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# BOUND STATES OF PERTURBED HAMILTONIANS IN THE STRONG COUPLING LIMIT

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## 1. Introduction

Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ be a non negative potential; the now classical Cwikel-Lieb-Rozenblum (CLR) inequality asserts that the number $N(V)$ of negative eigenvalues for the Schrödinger operator $-\Delta - V$ verifies

$$N(V) \le C_d \int V(x)^{d/2} dx,$$

when $d > 2$. More generally Egorov [7] proved that if the Laplacian is replaced by an elliptic positive differential operator of order $2l$, the CLR inequality remains valid when $d > 2l$ with $d/2$ replaced by $d/(2l)$. Recently Rozenblum and Solomiak [13] obtained a general result for a wide class of perturbed hamiltonians $H_0 - V$.
When the RHS in the CLR inequality is finite we can define the number $N(\lambda V)$ of negative eigenvalues for $H_0 - \lambda V$. The principal term of the asymptotic for $N(\lambda V)$ when $\lambda \to +\infty$ was found independently by Birman and Borzov [3] and Martin [10] in the case $d \ge 3$:

$$N(\lambda V) \sim \lambda^{d/2} \int V(x)^{d/2} dx\ .$$

Tamura [15] obtained later on a sharp remainder estimate for $d = 3$:

$$N(\lambda V) = \lambda^{3/2} \int V(x)^{3/2} dx + O(\lambda), \ \text{ when } \lambda \to +\infty,$$

for very regular potential like $\langle x \rangle^{-m}$ with $m > 2$. More recently Birman and Solomiak [4] obtained asymptotics for $H_0 = (-\Delta)^l$ and $d > 2l$ (and also for $d < 2l$ but we do not consider this case here).

The aim of this paper is to obtain similar Weyl asymptotics when $H_0$ is a pseudodifferential operator. Global ellipticity is not required; roughly speaking, we suppose that the complete symbol of $H_0$ is hypoelliptic and satisfy a condition which looks like ellipticity near $\xi = 0$, see the complete assumptions in 2.1. A typical example is for instance the relativistic Laplacian $H_0 = (-\Delta + 1)^{1/2} - 1$. The class of potentials $V$ considered here is less restrictive than those considered in [15].

Here is the plan of our work: in Section 2 we specify the different assumptions on the unperturbed hamiltonian $H_0$ and on the potential $V$ and we state the main theorem. In Section 3, following [15], we use Birman-Schwinger principle: this leads to estimate the number of eigenvalues less than $\lambda$ for a selfadjoint operator $A(\kappa, \lambda)$ depending on the parameters $\kappa, \lambda$ with $0 < \kappa \leq 1$ and $\lambda \geq 1$. For exploiting this idea Tamura has showed that it is essential to provide adapted weight functions, in the terminology of Beals, and corresponding classes of symbols in which $A(\kappa, \lambda)$ admits a principal symbol satisfying uniform estimates with respect to $\kappa, \lambda$. In Section 4 we localize the spectral problem and decompose the phase space in three regions; each of these regions requires a different method. These methods are explicited respectively in Sections 5, 6, 7. In particular the results of [13] are used in Section 5. Sections 6 and 7 are an adaptation of [1] and the proofs are only sketched.

## 2. Assumptions and main results

**2.1.** Let $p(x, \xi) \in C^\infty(\mathbb{R}^{2d})$ be a non negative symbol satisfying the following assumptions: there is positive real numbers $l, m$, $1 \leq l \leq m$ such that for all $(x, \xi)$:

$$0 \leq p(x, \xi) \leq C|\xi|^m, \tag{1}$$

$$\xi.\nabla_\xi p(x, \xi) \geq p(x, \xi), \tag{2}$$

and for $\xi \neq 0$,

$$|\partial_\xi^\alpha D_x^\beta p(x, \xi)| \leq C_{\alpha,\beta}\, p(x, \xi)|\xi|^{-|\alpha|}\langle x\rangle^{-|\beta|}; \tag{3}$$

here we have used the standard notation $\langle x\rangle = (1 + |x|^2)^{1/2}$; when $|\beta| = 1$ we require more precisely that for some $\varepsilon > 0$:

$$|\nabla_x p(x, \xi)| \leq C p(x, \xi)\langle x\rangle^{-1-\varepsilon}; \tag{4}$$

we suppose that for $|\xi| \geq 1$

$$p(x, \xi) \geq C^{-1}|\xi|^l, \tag{5}$$

and for $|\xi| \leq 1$, we assume that

$$p(x, \xi) \geq C^{-1}|\xi|^m, \tag{6}$$

and furthermore that when $|\alpha| \leq m$,

$$|\partial_\xi^\alpha D_x^\beta p(x, \xi)| \leq C_{\alpha,\beta}|\xi|^{m-|\alpha|}\langle x\rangle^{-|\beta|}, \tag{7}$$

and when $|\alpha| \geq m$,

$$|\partial_\xi^\alpha D_x^\beta p(x, \xi)| \leq C_{\alpha,\beta}\langle x\rangle^{-|\beta|}; \tag{8}$$

lastly we suppose that the operator $p(x, D)$ with classical symbol $p(x, \xi)$ is symmetric positive on $\mathcal{S}$:

$$\forall u \in \mathcal{S}, (p(x, D)u, u) \geq 0\,, \tag{9}$$

where $(u, v)$ denotes $\int u\bar{v}dx$ as usual. With the assumptions (1) to (9), the operator $p(x, D)$ defined on $\mathcal{S}$ is essentially selfadjoint; let $H_0$ be its selfadjoint realization; then $H_0 \geq 0$. For each $t > 0$, $e^{-tH_0}$ is an integral operator with $C^\infty$ bounded kernel $Q(t; x, y)$. Let

$$M(t) = ||Q(t; x, x)||_{L^\infty(\mathbb{R}^d)},$$

then we suppose that

$$\forall t > 0,\ M(t) \leq C_1 t^{-d/l} + C_2 t^{-d/m}\ . \tag{10}$$

**2.2.** We make some comments: assumptions (1), (3) and (5) are the classical assumptions of hypoellipticity, but (3) must be verified for all $\xi \neq 0$, not only for $|\xi| \geq 1$ as usual; (6), (7), (8) specify the behaviour of $p(x, \xi)$ near $\xi = 0$ and are rather restrictive (ellipticity near $\xi = 0$); the case $m$ is different from $l$ occurs for instance for the relativistic Schrödinger operator (see 2.5); (4) is a technical assumption, see Lemma 7; lastly, the estimate (10) is well known for $t$ sufficiently small; but in our case, this estimate must be verified for all $t > 0$, especially near $t = +\infty$; using only the assumptions (1) to (9), we are able to prove a weaker estimate:

$$M(t) \leq (C_1 t^{-d/l} + C_2 t^{-d/m})e^{t\delta},$$

for any $\delta > 0$: this can be done by writing $e^{-tH_0}$ as a Cauchy's integral and then constructing a parametrix for $H_0 - z$ (see [14]) but this is not sufficient for us, as we shall see later.

**2.3.** Let now $V$ be a positive potential; we assume that $V \in C^\infty(\mathbb{R}^d)$, $V(x) > 0$, and there is $\rho$, $0 < \rho < 1/m$, such that for all $\beta$

$$|D^\beta V(x)| \leq C_\beta V(x)^{1+\rho|\beta|}; \tag{11}$$

for $|\beta| = 1$ we require more precisely

$$C^{-1} V(x)^{1+\rho} \leq |\nabla V(x)| \leq C V(x)^{1+\rho}; \tag{12}$$

let $\phi(t) = \text{meas}\{x \in \mathbb{R}^d; V(x) > t\}$ be the volume function associated with $V$; we suppose that there is $C > 0$, such that for all $t > 0$

$$\phi(t) \leq C t^{-d\rho}\ . \tag{13}$$

From this last assumption it results that

$$V \in L^{d/m}(\mathbb{R}^d) \text{ and } V \in L^{d/l}(\mathbb{R}^d) : \tag{14}$$

in fact

$$\int V(x)^{d/m} dx = \int t^{d/m-1} \phi(t) dt$$

and

$$t^{d/m-1} \phi(t) t^{-1+d(1/m-\rho)} .$$

Let us denote by $V$ the multiplication operator defined by the potential $V(x)$; $V$ is bounded and $H_0$-compact. Kato-Rellich's theorem implies that $H_0 - V$ is selfadjoint on $D(H_0)$. Since $\sigma_{\text{ess}}(H_0) \subseteq [0, \infty)$, the negative spectrum consists of eigenvalues of finite multiplicity, possibly accumulating to 0. More precisely, we suppose that $d > m$. Under the assumptions (1) to (14), the negative spectrum of $H_0 - V$ is finite and if $N(V)$ denotes the number of negative eigenvalues, there is $C_d > 0$ such that

$$N(V) \leq C_d \left( \int V(x)^{d/m} dx + \int V(x)^{d/l} dx \right) ;$$

this follows easily from Theorem 2 in [13].

**2.4.** Replacing $V$ by $\lambda V$ for $\lambda > 0$, this theorem asserts that the number $N(\lambda V)$ of negative eigenvalues for $H_0 - \lambda V$ is finite, depending on $\lambda$. Let $N_0(\lambda V)$ be the volume function

$$N_0(\lambda V) = (2\pi)^{-d} \iint_{p(x,\xi) < \lambda V(x)} dx d\xi .$$

We suppose that, as $\lambda \to +\infty$

$$N_0(\lambda V) = C\lambda^{d/l} + O(\lambda^{d/l-(1/m)}) . \tag{15}$$

We have the following Weyl asymptotic formula:

**Theorem 1.** *Assume that the assumptions* (1) *to* (15) *are verified*; *then as* $\lambda \to +\infty$:

$$N(\lambda V) = N_0(\lambda V)[1 + O(\lambda^{-1/m})] .$$

**2.5.** We precise here the particular case of $(-\Delta + 1)^{1/2} - 1$ and $d = 3$: the non-homogeneous symbol $p(\xi) = (1 + |\xi|^2)^{1/2} - 1$ verifies (1) to (9) with $l = 1$, $m = 2$ and also the assumption (15) is verified since

$$N_0(\lambda V) = C\lambda^3 \int V(x)^3 dx + O(\lambda^{3/2}) .$$

We obtain then asymptotics for the relativistic hamiltonian of a spinless particle of unit mass:

$$N(\lambda V) = N_0(\lambda V)[1 + O(\lambda^{-1/2})] \ \ (\lambda \to +\infty) .$$

**2.6.** For an homogeneous symmetric differential operator of order 2 with symbol

$$p_0(x, \xi) = \sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k,$$

satisfying the assumptions

$$\sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k \geq C^{-1}|\xi|^2, \tag{16}$$

$$|D^\beta g^{jk}(x)| \leq C_\beta\langle x\rangle^{-|\beta|}, |\nabla g^{jk}(x)| \leq C\langle x\rangle^{-1-\varepsilon}, \tag{17}$$

the symbol $p_0(x, \xi)$ satisfies (1) to (9) with $l = m = 2$, and (10) is verified, see for instance [5]. If we set

$$\mu_g(x) = (2\pi)^{-d}\text{vol} \ \left\{\xi \in \mathbb{R}^d;\ \sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k < 1\right\}$$

we obtain

$$N_0(\lambda V) = \lambda^{d/2} \int V(x)^{d/2}\mu_g(x)dx,$$

and we can apply Theorem 1. When $p(x, \xi) = |\xi|^2$, we obtain the asymptotics of [15] with remainder estimates.

**2.7.** We can give a semi-classical version of Theorem 1 in the case of an homogeneous differential operator: let $p(x, D)$ be an homogeneous differential operator of order $l = m$; let us denote by $N_{-(\infty,0)}(\hbar)$ the (finite) number of negative eigenvalues for the perturbed hamiltonian $H(x, \hbar D) = p(x, \hbar D) - V$; with the preceding notations of 2.4, it results from Theorem 1 that as $\hbar \to 0$:

$$N_{(-\infty,0)}(\hbar) = N(\hbar^{-m}V) = N_0(\hbar^{-m}V))[1 + O(\hbar)].$$

Setting $H(x,\xi) = p(x,\xi) - V(x)$, we obtain a semi-classical Weyl formula

$$N_{(-\infty,0)}(\hbar) = (2\hbar)^{-d}\text{meas }(H^{-1}(-\infty,0))[1+O(\hbar)]\ . \tag{18}$$

It should be noted that the technics of Hellfer and Robert [8] cannot be used here since $H^{-1}(-\infty,0))$ has finite volume in the phase space but is not bounded.

**2.8.** Denote by $e(\lambda;x,y)$ the integral kernel of the spectral projector $E_\lambda$ corresponding to $H_0$. We suppose that there is $C>0$ such that

$$\forall\lambda\geq 0,\ e(\lambda;x,x)\leq C(\lambda^{d/m}+\lambda^{d/l}); \tag{19}$$

then the operator $H_0$ verifies the assumption (10): this follows immediately from the relation

$$Q(t;x,x) = \int_0^{+\infty} e^{-t\lambda}de(\lambda;x,x).$$

Estimations about $e(\lambda,;x,x)$ as above are well known for large $\lambda$; but here it must be assumed for all $\lambda\geq 0$, and this is more restrictive.

## 3. Birman-Schwinger technics

**3.1.** Let $V$ be a potential satisfying the assumptions stated in 2.3. We suppose furthermore that $V(x)\leq 1$; this assumption is not essential but will simplify some technical proofs. We will see that by means of a power of $V$, it is possible to define a convenient pair of weight functions. Let $q(x) = V(x)^\beta$ with $0<\beta<\rho$; then there is $C>0$ such that

$$|\nabla q(x)| \leq C q(x)^{1+\delta}, \tag{20}$$

with $\delta = \beta^{-1}\rho > 1$; of course: $0<q(x)\leq 1$. We now define

$$\phi(x,\xi) = (|\xi|^2 + q(x)^2)^{1/2},\ \varphi(x) = q(x)^{-\delta} = V(x)^{-\rho}\ . \tag{21}$$

Following the terminology of [9], we have to verify that the Riemannian metric $g$ defined on $\mathbb{R}^{2d}$ by $(\phi,\varphi)$:

$$g_{x,\xi}(z,\zeta) = \frac{|z|^2}{\varphi(x)^2} + \frac{|\zeta|^2}{\phi(x,\xi)^2},$$

will satisfy the conditions required for a global pseudodifferential calculus. First we note that

$$\phi(x,\xi)^{-1}\varphi(x)^{-1} \leq \langle\xi\rangle^{-1}V(x)^{\beta(\delta-1)}$$

therefore by (14), there is $N_1 > 0$ such that

$$(\phi\varphi)^{-N_1} \in L^1(\mathbb{R}^{2d}) . \tag{22}$$

Now it is easy to verify that $|\partial\phi/\partial\xi_j| \leq 1$, $|\partial\phi/\partial x_j| \leq C\phi\varphi^{-1}$, and $|\partial\varphi/\partial x_j| \leq C$, which proves that $g$ is slowly varying. In order to prove that $g$ is $\sigma$ $g$-temperate, we need the auxiliary result

**Lemma 1.** *There is $C > 0$ and $N > 0$ such that for all $x, y \in \mathbb{R}^d$:*

$$\frac{q(x)}{q(y)} + \frac{q(y)}{q(x)} \leq C(1 + q(y)|x - y|)^N .$$

Proof. As in [12], let $\varphi(t)$ be defined by

$$\varphi(t) = V\left(x + \frac{t(y-x)}{|y-x|}\right), \text{ for } 0 \leq t \leq |y - x| ;$$

by (12), there is $\gamma > 0$ such that $\varphi'(t) \geq -\gamma\varphi(t)^{1+\rho}$; therefore

$$\varphi(t)^{-\rho} \leq \varphi(0)^{-\rho} + \frac{\gamma}{\rho}t ;$$

in particular for $t = |x - y|$:

$$V(y)^{-\rho} \leq V(x)^{-\rho}\left[1 + \frac{\gamma}{\rho}|x - y|V(x)^\rho\right] ;$$

but, since $0 < \beta < \rho$ and $0 < V(x) \leq 1$, we remark that $V(x)^\rho \leq q(x)$, and consequently

$$V(y)^{-1} \leq CV(x)^{-1}(1 + |x - y|q(x))^{1/\rho},$$

and finally

$$q(x) = V(x)^\beta \leq Cq(y)(1 + |x - y|q(x))^{\beta/\rho},$$

that is, by permutting $x$ and $y$:

$$\frac{q(y)}{q(x)} \leq C(1 + |x - y|q(y))^{\beta/\rho} .$$

Since the metric $g$ is slowly varying, there exists $\varepsilon > 0$ and $C > 1$ such that $|x - y| < \varepsilon q(y)^{-\delta}$ implies $C^{-1}q(y) \leq q(x) \leq Cq(y)$, which proves that

$$\frac{q(y)}{q(x)} \leq C(1 + q(y)|x - y|)$$

in this case. And when $|x-y| \geq \varepsilon q(y)^{-\delta}$, we remark that, since $\delta > 1$, there is $N > 0$ such that $(\delta - 1)N \geq 1$, and then

$$q(y)^{-1} \leq (q(y)^{-\delta+1})^N$$

and therefore

$$q(y)^{-1} \leq (1 + \varepsilon^{-1}|x - y|q(y))^N,$$

and also

$$\frac{q(x)}{q(y)} \leq C(1 + |x - y|q(y))^N,$$

and this ends the proof of the lemma. □

Let now $g^\sigma$ be the dual metric associated with $g$:

$$g^\sigma_{x,\xi}(z, \zeta) = (|\xi|^2 + q(x)^2)|z|^2 + q(x)^{-2\delta}|\zeta|^2 \ ;$$

**Lemma 2.** *The metric $g$ is $\sigma$ $g$-temperate: there is $C > 0$ and $N > 0$ such that*

$$g^\sigma_{x,\xi}(z, \zeta) \leq C g^\sigma_{y,\eta}(z, \zeta)[1 + g^\sigma_{y,\eta}(x - y, \xi - \eta)]^N \ .$$

Proof. From Lemma 1 it results that

$$\frac{\varphi(x)^2}{\varphi(y)^2} \leq C(1 + q(y)^2|x - y|^2)^{\delta N},$$

and this implies

$$\varphi(x)^2 \leq C\varphi(y)^2[1 + g^\sigma_{y,\eta}(x - y, \xi - \eta)]^{\delta N} \ . \tag{23}$$

Similarly it follows from Lemma 1 that

$$q(x)^2 \leq Cq(y)^2(1 + q(y)^2|x - y|^2)^N,$$

therefore

$$\phi(x, \xi)^2 \leq C\phi(y, \xi)^2[1 + g^\sigma_{y,\eta}(x - y, \xi - \eta))^N \ . \tag{24}$$

Now, if we set $\xi = \xi' q(y)$ and $\eta = \eta' q(y)$:

$$|\xi|^2 + q(y)^2 = q(y)^2\langle\xi'\rangle^2 \leq 2q(y)^2\langle\eta'\rangle^2\langle\xi' - \eta'\rangle^2,$$

but $1 + |\xi' - \eta'|^2 = 1 + q(y)^{-2}|\xi - \eta|^2$ and, since $\delta > 1$, $q(y)^2 \leq \varphi(y)^2$; consequently

$$\phi(y,\xi)^2 \leq C\phi(y,\eta)^2[1 + g^{\sigma}_{y,\eta}(x - y, \xi - \eta)]^N \ . \tag{25}$$

Lemma 2 follows from (23), (24) and (25). □

**Lemma 3.** *There is $C > 0$ such that for all $x \in \mathbb{R}^d$:*

$$\langle x\rangle^{-1} \leq C\varphi(x)^{-1} \ .$$

Proof. Let $\omega \in S^{n-1}$ and $t \geq 0$; for $x = t\omega$, we set $\varphi(t) = V(x)$; since $\varphi'(t) = \omega.\nabla V(t\omega)$, the assumption (12) on V implies

$$-\varphi'(t) \leq C\varphi(t)^{1+\rho},$$

which in turn leads to

$$\varphi(t)^{-\rho} \leq \varphi(0)^{-\rho} + C\delta t,$$

or equivalently

$$V(x)^{-\rho} \leq V(0)^{-\rho} + C\delta|x| \leq C\langle x\rangle \ ,$$

and therefore

$$\langle x\rangle^{-1} \leq CV(x)^{\rho} = C\varphi(x)^{-1} \ . \qquad \square$$

**3.2.** Let $p(x,\xi), V(x), q(x)$ be defined as in 3.1, and $q(x,\lambda) \in C^{\infty}(\mathbb{R}^d)$ such that

$$|D^{\beta}q(x,\lambda)| \leq C_{\beta}q(x)^{m+\delta|\beta|}, \quad 0 \leq q(x,\lambda) \leq q(x)^m, \tag{26}$$

$$q(x,\lambda) = 0 \text{ for } \lambda V(x) < q(x)^m/2, \tag{27}$$

$$q(x,\lambda) = q(x)^m \text{ for } \lambda V(x) > q(x)^m, \tag{28}$$

and $V(x,\lambda)$ defined by

$$\lambda V(x) + q(x,\lambda) = \lambda V(x,\lambda) \tag{29}$$

which verifies in particular

$$V(x,\lambda) \in C^{\infty}(\mathbb{R}^d), \ 0 < V(x) \leq V(x,\lambda) \leq CV(x) \ . \tag{30}$$

We can then rewrite $H_0 - \lambda V$ as $H_0 + q(x,\lambda) - \lambda V(x,\lambda)$. For $\kappa > 0$ let $N_{\kappa}(\lambda V)$ be the number of eigenvalues less than $-\kappa$ for the operator $H_0 - \lambda V$. By the Birman-Schwinger principle, $N_{\kappa}(\lambda V)$ is equal to the number of eigenvalues less than $\lambda$ for

$$A(\kappa,\lambda) = V(x,\lambda)^{-1/2}(H_0 + \kappa + q(x,\lambda))V(x,\lambda)^{-1/2} \ .$$

This operator is selfadjoint positive with compact resolvent. Let $\mu_j = \mu_j(\kappa, \lambda)$ and $u_j = u_j(\kappa, \lambda) \in \mathcal{S}$ be respectively the eigenvalues and eigenfunctions of $A(\kappa, \lambda)$, and let us denote by $a_0$ the symbol

$$a_0 = a_0(x, \xi; \kappa, \lambda) = V(x, \lambda)^{-1}(p(x, \xi) + \kappa + q(x, \lambda)) . \tag{31}$$

The accuracy of the weights $\phi, \varphi$ is proved by the following lemma; its conclusion is no longer true for classical weight $\langle \xi \rangle$:

**Lemma 4.** *For all multiindices $\alpha$ and $\beta$, there is $C_{\alpha\beta}$ independent of $\kappa$, such that*

$$|\partial_\xi^\alpha D_x^\beta p(x, \xi)| \le C_{\alpha\beta}(p(x, \xi) + \kappa + q(x)^m)\phi(x, \xi)^{-|\alpha|}\varphi(x)^{-|\beta|} .$$

Proof. For $|\xi| \ge q(x)$, then $\phi(x, \xi) \approx |\xi|$ and the inequality follows from (3) and Lemma 3. And if $|\xi| \le q(x)$, then in particular $|\xi| \le 1$; therefore for $|\alpha| \le m$, (6), (7) and Lemma 3 imply

$$\begin{aligned} |\partial_\xi^\alpha D_x^\beta p(x, \xi)| &\le C_{\alpha\beta}(|\xi| + q(x))^{m-|\alpha|}\varphi(x)^{-|\beta|} \\ &\le C_{\alpha\beta}(|\xi|^m + q(x)^m)\phi(x, \xi)^{-|\alpha|}\varphi(x)^{-|\beta|} \\ &\le C_{\alpha\beta}(p(x, \xi) + \kappa + q(x)^m)\phi(x, \xi)^{-|\alpha|}\varphi(x)^{-|\beta|}; \end{aligned}$$

now for $|\alpha| \ge m$, then by (8) and Lemma 3:

$$|\partial_\xi^\alpha D_x^\beta p(x, \xi)| \le C_{\alpha\beta}\varphi(x)^{-|\beta|} C_{\alpha\beta} q(x)^{m-|\alpha|}\varphi(x)^{-|\beta|};$$

but when $|\xi| \le q(x)$, then $\phi(x, \xi) \approx q(x)$, and therefore

$$\begin{aligned} |\partial_\xi^\alpha D_x^\beta p(x, \xi)| &\le C_{\alpha\beta} q(x)^m \phi(x, \xi)^{-|\alpha|}\varphi(x)^{-|\beta|}, \\ &\le C_{\alpha\beta}(p(x, \xi) + \kappa + q(x)^m)\phi(x, \xi)^{-|\alpha|}\varphi(x)^{-|\beta|} . \end{aligned}$$ □

**Lemma 5.** *For all multiindices $\alpha$ and $\beta$, there is $C_{\alpha\beta}$ such that for $(x, \xi) \in \mathbb{R}^{2d}$ such that $\lambda V(x) > q(x)^m$:*

$$|\partial_\xi^\alpha D_x^\beta a_0(x, \xi; \kappa, \lambda)| \le C_{\alpha\beta} a_0(x, \xi; \kappa, \lambda)\phi(x, \xi)^{-|\alpha|}\varphi(x)^{-|\beta|} .$$

Proof. Setting $q_0 = q_0(x, \xi; \kappa, \lambda) = p(x, \xi) + \kappa + q(x, \lambda)$, we first prove that

$$|\partial_\xi^\alpha D_x^\beta q_0| \le C_{\alpha\beta} q_0 \phi^{-|\alpha|}\varphi^{-|\beta|} . \tag{32}$$

For $\alpha$ different from 0: $\partial_\xi^\alpha D_x^\beta q_0 = \partial_\xi^\alpha D_x^\beta p$; therefore, by Lemma 4 and (27), the inequality (32) follows immediately. Now for $\alpha = 0$, by (27):

$$|D_x^\beta q_0(x, \xi; \kappa, \lambda)| \leq C_1 q_0(x, \xi; \kappa, \lambda)\varphi(x)^{-|\beta|} + C_2 q(x)^m \varphi(x)^{-|\beta|} \ .$$

Finally, since $V(x, \lambda) = V(x) + \lambda^{-1} q(x)^m$, the potential $V(x, \lambda)$ verifies

$$|D_x^\beta V(x, \lambda)| \leq C_\beta V(x, \lambda)\varphi(x)^{-|\beta|},$$

which implies (32) by use of Leibniz formula. □

**Lemma 6.** *The symbol $a_0$ verifies $(\phi\varphi)^l \leq a_0 \leq (\phi\varphi)^{1/(\rho-\beta)}$.*

Proof. Since $V(x, \lambda) \geq V(x) > 0$, assumption (1) on $p(x, \xi)$ implies

$$0 < a_0(x, \xi; \kappa, \lambda) \leq C V(x)^{-1} \langle \xi \rangle^m = C q(x)^{-1/\beta} \langle \xi \rangle^m$$

On the other hand:

$$\phi(x, \xi)\varphi(x) \geq \langle \xi \rangle q(x)^{1-\delta},$$

but $\delta - 1 = (\rho - \beta)/\beta$, therefore, since $0 < \beta < \rho < 1/m$,

$$\phi(x, \xi)\varphi(x) \geq ([q(x)^{-1}]^{1/\beta})^{\rho-\beta} (\langle \xi \rangle^m)^{\rho-\beta}$$

or equivalently

$$\phi\varphi \geq C a_0^{\rho-\beta} \ ,$$

and this proves the right hand side of the lemma. For the left hand side, it suffices to remark that by (5) and (6):

$$a_0(x, \xi; \kappa, \lambda) \geq \varphi(x)^{1/\rho} \phi(x, \xi)^m \ \text{for} \ |\xi| \leq 1,$$

and similarly

$$a_0(x, \xi; \kappa, \lambda) \geq \varphi(x)^{1/\rho} \phi(x, \xi)^l \ \text{for} \ |\xi| \geq 1 \ .$$

Since $\varphi(x) \geq 1$ and $1/\rho > m \geq l$, these inequalities imply

$$a_0(x, \xi; \kappa, \lambda) \geq (\phi(x, \xi)\varphi(x))^l$$

in all the cases. □

**3.3.** The next lemma will be essential in Section 7.1 to solve locally the characteristic equation relative to the symbol $a_0$.

**Lemma 7.** *There exist positive constants $C_1$ and $C_2$ such that when $\lambda V(x) > q(x)^m$ and $a_0 > C_1$*

$$\phi(x,\xi)|\nabla_\xi a_0| + \varphi(x)|\nabla_x a_0| \geq C_2 a_0 \ .$$

Proof. Recall that by definition, when $\lambda V(x) > q(x)^m$

$$a_0(x,\xi;\kappa,\lambda) = V(x)^{-1}(p(x,\xi) + \kappa + q(x)^m),$$

such that by (2)

$$\phi(x,\xi)|\nabla_\xi a_0| \geq \nabla_\xi a_0.\xi \geq V(x)^{-1} p(x,\xi) \ . \tag{33}$$

Now

$$\nabla_x a_0 = \nabla(V^{-1})(p+\kappa) + V^{-1}\nabla_x p + \nabla(V^{\beta-1}),$$

which can be rewritten

$$\nabla_x a_0 = -(V^{-1}\nabla V)[V^{-1}(p+\kappa) + (1-\beta)V^{\beta-1}] + V^{-1}\nabla_x p;$$

we remark now that by (12)

$$\varphi\ .(V^{-1}|\nabla V|)[V^{-1}(p+\kappa) + (1-\beta)V^{\beta-1}] \geq (1-\beta)|V^{-1}\nabla V|a_0\varphi \geq (C_3)a_0; \tag{34}$$

on the other hand, by (4)

$$\varphi(x)|V(x)^{-1}\nabla_x p(x,\xi)| \leq C_4 V(x)^{-1} p(x,\xi)\langle x\rangle^{-\varepsilon} \ . \tag{35}$$

Then, by (33), (34) and (35)

$$\phi(x,\xi)|\nabla_\xi a_0| + \varphi(x)|\nabla_x a_0| \geq V(x)^{-1} p(x,\xi)(1 - C_4\langle x\rangle^{-\varepsilon}) + C_3 a_0(x,\xi;\kappa,\lambda);$$

but there is $R > 0$ such that $|x| \geq R > 0$ implies $1 - C_4\langle x\rangle^{-\varepsilon} \geq 0$, which proves the lemma in this case. Now, when $|x| \leq R$, $V(x)^{-1} \leq C_5$ and for $a_0 > 3C_5$

$$2C_5 < a_0(x,\xi;\kappa,\lambda) \leq C_5 p(x,\xi) + 2C_5$$

therefore

$$p(x,\xi) \geq C_5 \geq V(x)^{-1} \geq q(x)^m$$

and by (33)

$$\phi(x,\xi)|\nabla_\xi a_0| \geq V(x)^{-1}(p(x,\xi) + \kappa + q(x)^m) = a_0(x,\xi;\kappa,\lambda),$$

and the lemma is proved. □

## 4. Localization in the phase space

In this section we prove that a direct study of $N_\kappa(\lambda V)$ can be replaced by estimations on localized counting functions (see the precise definition below). This idea is not new, see [15], [11], [1], but the presence of various parameters leads to be very careful.

**4.1.** Let $\theta$ be a positive real number, which will be fixed later on, and $V = V(x)$ be a potential as in Section 4. We define a covering of the phase space by

$$\begin{aligned} U &= \{(x,\xi); \lambda^\theta V(x) < 2\}, \qquad V = \{(x,\xi); \lambda^\theta V(x) > 1\} \\ V_1 &= \left\{(x,\xi); \lambda^\theta V(x) > 1 \text{ and } a_0(x,\xi;\kappa,\lambda) \notin \left(\frac{\lambda}{2}, 2\lambda\right)\right\}, \\ V_2 &= \left\{(x,\xi); \lambda^\theta V(x) > 1 \text{ and } a_0(x,\xi;\kappa,\lambda) \in \left(\frac{\lambda}{4}, 4\lambda\right)\right\}. \end{aligned}$$

**Lemma 8.** *There is $\omega = \omega(x;\lambda), \theta = \theta(x;\lambda) \in S(1;\phi,\varphi)$, uniformly with respect to $\kappa$ such that* $\operatorname{Supp}\omega \subseteq U, \operatorname{Supp}\theta \subseteq V$, *and* $\omega(x;\lambda) = 1$ *on* $\{(x,\xi); \lambda^\theta V(x) \leq 1\}$ *and such that* $\omega^2 + \theta^2 = 1$.

Proof. Let $\chi \in C_0^\infty((-1,2)), \chi(t) = 1$ when $0 \leq t \leq 1$ and $0 \leq \chi(t) \leq 1$; we define

$$\tilde\omega = \tilde\omega(x;\lambda) = [\chi(\lambda^\theta V(x))]^2;$$

$\tilde\omega \in S(1;\phi,\varphi)$, uniformly with respect to $\lambda$. Now let

$$\psi = \tilde\omega^2 + (1-\tilde\omega)^2 :$$

then: $\psi \in S(1;\phi,\varphi)$ and $\psi \geq 1/2$. Therefore, if we set

$$\theta = (1-\tilde\omega)\psi^{-1/2}, \omega = \tilde\omega\psi^{-1/2},$$

the pair $(\omega,\theta)$ will satisfy the conditions required in the lemma. □

**4.2.** We consider now a partition of unity as in [9]: let $g$ be the Riemannian metric associated with the weights $\phi,\varphi$ defined in Section 4; for $v = (x,\xi) \in \mathbb{R}^{2d}$ and $\varepsilon > 0$, we set

$$U_\varepsilon(v) = \{w = (z,\zeta);\ g_v(w-v) < \varepsilon\}.$$

There is $(v_k)$ et $(\chi_k)$ satisfying:
(i) $(\chi_k)$ is bounded in $S(1;\phi,\varphi)$;
(ii) $0 \leq \chi_k \leq 1$ and $\operatorname{Supp}\chi_k \subseteq U_\varepsilon(v_k)$;

(iii) there is $N > 0$ such that each point in $\mathbb{R}^{2d}$ lies in at most $N$ balls $U_k$;
(iv) $\sum_k \chi_k = 1$.
From now on we denote by $U_k$ the ball $U_\varepsilon(V_k)$. Let $J$ be a subset of $\mathbf{N}$, finite or infinite, and

$$\chi_J = \sum_{k\in J} \chi_k \ .$$

By (iii), $\chi_J$ is well defined and $(\chi_J)$ is bounded in $S(1;\phi,\varphi)$. Let $v \in \mathbb{R}^{2d}$: there is $k$ such that $\chi_k(v) \geq 1/N$: otherwise (iii) implies that $\sum_k \chi_k(v) < 1$. Let us define $\psi_J$ by

$$\psi_J = \sum_{k\in J} \chi_k^2 + \left(\sum_{k\notin J} \chi_k\right)^2 \ ;$$

the family $(\psi_J)$ is bounded in $S(1;\phi,\varphi)$ and $\psi_J N^{-2}$. Finally we set $\mathcal{J} = J \cup \{\infty\}$ and

$$\varphi_{k,J} = \psi_J^{-1/2} \chi_k \text{ for } k \in J \text{ and } \varphi_{\infty,J} = \psi_J^{-1/2} \sum_{k\notin J} \chi_k \ .$$

The family $(\varphi_{k,J})$ for $J \subseteq \mathbf{N}$ and $k \in \mathcal{J}$ is bounded in $S(1;\phi,\varphi)$ and

$$\sum_{k\in\mathcal{J}} \varphi_k^2 = 1 \ .$$

From this partition of unity, we construct a (reduced) pseudodifferential partition of unity. For this, we need an auxiliary and well known result, the proof of which we omit:

**Lemma 9.** *Let $a \in S(1;\phi,\varphi)$ be a symbol such that $a \geq 1$. For each $N \in \mathbf{N}$ there is $b \in S(1;\phi,\varphi)$ and $t \in S((\phi\varphi)^{-N};\phi,\varphi)$ such that*

$$Op^W a = [Op^W b]^2 + Op^W t.$$

Now we can state

**Lemma 10.** *For all $N \in \mathbf{N}$, there is $(\omega_k), k \in \mathcal{J}$ and $\rho$, $\omega_k \in S(1;\phi,\varphi)$ and $\rho \in S((\phi\varphi)^{-N};\phi,\varphi)$ such that*

$$\sum_{k\in\mathcal{J}} [Op^W \omega_k]^2 = I + Op^W \rho;$$

*furthermore the symbols $\omega_k$ and $\rho$ are in bounded sets of $S(1;\phi,\varphi)$ and $S((\phi\varphi)^{-N};\phi,\varphi)$ respectively.*

**4.3.** We apply these general results to our particular case; let us set

$$J = J(\kappa, \lambda) = \left\{ k \in \mathbf{N}; \frac{\lambda}{4} < a_0(v_k; \kappa, \lambda) < \lambda \right\} .$$

We can reformulate Lemma 10:

**Lemma 11.** *Let $\omega = \omega(x; \lambda)$ be the symbol defined in* Lemma 8; *there exists symbols $\omega_k(\kappa, \lambda)$ and $\rho(\kappa, \lambda)$ respectively in bounded sets of $S(1; \phi, \varphi)$ and $S((\phi\varphi)^{-N}; \phi, \varphi)$ such that*

$$[Op^W \omega]^2 + \sum_{k \in \mathcal{J}} [Op^W \omega_k]^2 = I + Op^W \rho$$

*and*:

$$\operatorname{Supp} \omega \subseteq U, \operatorname{Supp} \omega_k \subseteq U_k \subseteq V_2 \text{ for } k \in J , \operatorname{Supp} \omega_\infty \subseteq V_1 .$$

In Section 3 we considered the operator

$$A(\kappa, \lambda) = V(x, \lambda)^{-1/2}(H_0 + \kappa + q(x, \lambda))V(x, \lambda)^{-1/2} ;$$

let $\mu_j = \mu_j(\kappa, \lambda)$ and $u_j = u_j(\kappa, \lambda) \in \mathcal{S}$ be respectively the eigenvalues and eigenfunctions of $A(\kappa, \lambda)$; for a bounded symbol $\omega$, which may depends on various parameters, we define

$$N(\mu, \omega) = \sum_{\mu_j < \mu} ||Op^W \omega . u_j||^2 . \tag{36}$$

For further reference, we remark that when $Op^W \omega$ is an Hilbert-Schmidt operator

$$N(\mu, \omega) \leq ||Op^W \omega||^2_{HS} . \tag{37}$$

Let us recall the definition of $N_\kappa(\lambda V)$ stated at the beginning of the Section 3.2; from the preceding remark and from Lemma 11 it follows that (see [11]):

**Proposition 1.** *With the notations of* Lemma 10, *there is $C > 0$, such that*

$$\left| N_\kappa(\lambda V) - N(\lambda, \omega) - \sum_{k \in \mathcal{J}} N(\lambda, \omega_k) \right| C .$$

## 5. Bounds for $N(\lambda, \omega)$ with CLR inequality

### 5.1. Preliminary estimates.

**Lemma 12.** *Let* $\rho \in S((\phi\varphi)^{-1}; \phi, \varphi)$ *uniformly with respect to* $(\kappa, \lambda)$; *then*

$$N(\lambda, \rho) \leq C\lambda^{d/l-(1/l)} .$$

Proof. For $k \in \mathbf{N}$ sufficiently large

$$\mathrm{Tr}[\rho(A(\lambda, \kappa)^k + \lambda^k)^{-1}\rho] = \int_0^\infty (\zeta^k + \lambda^k)^{-1} dN(\zeta, \rho) .$$

Since

$$N(\lambda, \rho) \leq \int_0^\lambda dN(\zeta, \rho) \leq 2\lambda^k \int_0^\infty (\zeta^k + \lambda^k)^{-1} dN(\zeta, \rho),$$

it is sufficient to prove that

$$\int_0^\infty (\zeta^k + \lambda^k)^{-1} dN(\zeta, \rho) = O(\lambda^{d/l-(1/l)}), \lambda \to \infty .$$

By a standard construction of parametrix, we are led to evaluate the integral

$$\iint (a_0^k + \lambda^k)^{-1}(\phi\varphi)^{-1} dx d\xi$$

with $a_0 = a_0(x, \xi; \kappa, \lambda) = V(x, \lambda)^{-1}(p(x, \xi) + \kappa + q(x, \lambda)) \geq V(x)p(x, \lambda)$. Cutting the integral into the integrals on the regions $|\xi| \leq 1$ and $|\xi| \geq 1$, using respectively the assumptions (5) and (6) on $p(x, \xi)$, the inequality $\phi(x, \xi) \geq |\xi|$ and the definition of $\varphi(x)$, we have to estimate the integral

$$\iint_{|\xi|\leq 1} ((|\xi|^l V(x)^{-1})^k + \lambda^k)^{-1} |\xi|^{-1} V(x)^\rho dx d\xi$$

and the similar integral with $l$ replaced by $m$. Using the change of variables defined by

$$\xi = V(x)^{1/l}\lambda^{1/l}\zeta$$

the preceding integral is bounded by

$$\lambda^{-k+(d/l)-(1/l)} \int V(x)^{-(1/l)+\rho+(d/l)} dx \int (|\zeta|^{kl} + 1)^{-1} |\zeta|^{-1} d\zeta$$

with

$$\int V(x)^{-(1/l)+\rho+(d/l)} dx$$

finite, and similarly for the other integral. □

**Lemma 13.** *Let $\theta$ be a real number; for $\theta \geq 1/(md(1/l-\rho))$, there is $C > 0$ such that*

$$\int_{\lambda^\theta V(x)<2} V(x)^{d/l} dx \leq C\lambda^{-1/m} .$$

Proof. Let us denote by $\phi(t)$ the volume function associated with the potential $V$; then

$$\int_{\lambda^\theta V(x)<2} V(x)^{d/l} dx = -\int_0^{2\lambda^{-\theta}} t^{d/l} d\phi(t);$$

and since $\phi$ verifies (13), we deduce

$$\int_{\lambda^\theta V(x)<2} V(x)^{d/l} dx \leq C\lambda^{-\theta d(1/l-\rho)} .$$ □

**Lemma 14.** *For $\theta \leq 1/(m(1/m-\beta))$, the condition $\lambda^\theta V(x) > 1$ implies $\lambda V(x) > q(x)^m$.*

Proof. Recall that $q(x) = V(x)^\beta$. Since $\lambda > V(x)^{-1/\theta}$ implies $\lambda V(x) > V(x)^{1-1/\theta}$, it is sufficient to require

$$\frac{\theta - 1}{\theta} \leq m\beta,$$

which proves the lemma. □

**5.2.** Bound for $N(\lambda, \omega)$.

**Proposition 2.** *Let $\omega \in S(1; \phi, \varphi)$ such that* $\operatorname{Supp} \omega \subseteq \{x; \lambda^\theta V(x) < \delta\}$; *then*

$$N(\lambda, \omega) \leq C\lambda^{d/l-(1/m)} .$$

Proof. As above we denote by $\mu_j = \mu_j(\kappa, \lambda)$ and $u_j = u_j(\kappa, \lambda) \in \mathcal{S}$ respectively the eigenvalues and eigenfunctions of $A = A(\kappa, \lambda)$. Then

$$A.(\omega u_j) = \omega\ A.u_j + [A, \omega].u_j$$

consequently

$$\omega u_j = \mu_j A^{-1}.(\omega u_j) + A^{-1}[A, \omega].u_j;$$

then, multiplying by $\omega$:

$$\omega^2 u_j = \mu_j(\omega A^{-1}\omega).u_j + \omega A^{-1}[A,\omega].u_j .$$

This implies

$$||\omega^2 u_j||^2 \leq 2\mu_j^2 ||(\omega A^{-1}\omega).u_j||^2 + 2||\omega A^{-1}[A,\omega].u_j||^2 ,$$

and summing with respect to $j$ such that $\mu_j < \lambda$:

$$N(\lambda,\omega^2) \leq 2 \sum_{\mu_j<\lambda} ||(\lambda^{1/2}\omega A^{-1}\omega\lambda^{1/2}).u_j||^2 + 2N(\lambda, \omega A^{-1}[A,\omega]) \ . \tag{38}$$

We remark now that, setting $W(x,\lambda) = \lambda\omega^2(x,\lambda)V(x,\lambda)$:

$$\lambda^{1/2}\omega A^{-1}\omega\lambda^{1/2} = W(x,\lambda)^{1/2}[H_0 + \kappa + q(x,\lambda)]W(x,\lambda)^{1/2}$$

and therefore

$$\lambda^{1/2}\omega A^{-1}\omega\lambda^{1/2} \leq W(x,\lambda)^{1/2}(H_0+\kappa)^{-1}W(x,\lambda)^{1/2};$$

let us denote by $T = T(\kappa,\lambda)$ this last operator, we have to estimate

$$\sum_{\mu_j<\lambda} ||T.u_j||^2 = N(T,\lambda) \ .$$

In the particular case $d = 3$, $l = m = 2$ this operator is Hilbert-Schmidt and the conclusion is easy; but this property remains no longer true in the general case, which leads to some difficulties. First we note that

$$\sum_{\mu_j<\lambda} ||T.u_j||^2 = \mathrm{Tr}[TE_\lambda T],$$

where $(E_\lambda)$ is the spectral family associated with $A(\kappa,\lambda)$; so, if $(v_j)$ denotes the system of eigenfunctions corresponding to $T$, and $(\nu_j)$ the corresponding eigenvalues:

$$\begin{aligned}\mathrm{Tr}[TE_\lambda T] &= \sum_j \langle TE_\lambda T.v_j, v_j\rangle \\ &= \sum_j \nu_j^2 \langle E_\lambda v_j, v_j\rangle \ ;\end{aligned}$$

we apply now the Hölder inequality: the last sum is majorized by

$$\left(\sum_j \nu_j^{2p}\right)^{1/p} \left(\sum_j \langle E_\lambda v_j, v_j\rangle^q\right)^{1/q} ,$$

where $1/p + 1/q = 1$; but since $E_\lambda$ is a projector and $||v_j|| = 1$:

$$0 \leq \langle E_\lambda v_j, v_j \rangle \leq 1 \ ,$$

therefore

$$\sum_j \langle E_\lambda v_j, v_j \rangle^q \leq \sum_j \langle E_\lambda v_j, v_j \rangle = \mathrm{Tr}[E_\lambda],$$

but $\mathrm{Tr}[E_\lambda]$ is equal to the number of eigenvalues less than $\lambda$ for $A = A(\kappa, \lambda)$; consequently, by the CLR inequality:

$$\mathrm{Tr}[E_\lambda] \leq C_d \left( \lambda^{d/l} \int V(x)^{d/l} dx + \lambda^{d/m} \int V(x)^{d/m} dx \right)$$

and

$$\left( \sum_j \langle E_\lambda v_j, v_j \rangle^q \right)^{1/q} \leq C_d \lambda^{d/(lq)} \left[ \int V(x)^{d/m} dx \right]^{1/q} \ .$$

On the other hand

$$\left( \sum_j \nu_j^{2p} \right)^{1/p} = ||T||^2_{\mathcal{S}_{2p}} = \left[ - \int_0^\infty t^{2p} dn(t, T) \right]^{1/p} ,$$

(here $n(t, T)$ denotes the number of eigenvalues of $T$ greater than $t$); but $T = T(\kappa, \lambda)$ is the Birman-Schwinger operator associated with $H_0 - W(x, \lambda)$; consequently the Birman-Schwinger principle and the CLR inequality imply that

$$\begin{aligned} n(t, T) &\leq N(t^{-1} W(x, \lambda)) \\ &\leq C_d \left( t^{-d/l} \int W(x, \lambda)^{d/l} dx + t^{-d/m} \int W(x, \lambda)^{d/m} dx \right) . \end{aligned}$$

By definition $W(x, \lambda) = \lambda \omega(x, \lambda)^2 V(x, \lambda)$, and by (29) and the assumption on $\mathrm{Supp}\, \omega$:

$$n(t, T) \leq C_{d,p} \left( t^{-d/l} + t^{-d/m} \right) \lambda^{d/l} \int_{\lambda^\theta V(x) < 2} V(x)^{d/m} dx$$

for $2p - (d/l) > 0$, and then

$$\left( \sum_j \nu_j^{2p} \right)^{1/p} \leq C \lambda^{d/(lp) - (\theta d/p)(1/m - \rho)} \ .$$

We fix now the values for $\theta$ and then for $p$: let

$$\theta_0 = \frac{1}{m(1/m - \rho)};$$

we remark that for

$$\frac{\theta_0}{d} \leq \theta = \frac{1}{m(1/m - \beta)} < \theta_0,$$

the conclusion of Lemmas 13 and 14 remains valid, and that

$$\theta_0 d \left( \frac{1}{m} - \rho \right) = \frac{d}{m} > \frac{d}{2l} . \frac{1}{m}$$

since $l \geq 1$; therefore there exists $\theta = 1/(m(1/m - \beta))$ such that $\theta_0/d \leq \theta < \theta_0$ and

$$\frac{d}{2l} . \frac{1}{m} < \theta d \left( \frac{1}{m} - \rho \right) < \theta_0 d \left( \frac{1}{m} - \rho \right)$$

and lastly there is $p > d/2l$ such that

$$\theta d \left( \frac{1}{m} - \rho \right) = \frac{p}{m}$$

which implies

$$N(\lambda, T) \leq C \lambda^{d/l - (1/m)}.$$

Finally the operator $\omega A^{-1}[A, \omega]$ in the right side of (37) verifies

$$\omega A^{-1}[A, \omega] = Op\ \rho$$

with $\rho \in S((\phi\varphi)^{-1}; \phi\varphi)$ since $\omega = 1$ when $\lambda^\theta V(x) < 1$, and we can apply Lemma 12 for estimating $N(\lambda, \rho)$. □

**Proposition 3.** *Let* $\omega \in S(1; \phi, \varphi)$ *such that* $\operatorname{Supp} \omega \subseteq \{x; \lambda^\theta V(x) < \delta\}$; *then*

$$\left| N(\lambda, \omega) - (2)^{-n} \iint_{a_0 < \lambda} \omega^2 dx d\xi \right| \leq C \lambda^{d/l - (1/m)} .$$

Proof. By the preceding lemma, it suffices to verify that

$$\iint_{a_0 < \lambda} \omega^2 dx d\xi \leq C \lambda^{d/l - (1/m)} ;$$

but this inequality results from the inequalities

$$\iint_{a_0 < \lambda} \omega^2 dx d\xi \leq C_1 \iint_{|\xi| \geq 1,\ |\xi|^l \leq C\lambda V(x)} dx d\xi + C_2 \iint_{|\xi| \leq 1,\ |\xi|^m \leq C\lambda V(x)} dx d\xi . \quad \square$$

## 6. Functional calculus

**6.1.** The aim of this section is to estimate $N(\lambda, \omega)$ when the support of $\omega$ is such that $\mathrm{Supp}\,\omega \subseteq U_\varepsilon(v) \subseteq V_1$, with $V_1$ defined at the beginning of Section 4.1. Let us denote by $\theta = \theta(x, \xi; \kappa, \lambda)$ the symbol in $S(1; \phi, \varphi)$ given by the composition of pseudodifferential operators:

$$[Op^W \omega]^2 = Op^W \theta + Op^W \rho \tag{39}$$

with $\rho \in S((\phi\varphi)^{-N}; \phi\varphi)$ for $N$ sufficiently large. Let $(E_\lambda)$ be the spectral family associated with $A = A(\kappa, \lambda)$; since $N(\lambda, \omega) = \mathrm{Tr}[\omega E_\lambda \omega]$, we will approximate $N(\lambda, \omega)$ by $\mathrm{Tr}[\omega f(A)\omega]$ for suitable $f \in \mathcal{S}$. This leads first to develop a functional calculus for $A = A(\kappa, \lambda)$ in the spirit of [8].

**6.2.** We do not go into the details of the proofs because this procedure is well known and we content ourselves to indicate the different steps. The first step is the construction of a local right parametrix $B^\omega_{z,N}$ for $A - z$, that is a pseudodifferential operator satisfying

$$(A - z)B^\omega_{z,N} = Op^W \omega + R^\omega_{z,N},$$

with $R^\omega_{z,N}$ of trace-class. Let us set

$$B^\omega_{z,N} = \sum_{j=0}^{n} Op^W b_{z,j}$$

where the symbols $b_{z,j}$ are defined inductively by

$$b_{z,0}(a_o - z) = \omega,$$

and for $j \geq 1$ by

$$b_{z,j}(a_0 - z) + \sum \Gamma(\alpha, \beta)\partial^\alpha_\xi D^\beta_x a_0 \partial^\alpha_\xi D^\beta_x b_{z,k} = 0,$$

with $0 \leq k < j$ and $k + |\alpha| + |\beta| = j$. Since $\mathrm{Supp}\, b_{z,j} \subseteq \mathrm{Supp}\,\omega$ for all $j \in \mathbf{N}$, the symbol $a_0$ verifies the estimation explicited in Lemma 5. Consequently

$$(A - z)^{-1}\omega = B^\omega_{z,N} - (A - z)^{-1} R^\omega_{z,N} .$$

By composition with $\omega$:

$$\omega(A - z)^{-1}\omega = B_{z,N} + R_{z,N}$$

with $R_{z,N}$ of trace class.

**6.3.** This enable us to define $\omega A^s \omega$ for $s \in \mathbf{C}$ and then $\omega f(A)\omega$ for $f \in \mathcal{S}$ via the Mellin transform; more precisely (see [6]):

$$\omega f(A)\omega = \sum_{j=0}^{N} Op^W a_{f,j} + R_N.$$

On the remainder we have the following estimate

**Lemma 15.** *There is $N_1, N_2 > 0$ and $C_1, C_2 > 0$ such that for $N$ sufficiently large and $f \in \mathcal{S}$ such that $f^{(k)}(0) = 0$ for $k \geq 1$ and $f(0) = 1$:*

$$||R_N||_{\mathrm{Tr}} \leq C_1 \int_0^\infty t^{-\beta N + N_1} \left| \left( \frac{t\partial}{\partial t} \right)^{2N+N_2} f(t) \right| dt + C_2 \ .$$

Proof. We have only to follow [6] and to use in particular Lemma 6 above. □

**Lemma 16.** *Let $f \in C_0^\infty$ such that $f = 1$ on $[0, 2\lambda]$, $f(t) = 0$ when $t > 2\lambda$. Then*

$$\left| \mathrm{Tr}[\omega f(A)\omega] - (2\pi)^{-n} \iint \theta(x,\xi;\kappa,\lambda) f(a_0)(x,\xi;\kappa,\lambda) dx d\xi \right| \leq C \ .$$

Proof. See [1], Proposition 3 for the details. □

**Proposition 4.** *Let $\omega$ be a bounded symbol such that* $\mathrm{Supp}\,\omega \subseteq V_1$; *then*

$$\left| N(\lambda,\omega) - (2\pi)^{-n} \iint_{a_0(x,\xi;\kappa,\lambda)<\lambda} \theta(x,\xi;\kappa,\lambda) dx d\xi \right| \leq C \ .$$

Proof. Let $\theta$ be a real number such that $0 < \theta < \beta/2$ and

$$J(\lambda,\theta) = \left[\lambda - \lambda^{1-\theta}, \lambda + \lambda^{1-\theta}\right], \quad I(\lambda,\theta) = \frac{\left[\lambda - \lambda^{1-\theta}/2, \lambda + \lambda^{1-\theta}\right]}{2};$$

let $f_{\lambda,\theta}, g_{\lambda,\theta} \in C_0^\infty$ such that

$$\mathrm{Supp}\, g_{\lambda,\theta} \subseteq J(\lambda,\theta),\ g_{\lambda,\theta}(t) = 1 \text{ for } t \in I(\lambda,\theta),$$
$$\mathrm{Supp}\, f_{\lambda,\theta} \subseteq \left(0, \lambda + \frac{\lambda^{1-\theta}}{2}\right),$$

and

$$f_{\lambda,\theta}(t) = 1 \text{ for } t \notin I(\lambda,\theta), t < \lambda;$$

and we suppose the following assumptions on the derivatives: for all $k \in \mathbf{N}$, there is $C_k > 0$ such that

$$\left|\left(\frac{t\partial}{\partial t}\right)^k f_{\lambda,\theta}(t)\right| + \left|\left(\frac{t\partial}{\partial t}\right)^k g_{\lambda,\theta}(t)\right| \leq C_k \lambda^{k\theta} \ . \tag{40}$$

It results of the definition of $f_{\lambda,\theta}, g_{\lambda,\theta}$ that

$$|\chi_{(0,\lambda)} - f_{\lambda,\theta}| \leq g_{\lambda,\theta} \leq \chi_{J(\lambda,\theta)} \ . \tag{41}$$

But for $\lambda$ sufficiently large: $I(\lambda, \theta) \subseteq [\lambda/2, 2\lambda]$; therefore, using (39) and Lemma 15:

$$|R_N(\lambda,\theta)| \leq C \int_0^{2\lambda} t^{-\beta N + N_1 + (2N+N_2)\theta} dt \leq C \lambda^{-\beta N + N_1 + (2N+N_2)\theta} \ ;$$

the exponent can be rewritten: $N(2\theta - \beta) + \delta_0$ and since $0 < \theta < \beta/2$, it will be negative for $N$ sufficiently large. So

$$|R_N(\lambda,\theta)| \leq C \ .$$

Now

$$\begin{aligned}
&\left|N(\lambda,\omega) - (2\pi)^{-n} \iint_{a_0(x,\xi;\kappa,\lambda)<\lambda} \theta(x,\xi;\kappa,\lambda) dx d\xi\right| \\
&\quad \leq |\operatorname{Tr}[\omega^2 E_\lambda] - \operatorname{Tr}[\omega^2 f_{\lambda,\theta}(A)]| \\
&\qquad + \left|\operatorname{Tr}[\omega f_{\lambda,\theta}(A)\omega] - (2\pi)^{-n} \iint \theta(x,\xi;\kappa,\lambda) f(a_0(x,\xi;\kappa,\lambda)) dx d\xi\right| \\
&\qquad + \left|(2\pi)^{-n} \iint \theta(x,\xi;\kappa,\lambda) f(a_0(x,\xi;\kappa,\lambda)) dx d\xi \right. \\
&\qquad\quad \left. -(2\pi)^{-n} \iint_{a_0(x,\xi;\kappa,\lambda)<\lambda} \theta(x,\xi;\kappa,\lambda) dx d\xi\right| \ .
\end{aligned}$$

But, by (40), the first term of the right side is bounded by

$$\operatorname{Tr}[\omega^2 g(A)] \leq C \ .$$

For the second term, we use Lemma 16; finally we remark that

$$\iint \theta(x,\xi;\kappa,\lambda) f(a_0(x,\xi;\kappa,\lambda)) dx d\xi = \iint_{a_0(x,\xi;\kappa,\lambda)<\lambda} \theta(x,\xi;\kappa,\lambda) dx d\xi \ . \qquad \square$$

## 7. Tauberian technics

In this section we estimate $N(\lambda,\omega)$ when $\operatorname{Supp}\omega \subseteq U_\varepsilon(v) \subseteq V_2$, where $V_2$ is defined in 4.1. For this region of the phase space, we use a tauberian method: more precisely we adapt the Hörmander-Levitan method to our case. This have been made in [15], [11], and in our paper [1].

**7.1.** Approximation of the unitary group.
Using convenient classes of Fourier integral operator we can prove

**Proposition 5.** *There is* $\rho > 0$ *such that, setting* $\rho_v = \rho(\phi\varphi)(v)$, *then for* $N \in \mathbf{N}$ *sufficiently large and* $t \in (-\rho_v, \rho_v)$

$$\omega e^{-itA}\omega = F_N^{(0)}(t;\kappa,\lambda) + F_N^{(1)}(t;\kappa,\lambda)\,, \tag{42}$$

*wher* $F_N^{(0)}(t;\kappa,\lambda)$ *is a Fourier integral operator and* $F_N^{(1)}(t;\kappa,\lambda)$ *is a trace class operator such that*: *there is* $C > 0$ *such that for* $j = 0, 1, 2$

$$\| \partial_t^j F_N^{(1)}(t;\kappa,\lambda) \|_{\mathrm{Tr}} \leq C(\phi\varphi)(v)^{-N}\ . \tag{43}$$

Proof. We follow the proof of Proposition 9 in [1], which is rather long and technical; we need in particular the result of Lemma 7. □

**7.2.** Estimation of $N(\lambda,\omega)$.
For a bounded symbol $\omega$, let $\theta$ be the symbol defined as in (38).

**Proposition 6.** *Let* $\omega$ *be a bounded symbol such that* $\mathrm{Supp}\,\omega \subseteq U_\varepsilon(v) \subseteq V_2$; *there is* $C_1, C_2 > 0$ *and* $\delta > 0$ *such that*

$$\left| N(\lambda,\omega) - (2\pi)^{-d} \iint_{a_0(x,\xi;\kappa,\lambda)<\lambda} \theta(x,\xi;\kappa,\lambda)dxd\xi \right| \leq$$
$$C_1 \iint_{\left\{\substack{\delta^{-1}\lambda < a_0(x,\xi;\kappa,\lambda) < \delta\lambda \\ (x,\xi)\in U_k(3\varepsilon)}\right.} (\phi\varphi)(x,\xi)^{-1}dxd\xi + C_2(\phi\varphi)(v)^{-N}\ .$$

Proof. We follow the proof of Proposition 11 in [1]. □

## 8. Proof of Theorem 1

Let $N$ sufficiently large such that

$$\sum_k (\phi\varphi)^{-N}(v_k) < \infty \text{ and } \iint (\phi\varphi)^{-N}(x,\xi)dxd\xi < \infty\ .$$

From the definitions of the functions $\omega_k(\lambda)$ and $\theta_k(\lambda)$ it results that

$$\left| 1 - \omega - \sum_{k\in\mathcal{J}(\lambda)} \theta_k(x,\xi;\lambda) \right| \leq C(\phi\varphi)^{-2N}(x,\xi)\ .$$

The conclusion is obtained by summation with respect to $k$ by using the property that each point in $\mathbf{R}^{2d}$ belongs to at most $N$ balls $U_k(3\varepsilon)$:

$$\sum_k \iint_{\left\{\substack{\delta^{-1}\lambda<a_0(x,\xi;\kappa,\lambda)<\delta\lambda \\ (x,\xi)\in U_k(3\varepsilon)}\right.} (\phi\varphi)^{-1}(x,\xi)dxd\xi \;\leq$$
$$C \iint_{\delta^{-1}\lambda<a_0(x,\xi;\kappa,\lambda)<\delta\lambda} (\phi\varphi)^{-1}(x,\xi)dxd\xi \;.$$

Then

$$\left| N_\kappa(\lambda V) - (2\pi)^{-d} \iint_{a_0<\lambda} dxd\xi \right| \leq C_1 + C_2 \iint_{\delta^{-1}\lambda<a_0<\delta\lambda} (\phi\varphi)^{-1} dxd\xi.$$

Now we let $\kappa$ tend to 0, and we apply the Lebesgue dominated convergence theorem and the estimation used in the proof of Lemma 12.

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