constant involved in $O(1 - |\xi \cdot \eta|)$ is easily seen to depend only on n and m. The proof is complete. $\square$

Proof of (2) $\implies$ (1). Assume (2) holds. First, fix a large positive integer N such that $C_\omega < 2^N$. Let $\zeta \in \partial B$. For $0 < \delta < 1$, put $f_\delta = \mathcal{D}^N \varphi_\delta$ where $\varphi_\delta = \varphi_{\delta,\zeta,\zeta}$. Then f_δ is harmonic on $\overline{B}$. We have by assumption and Lemma 3.3

$$
\int_B |\mathcal{D}^m f_\delta|^p \, d\mu \lesssim \int_{\partial B} |f_\delta|^p \omega \, d\sigma \lesssim \frac{\omega(S_\delta)}{\delta^{p(n+N-2)}}. \tag{3.9}
$$

Now, consider $y \in \widehat{S}_{\epsilon\delta}$ where $\epsilon < 1/2$ is a small positive number to be chosen in a moment. Note that

$$
D^m_{y/|y|} f_\delta(y) = |y|^{-m} \sum_{|\gamma|=m} \frac{m!}{\gamma!} y^\gamma \partial^\gamma f_\delta(y).
$$

Also, note

$$
\mathcal{D}^m f_\delta(y) = \sum_{|\gamma|=m} \frac{m!}{\gamma!} y^\gamma \partial^\gamma f_\delta(y) + E_m f_\delta(y)
$$

for some differential operator E_m of order $(m-1)$ with smooth coefficients. Therefore, we have

$$
\mathcal{D}^m f_\delta(y) = |y|^m D^m_{y/|y|} f_\delta(y) + E_m f_\delta(y). \tag{3.10}
$$

We first estimate the first term of the right side of (3.10). Put $z = y - \zeta - \delta\zeta$ and $\xi = y/|y| \in S_{\epsilon\delta}$. Then, $\delta(1-\epsilon) < |z| < \delta(1+\epsilon)$ and thus $|z| \approx \delta$. Therefore we have

$$
\xi \cdot z = (\xi - \zeta) \cdot (\xi - \delta\zeta) - \delta < \delta(2\epsilon - 1) < 0
$$

and

$$\left|\xi \cdot \frac{z}{|z|}\right| \gtrsim 1 - 2\epsilon.$$

Note $\mathcal{D}_\xi^k \varphi_\delta(y) = D_\xi^k \varphi(y - \zeta - \delta\zeta)$ for any integer $k \geq 1$. Hence, by Lemma 3.4, we have

$$D_\xi^m f_\delta(y) = D_\xi^{N+m} \varphi(y - \zeta - \delta\zeta) = \frac{C_1}{\delta^{n+N+m-2}}\big[1 + O(\epsilon)\big].$$

Recall that $C_1 \neq 0$ is a constant depending only on n and $m + N$ and the same is true for the constant involved in $O(\epsilon)$. Next, for the second term of the right side of (3.10), it is straightforward to see from (3.6) that

$$|E_m f_\delta(y)| \lesssim \delta^{-(n+N+m-3)}. \tag{3.11}$$

Since $|y| \approx 1$, combining these estimates, we have by (3.10)

$$|\mathcal{D}^m f_\delta(y)| \approx \frac{1 + O(\epsilon) + O(\delta)}{\delta^{n+N+m-2}}$$

so that we can fix ϵ and δ_0 sufficiently small such that

$$|\mathcal{D}^m f_\delta(y)| \approx \frac{1}{\delta^{n+N+m-2}}, \qquad y \in \widehat{S}_{\epsilon\delta}, \quad \delta \leq \delta_0, \tag{3.12}$$

which is a uniform estimate independent of ζ and δ. Thus, for $\delta \leq \delta_0$, we obtain from (3.12) and (3.9)

$$\frac{\mu(\widehat{S}_{\epsilon\delta})}{\delta^{p(n+N+m-2)}} \approx \int_{\widehat{S}_{\epsilon\delta}} |\mathcal{D}^m f_\delta|^p \, d\mu \lesssim \frac{\omega(S_\delta)}{\delta^{p(n+N-2)}}$$

so that

$$\mu(\widehat{S}_{\epsilon\delta}) \lesssim \delta^{mp}\omega(S_\delta) \lesssim \delta^{mp}\omega(S_{\epsilon\delta})$$

where the second inequality holds by doubling property. Consequently, for $\delta \leq \epsilon\delta_0$, we have

$$\mu(\widehat{S}_\delta) \lesssim \delta^{mp}\omega(S_\delta) \tag{3.13}$$

and this estimate is independent of ζ. Note that the above argument shows that $\mu(\widehat{S_{\epsilon\delta_0}}) < \infty$ for all $\zeta \in \partial B$. Since $d\mu$ is locally finite, it follows that $d\mu$ is a finite measure. Thus, for $\delta > \epsilon\delta_0$, we also have (3.13) by doubling property. This completes the proof. $\square$

Proof of (3) $\Longrightarrow$ (1). Assume (3) holds. As above, fix a large positive integer N such that $C_\omega < 2^N$. Let $\zeta \in \partial B$ and choose $\xi \in \partial B$ such that $\zeta \cdot \xi = 0$. Let $\eta = \epsilon\zeta + \sqrt{1-\epsilon^2}\,\xi$ where $\epsilon < 1/2$ is a small positive number to be chosen later. For $0 < \delta < 1$, this time we let $f_\delta = D_\xi^N \varphi_\delta$ where $\varphi_\delta = \varphi_{\delta,\zeta,\eta}$. Then f_δ is harmonic on $\overline{B}$. Since $\zeta \cdot \eta = \epsilon$, we have by assumption and Lemma 3.3

$$\int_B |\mathcal{D}^m f_\delta|^p \, d\mu \lesssim \int_{\partial B} |f_\delta|^p \omega \, d\sigma \lesssim \frac{\omega(S_\delta)}{(\epsilon\delta)^{p(n+N-2)}}. \tag{3.14}$$

Consider $y \in \widehat{S}_{\epsilon\delta}$ and put $z = y - \zeta - \delta\eta$. Then, $\delta(1-\epsilon) < |z| < \delta(1+\epsilon)$ and thus $|z| \approx \delta$. Since $\eta \cdot \xi = \sqrt{1-\epsilon^2} > 1 - \epsilon$, we have

$$\xi \cdot z = (y - \zeta)\cdot \xi - \delta\eta\cdot\xi < \delta(2\epsilon - 1) < 0$$

and

$$\left| \xi \cdot \frac{z}{|z|} \right| \gtrsim 1 - 2\epsilon.$$

Therefore, by Lemma 3.4, we have

$$D_\xi^m f_\delta(y) = D_\xi^{N+m} \varphi(y - \zeta - \delta\eta) = \frac{C_1}{\delta^{n+N+m-2}}\left[1 + O(\epsilon)\right] \tag{3.15}$$

where $C_1 \neq 0$ is a constant depending only on n and $m + N$ and the same is true for the constant involved in $O(\epsilon)$. Now, since $\zeta \cdot \xi = 0$, we can find coefficients $c_{ij} = c_{ij}(\zeta, \xi)$ such that

$$D_\xi f_\delta(\zeta) = T f_\delta(\zeta), \quad T = \sum c_{ij}\mathcal{T}_{ij}.$$

Moreover, as in the proof of Proposition 5.2 of [2], we may find those coefficients in such a way that $\sup_{\zeta,\xi} |c_{ij}(\zeta, \xi)| < \infty$. Let

$$a_j(x, \xi) = \sum_{i<j} c_{ij} x_i - \sum_{i>j} c_{ji} x_i$$

for each j. Put $a = (a_1, \ldots, a_n)$. Then, we have

$$\begin{aligned}
T^m f_\delta(y) &= \left(\sum a_j(y,\xi)\partial_j\right)^m f_\delta(y) \\
&= \sum_{|\beta|=m} \frac{m!}{\beta!} a^\beta(y,\xi)\partial^\beta f_\delta(y) + E_m f_\delta(y) \\
&= D_\xi^m f_\delta(y) + \sum_{|\beta|=m} \frac{m!}{\beta!}\left[a^\beta(y,\xi) - \xi^\beta\right]\partial^\beta f_\delta(y) + E_m f_\delta(y)
\end{aligned}$$

where E_m is a differential operator of order $(m-1)$ with smooth coefficients. Note $a(\zeta,\xi) = \xi$. Thus, since c_{ij}'s are uniformly bounded, we have

$$|a(y,\xi)-\xi| \le C_2|y-\zeta| < C_2\epsilon\delta, \qquad y \in \widehat{S}_{\epsilon\delta}$$

for some constant C_2 depending only on n. Also, by (3.6) we have

$$\sum_{|\beta|=m} |\partial^\beta f_\delta(y)| \le \frac{C_3}{\delta^{n+N+m-2}}, \qquad y \in \widehat{S}_{\epsilon\delta}$$

where C_3 is a constant depending only on n and $m+N$. Thus, for $y \in \widehat{S}_{\epsilon\delta}$, we have

$$T^m f_\delta(y) = D^m_\xi f_\delta(y) + \frac{O(\epsilon\delta)}{\delta^{n+N+m-2}} + E_m f_\delta(y).$$

Therefore, by (3.15) and (3.11), we can fix $\epsilon > 0$ and $\delta_0 > 0$ such that

$$|T^m f_\delta(y)| \approx \frac{1}{\delta^{n+N+m-2}}, \quad y \in \widehat{S}_{\epsilon\delta}, \quad \delta \le \delta_0. \tag{3.16}$$

Now, for the rest of the proof, one may proceed as in the proof of (2) $\Longrightarrow$ (1) by using (3.14) and (3.16). The proof is complete. $\square$

Proof of (4) $\Longrightarrow$ (1). Assume (4) holds. By compactness of ∂B, it suffices to give local estimates. So, fix $\eta \in \partial B$ and assume $\zeta \in \partial B$, $\zeta\cdot\eta > 1/2$. Since φ is harmonic and not a polynomial, we can choose $\beta = \beta(\eta)$ such that $\partial^\alpha\partial^\beta\varphi(-\eta) \ne 0$ and $C_\omega < 2^{|\beta|}$. Let $N = |\beta|$. For $0<\delta<1$, put $f_\delta = \partial^\beta\varphi_\delta$ where $\varphi_\delta = \varphi_{\delta,\zeta,\eta}$ $f_\delta = \partial^\beta\varphi_{\delta,\zeta,\eta}$. Then f_δ is harmonic on $\overline{B}$. Since $\zeta\cdot\eta > 1/2$, we have by assumption and Lemma 3.3

$$\int_B |\partial^\gamma f_\delta|^p\, d\mu \lesssim \int_{\partial B} |f_\delta|^p \omega\, d\sigma \le C_1 \frac{\omega(S_\delta)}{\delta^{p(n+N-2)}} \tag{3.17}$$

where $C_1 = C_1(p,\eta)$ is a constant independent of ζ and δ.

Consider $y \in \widehat{S}_{\epsilon\delta}$ where $\epsilon = \epsilon(\eta) < 1/2$ is a small positive constant to be chosen in a moment. Put $z = (y-\zeta)/\delta$ and $\gamma = \alpha+\beta$. Then, by (3.5), we have

$$\partial^\alpha f_\delta(y) = \partial^\gamma\varphi(\delta z - \delta\eta) = \delta^{-(n+N+m-2)} \frac{g_\gamma(z-\eta)}{|z-\eta|^{n+2N+2m-2}}. \tag{3.18}$$

Note $g_\gamma(z-\eta) = g_\gamma(-\eta) + O(|z|)$ and $|z| < \epsilon$. Also, $g_\gamma(-\eta) \ne 0$, because $\partial^\gamma\varphi(-\eta) \ne 0$. Thus,

$$g_\gamma(z-\eta) = g_\gamma(-\eta)[1+O(\epsilon)], \qquad y \in \widehat{S}_{\epsilon\delta}.$$

Here, the constant involved in $O(\epsilon)$ is independent of ζ and δ. Note $|z-\eta| \approx 1$. Thus, by (3.18), we may fix ϵ sufficiently small such that

$$|\partial^\alpha f_\delta(y)| \approx \frac{1}{\delta^{n+N+m-2}} \qquad y \in \widehat{S}_{\epsilon\delta}, \quad \delta < 1 \tag{3.19}$$

and this estimate is independent of ζ and δ.

Now, using (3.17), (3.19) and imitating the argument of (2) $\Longrightarrow$ (1), we obtain

$$\mu(\widehat{S}_\delta) \leq C_\eta \delta^{mp} \omega(S_\delta), \qquad y \in \widehat{S}_\delta, \quad \delta > 0$$

which is an estimate independent of ζ and δ. The proof is complete. □

4. The case $1 < p < q < \infty$

In this section we give a proof of the main theorem for the case $1 < p < q < \infty$. Except for the implication (1) $\Longrightarrow$ (4), the arguments of other implications of the previous section are easily modified and thus details are left to the readers. For the implication (1) $\Longrightarrow$ (4), we make use of the idea of [7].

Lemma 4.1. *For $a > 1$ and $\omega \in A_p$, let $d\tau_a$ be a measure on B defined by*

$$d\tau_a(y) = r(y)^{-n}\, \omega\left[S_{r(y)}\left(\frac{y}{|y|}\right)\right]^a dy.$$

Then, we have

$$\tau_a\left[\widehat{S}_\delta(\zeta)\right] \leq C\, \omega\left[S_\delta(\zeta)\right]^a \tag{4.1}$$

for all $\zeta \in \partial B$ and $\delta > 0$.

The analogue of the above lemma is proved in [7, Lemma 2.3] on the unit ball of the complex n-space. Their idea is to use reverse Hölder's inequality for A_p-weights. More precisely, to each $\omega \in A_p$ there corresponds a constant $b > 1$ such that

$$\left(\frac{1}{|S|}\int_S \omega^a\, d\sigma\right)^{1/a} \lesssim \frac{1}{|S|}\int_S \omega\, d\sigma \tag{4.2}$$

holds for all $S = S_\delta(\zeta)$ and $0 < a \leq b$.

Proof. Let $S_\delta(\zeta)$ be given. First, assume $1 < a \leq b$ where b is chosen so that (4.2) holds for ω. Then, by integrating in polar coordinates, we have

$$\tau_a[\widehat{S}_\delta(\zeta)] \lesssim \int_0^\delta t^{-n} \int_{S_\delta(\zeta)} \omega[S_t(\eta)]^a\, d\sigma(\eta)\, dt. \tag{4.3}$$

Note that, for $\eta \in S_\delta(\zeta)$ and $0 < t < \delta$, we have $S_t(\eta) \subset S_{2\delta}(\zeta)$. Thus, letting $|S_t| = |S_t(\xi)|$ for any $\xi \in \partial B$, we have by Hölder's inequality, reverse Hölder's inequality and doubling property

$$\begin{aligned}\int_{S_\delta(\zeta)} \omega[S_t(\eta)]^a \, d\sigma(\eta) &\leq |S_t|^{a-1} \int_{S_{2\delta}(\zeta)} \int_{S_t(\eta)} \omega(\xi)^a \, d\sigma(\xi) \, d\sigma(\eta) \\ &\leq |S_t|^a \int_{S_{2\delta}(\zeta)} \omega^a \, d\sigma \\ &\lesssim |S_t|^a |S_{2\delta}|^{1-a} \omega[S_{2\delta}(\zeta)]^a \\ &\lesssim |S_t|^a |S_\delta|^{1-a} \omega[S_\delta(\zeta)]^a.\end{aligned}$$

Now, inserting this estimate into (4.3), we obtain (4.1).

Next, assume $a > b$. Note that we have by doubling property

$$\frac{\omega\left[S_{r(y)}(y/|y|)\right]}{\omega\left[S_\delta(\zeta)\right]} \leq \frac{\omega\left[S_{2\delta}(\zeta)\right]}{\omega\left[S_\delta(\zeta)\right]} \lesssim 1, \qquad y \in \widehat{S}_\delta(\zeta)$$

so that

$$\begin{aligned}\omega\left[S_\delta(\zeta)\right]^{-a} \tau_a\left[\widehat{S}_\delta(\zeta)\right] &= \int_{\widehat{S}_\delta(\zeta)} \left(\frac{\omega\left[S_{r(y)}(y/|y|)\right]}{\omega\left[S_\delta(\zeta)\right]} \right)^a r(y)^{-n} \, dy \\ &\lesssim \omega\left[S_\delta(\zeta)\right]^{-b} \tau_b\left[\widehat{S}_\delta(\zeta)\right].\end{aligned}$$

Thus, (4.1) follows from the previous case. The proof is complete. □

The following is a special case of [3, Theorem 5.2]. In fact, Gu [3] worked on the half-space and a straightforward modification gives the same on the ball.

Lemma 4.2. *Let $1 < p \leq q < \infty$. Assume $\omega \in A_p$ and $d\tau$ is a locally finite positive Borel measure on B. Then,*

$$\tau\left[\widehat{S}_\delta(\zeta)\right] \leq C\, \omega\left[S_\delta(\zeta)\right]^{q/p} \qquad \textit{for all} \quad \zeta \in \partial B \quad \textit{and} \quad \delta > 0$$

if and only if

$$||f||_{L^q(\tau)} \leq C ||f||_{L^p(\omega)} \qquad \textit{for all} \quad f \in h^p(\omega).$$

Now, we give the proof of (1) $\Longrightarrow$ (4). Assume (1) holds. Let $f \in h^p(\omega)$. One may proceed as in the case $p = q \geq 2$ to obtain the following estimates:

$$\int_B |\partial^\alpha f(x)|^q \, d\mu(x) \lesssim \int_B r(x)^{-mq-n} \int_{B(x)} |f(y)|^q \, dy \, d\mu(x)$$

$$\lesssim \int_B |f(y)|^q r(y)^{-mq-n}\, \mu\left[\widehat{S}_{2r(y)}\left(\frac{y}{|y|}\right)\right] dy$$
$$\lesssim \int_B |f(y)|^q \, d\tau(y)$$

where $d\tau = d\tau_{q/p}$ is the measure defined as in Lemma 4.1. Therefore, we conclude (4) by Lemma 4.1 and Lemma 4.2.

5. Compactness

Recall that a linear operator Λ from a Banach space into another is called compact if Λ maps bounded sets onto relatively compact sets. In the following we let $\widehat{\mu}_\delta = \widehat{\mu}_{\delta,m,p,q}$ denote the function defined by

$$\widehat{\mu}_\delta(\zeta) = \frac{\mu\left[\widehat{S}_\delta(\zeta)\right]}{\omega\left[S_\delta(\zeta)\right]^{q/p}\delta^{mp}}, \qquad \zeta \in \partial B, \quad \delta > 0$$

and let

$$\mathcal{T}^m = \sum_{|\beta|=m} |\mathcal{T}^\beta|.$$

Theorem 5.1. *Assume* $2 \le p = q < \infty$ *or* $1 < p < q < \infty$. *Let* $\omega \in A_p$ *and* α *be a multi-index of order* $m \ge 1$. *Then, for a locally finite positive Borel measure* $d\mu$ *on* B, *the following are equivalent.*

(1) $\widehat{\mu}_\delta(\zeta) = o(1)$ *uniformly in* $\zeta \in \partial B$ *as* $\delta \to 0$.
(2) $\mathcal{D}^m : h^p(\omega) \to L^q(\mu)$ *is compact.*
(3) $\mathcal{T}^m : h^p(\omega) \to L^q(\mu)$ *is compact.*
(4) $\partial^\alpha : h^p(\omega) \to L^q(\mu)$ *is compact.*

Proof of (1) $\implies$ (4). Assume (1) holds. For $0 < t < 1$, let B_t be the ball centered at the origin with radius t and $\mathcal{X}_t$ be the characteristic function of B_t. Define $\Lambda_t : h^p(\omega) \to L^q(\mu)$ by $\Lambda_t f = \mathcal{X}_t \partial^\alpha f$. First, we show that each Λ_t is compact. Let U be a given bounded set in $h^p(\omega)$ and fix $f \in U$. Then, for each $x \in B$, we have

$$f(x) = \int_{\partial B} P(x,\zeta) f(\zeta)\, d\sigma(\zeta)$$

where P is the Poisson kernel for B. Hence, by Hölder's inequality, we have

$$|f(x)| \le \int_{\partial B} P(x,\zeta)|f(\zeta)|\, d\sigma(\zeta)$$
$$\le \frac{1}{(1-|x|)^{n-1}} \int_{\partial B} |f|\omega^{-1} \cdot \omega\, d\sigma$$

$$\lesssim \frac{||f||_{h^p(\omega)}||\omega^{-1}||_{L^{p'}(\omega)}}{(1-|x|)^{n-1}}$$

where p' is conjugate index of p. Note $\omega^{-1} \in L^{p'}(\omega)$, because $\omega \in A_p$. Thus U is locally uniformly bounded and thus is a normal family. Therefore, there is a sequence $\{f_j\}$ in U which converges uniformly on every compact subset of B. Let $f = \lim f_j$. Now, since $\partial^\alpha f_j \to \partial^\alpha f$ uniformly on B_t, we have $\Lambda_t f_j \to \mathcal{X}_t \partial^\alpha f$ in $L^q(\mu)$. Hence, $\Lambda_t : h^p(\omega) \to L^q(\mu)$ is compact.

For the case $p < q$, one may follow the argument (using Lemma 4.1) in the previous section to obtain

$$\int_{B\setminus B_t} |\partial^\alpha f|^q \, d\mu \lesssim \left(\sup_{\substack{\delta \lesssim 1-t \\ \zeta \in \partial B}} \widehat{\mu}_\delta(\zeta) \right) \left(\int_{\partial B} |f|^p \omega \, d\sigma \right)^{q/p}$$

for functions $f \in h^p(\omega)$. Also, one may follow the arguments of (3.1), (3.2) and (3.3) to see the same for the case $p = q \geq 2$. Hence, in either case, we have

$$||\Lambda_t - \partial^\alpha|| \lesssim \left(\sup_{\substack{\delta \lesssim 1-t \\ \zeta \in \partial B}} \widehat{\mu}_\delta(\zeta) \right)^{1/q} \to 0 \qquad \text{as} \quad t \to 1$$

so that ∂^α is compact, as desired. This completes the proof. $\square$

Now, the implication (1) $\Longrightarrow$ (2) + (3) easily follows from (1) $\Longrightarrow$ (4).

Proof of (2) $\Longrightarrow$ (1). Assume (2) holds. Let $\zeta \in \partial B$. We continue using the notations defined in the proof of (2) $\Longrightarrow$ (1) of Section 3. For $\delta > 0$, let

$$h_\delta = \delta^{n+N-2} \omega(S_\delta)^{-1/p} f_\delta.$$

Note $||h_\delta||_{h^p(\omega)} \lesssim 1$ by Lemma 3.3. First, we show that

$$\int_B |\mathcal{D}^m h_\delta|^q \, d\mu \to 0 \qquad \text{as} \quad \delta \to 0. \tag{5.1}$$

Suppose not. Then there exists a sequence $\delta_j \to 0$ such that

$$\inf_j \int_B |\mathcal{D}^m h_{\delta_j}|^q \, d\mu > 0$$

Since h_δ's are bounded in $h^p(\omega)$, by using the compactness of $\mathcal{D}^m$, we may assume $\mathcal{D}^m h_{\delta_j} \to h$ in $L^q(\mu)$ for some $h \in L^q(\mu)$. Note $||h||_{L^q(\mu)} > 0$. On the other hand, since f_δ's are locally uniformly bounded by (3.6) and

$$\omega(\partial B) \le 2^{N(\log_2 \delta^{-1}+2)}\omega(S_\delta) = 2^{2N}\delta^{-N}\omega(S_\delta)$$

by doubling property, we see that h_{δ_j} converges to 0 uniformly on every compact subset of B, and so is $\mathcal{D}^m h_{\delta_j}$. It follows that $h = 0$ in $L^q(\mu)$, which is a contradiction. Thus, (5.1) holds.

Now, we have by (3.12)

$$|\mathcal{D}^m h_\delta(y)|^q \approx \delta^{-mq}\omega(S_\delta)^{-q/p}, \qquad y \in \widehat{S}_{\epsilon\delta}, \quad \delta \le \delta_0$$

and thus

$$\mu(\widehat{S}_{\epsilon\delta}) \lesssim \delta^{mq}\omega(S_\delta)^{q/p}\int_B |\mathcal{D}^m h_\delta|^q \, d\mu$$

for all δ sufficiently small. Thus, we have by doubling property

$$\widehat{\mu}_{\epsilon\delta}(\zeta) \lesssim \left(\frac{\omega(S_\delta)}{\omega(S_{\epsilon\delta})}\right)^{q/p}\int_B |\mathcal{D}^m h_\delta|^q \, d\mu \lesssim \int_B |\mathcal{D}^m h_\delta|^q \, d\mu$$

and this is an estimate independent of ζ and δ small. Thus, we conclude (1) by (5.1). The proof is complete. $\square$

Proofs of the implications (3) $\implies$ (1) and (4) $\implies$ (1) are also easy modifications of corresponding ones in the previous section.

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