Masuda, K.
Osaka J. Math.
38 (2001), 501–506

MODULI OF ALGEBRAIC SL_3-VECTOR BUNDLES OVER ADJOINT REPRESENTATION

KAYO MASUDA

(Received June 21, 1999)

1. Introduction and result

Let G be a reductive complex algebraic group and P a complex G-module. We consider algebraic G-vector bundles over P. An algebraic G-vector bundle E over P is an algebraic vector bundle $p : E \to P$ together with a G-action such that the projection p is G-equivariant and the action on the fibers is linear. We assume that G is non-abelian since every G-vector bundle over P is isomorphic to a trivial G-bundle $P \times Q \to P$ for a G-module Q when G is abelian by Masuda-Moser-Petrie [12]. We denote by $\mathrm{VEC}_G(P, Q)$ the set of equivariant isomorphism classes of algebraic G-vector bundles over P whose fiber over the origin is a G-module Q. The isomorphism class of a G-vector bundle E is denoted by $[E]$. The set $\mathrm{VEC}_G(P, Q)$ is a pointed set with a distinguished class $[\mathbf{Q}]$ where $\mathbf{Q}$ is the trivial G-bundle $P \times Q$, and can be non-trivial when the dimension of the algebraic quotient space $P/\!/G$ is greater than 0 ([15], [2], [13], [11]). In fact, Schwarz ([15], cf. Kraft-Schwarz [5]) showed that $\mathrm{VEC}_G(P, Q)$ is isomorphic to an additive group $\mathbb{C}^p$ for a nonnegative integer p determined by P and Q when $\dim P/\!/G = 1$. When $\dim P/\!/G \geq 2$, $\mathrm{VEC}_G(P, Q)$ is not necessarily finite-dimensional. In fact, $\mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q) \cong (\mathbb{C}[y_1, \cdots, y_m])^p$ for a G-module P with one-dimensional quotient [9]. Furthermore, Mederer [14] showed that $\mathrm{VEC}_G(P, Q)$ can contain a space of uncountably-infinite dimension. He considered the case where G is a dihedral group $D_m = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/m\mathbb{Z}$ and P is a two-dimensional G-module V_p, on which $\mathbb{Z}/m\mathbb{Z}$ acts with weights p and $-p$ and the generator of $\mathbb{Z}/2\mathbb{Z}$ acts by interchanging the weight spaces. Mederer showed that $\mathrm{VEC}_{D_3}(V_1, V_1)$ is isomorphic to $\Omega^1_{\mathbb{C}}$ which is the universal Kähler differential module of $\mathbb{C}$ over $\mathbb{Q}$. In this article, we show that under some conditions there exists a surjection from $\mathrm{VEC}_G(P, Q)$ to $\mathrm{VEC}_{D_3}(V_1, V_1) \cong \Omega^1_{\mathbb{C}}$. It is induced by taking a H-fixed point set E^H for $[E] \in \mathrm{VEC}_G(P, Q)$ where H is a reductive subgroup of G (cf. Proposition 2.3). In particular, we obtain the first example of a moduli space of uncountably-infinite dimension for a connected group.

2000 *Mathematics Subjects Classification*. Primary: 14R20; Secondary: 14L30, 14D20.
algebraic G-vector bundles, Equivariant Serre Problem.

Theorem 1.1. *Let $G = SL_3$ and let $\mathfrak{sl}_3$ be the Lie algebra with adjoint action. Then for any G-module R, there exists a surjection from $\mathrm{VEC}_G(\mathfrak{sl}_3 \oplus R, \mathfrak{sl}_3)$ onto $\Omega^1_{\mathbb{C}}$. Hence $\mathrm{VEC}_G(\mathfrak{sl}_3 \oplus R, \mathfrak{sl}_3)$ contains an uncountably-infinite dimensional space.*

At present, G-vector bundles over P are not yet classified for general G-modules P with $\dim P/\!/G \geq 2$ (cf. [10]). Theorem 1.1 suggests that the moduli space $\mathrm{VEC}_G(P, Q)$ is huge when $\dim P/\!/G \geq 2$.

I am thankful to M. Miyanishi for his help and encouragement. I thank the referees for giving advices to the previous version of this paper.

2. Proof of Theorem 1.1

Let G be a reductive algebraic group and let P and Q be G-modules. Let $\pi_P : P \to P/\!/G$ be the algebraic quotient map. By Luna's slice theorem [6], there is a finite stratification of $P/\!/G = \cup_i V_i$ into locally closed subvarieties V_i such that $\pi_P|_{\pi_P^{-1}(V_i)} : \pi_P^{-1}(V_i) \to V_i$ is a G-fiber bundle (in the étale topology) and the isotropy groups of closed orbits in $\pi_P^{-1}(V_i)$ are all conjugate to a fixed reductive subgroup H_i. The unique open dense stratum of $P/\!/G$, which we denote by U, is called the principal stratum and the corresponding isotropy group, which we denote by H, is called a principal isotropy group. We denote by $\mathrm{VEC}_G(P, Q)_0$ the subset of $\mathrm{VEC}_G(P, Q)$ consisting of elements which are trivial over $\pi_P^{-1}(U)$ and $\pi_P^{-1}(V)$ for $V := P/\!/G - U$. When $\dim P/\!/G = 1$, it is known that $\mathrm{VEC}_G(P, Q) = \mathrm{VEC}_G(P, Q)_0$ ([15], [5]). We assume that the dimension of $Y := P/\!/G$ is greater than 1 and the ideal of V is principal. We denote by $\mathcal{O}(P)$ the $\mathbb{C}$-algebra of regular functions on P and by $\mathcal{O}(P)^G$ the subalgebra of G-invariants of $\mathcal{O}(P)$. Let f be a polynomial in $\mathcal{O}(Y) = \mathcal{O}(P)^G$ such that the ideal (f) defines V.

Lemma 2.1. *Let $[E] \in \mathrm{VEC}_G(P, Q)_0$. Then E is trivial over $P_h := \{x \in P \mid h(x) \neq 0\}$ where h is a polynomial in $\mathcal{O}(Y)$ such that $h - 1 \in (f)$.*

Proof. Since $E|_{\pi_P^{-1}(V)}$ is, by the assumption, isomorphic to a trivial bundle, it follows from the Equivariant Nakayama Lemma [1] that the trivialization $E|_{\pi_P^{-1}(V)} \to \pi_P^{-1}(V) \times Q$ extends to a trivialization over a G-stable open neighborhood $\tilde{U}$ of $\pi_P^{-1}(V)$. Let $\tilde{V}$ be the complement of $\tilde{U}$ in P. Since $\tilde{V}$ is a G-invariant closed set, $\pi_P(\tilde{V})$ is closed in Y [4]. Note that $V \cap \pi_P(\tilde{V}) = \emptyset$ since $\pi_P^{-1}(V) \cap \tilde{V} = \emptyset$. Let $\mathfrak{a} \subset \mathcal{O}(Y)$ be the ideal which defines $\pi_P(\tilde{V})$. Then $(f) + \mathfrak{a} \ni 1$ since $V \cap \pi_P(\tilde{V}) = \emptyset$. Hence there exists an $h \in \mathfrak{a}$ such that $h - 1 \in (f)$. Since $P_h \subset \tilde{U}$, E is trivial over P_h. □

We define an affine scheme $\tilde{Y} = \mathrm{Spec}\, \tilde{A}$ by

$$\tilde{A} = \{h_1/h_2 | h_1, h_2 \in \mathcal{O}(Y), h_2 - 1 \in (f)\}.$$

Set $\tilde{Y}_f := Y_f \times_Y \tilde{Y}$, $\tilde{P} := \tilde{Y} \times_Y P$ and $\tilde{P}_f := \tilde{Y}_f \times_Y P$. The group of morphisms from P to $M := \mathrm{GL}(Q)$ is denoted by $\mathrm{Mor}\,(P, M)$ or $M(P)$. The group G acts on M by conjugation and on $M(P)$ by $(g\mu)(x) = g \cdot (\mu(g^{-1}x))$ for $g \in G$, $x \in P$, $\mu \in M(P)$. The group of G-invariants of $M(P)$ is denoted by $\mathrm{Mor}(P, M)^G$ or $M(P)^G$. Let $[E] \in \mathrm{VEC}_G(P, Q)_0$. Then by definition of $\mathrm{VEC}_G(P, Q)_0$, E has a trivialization over $\pi_P^{-1}(U) = P_f$. By Lemma 2.1, E has a trivialization also over an open neighborhood of $\pi_P^{-1}(V)$, i.e., P_h for some $h \in \mathcal{O}(Y)$ with $h - 1 \in (f)$. Hence, E is isomorphic to a G-vector bundle obtained by glueing two trivial G-vector bundles $P_f \times Q$ and $P_h \times Q$ over P_{fh}. Note that the transition function of E is an element of $M(P_{fh})^G \subset M(\tilde{P}_f)^G$. Conversely, if $\phi \in M(\tilde{P}_f)^G$ is given, then $\phi \in M(P_{fh})^G$ for some $h \in \mathcal{O}(Y)$ with $h - 1 \in (f)$ and we obtain a G-vector bundle $[E] \in \mathrm{VEC}_G(P, Q)_0$ by glueing together trivial bundles $P_f \times Q$ and $P_h \times Q$ by ϕ. Since $[E]$ is determined by the transition function $\phi \in M(P_{fh})^G$ up to automorphisms of trivial G-bundles $P_f \times Q$ and $P_h \times Q$, we have a bijection to a double coset (cf. [8, 3.4])

$$\mathrm{VEC}_G(P, Q)_0 \cong M(P_f)^G \backslash M(\tilde{P}_f)^G / M(\tilde{P})^G.$$

The inclusion $P^H \hookrightarrow P$ induces an isomorphism $P^H /\!/ N(H) \xrightarrow{\sim} P /\!/ G$ where $N(H)$ is the normalizer of H in G. The stratification of $P /\!/ G$ coincides with the one induced by $P^H /\!/ N(H)$ [7]. Set $W := N(H)/H$. When we consider P^H as a W-module, we denote it by B. Let $L := \mathrm{GL}(Q)^H$. By an observation similar to the case of $\mathrm{VEC}_G(P, Q)_0$, we have

$$\mathrm{VEC}_{N(H)}(P^H, Q)_0 \cong L(B_f)^W \backslash L(\tilde{B}_f)^W / L(\tilde{B})^W.$$

Let $\beta : M(P)^G \to L(B)^W$ be the restriction map. We say P *has generically closed orbits* if $\pi_P^{-1}(\xi)$ for any $\xi \in Y_f$ consists of a closed orbit, i.e. $\pi_P^{-1}(\xi) \cong G/H$. When P has generically closed orbits, $P_f = G P_f^H$. Hence $M(P_f)^G = \mathrm{Mor}\,(G P_f^H, \mathrm{GL}(Q))^G \cong L(B_f)^W$, i.e. β is an isomorphism over Y_f.

Let $[E] \in \mathrm{VEC}_G(P, Q)$. The H-fixed point set E^H is equipped with a W-vector bundle structure over B. The fiber of E^H over the origin is a W-module Q^H. Hence there is a map

$$r_H : \mathrm{VEC}_G(P, Q) \ni [E] \mapsto [E^H] \in \mathrm{VEC}_W(B, Q^H).$$

Note that r_H factors through $\mathrm{VEC}_{N(H)}(P^H, Q)$ since the restricted bundle $[E|_{P^H}] \in \mathrm{VEC}_{N(H)}(P^H, Q)$ splits to a Whitney sum of trivial H-bundles [3] and $(E|_{P^H})^H = E^H$. Note also that r_H maps $\mathrm{VEC}_G(P, Q)_0$ to $\mathrm{VEC}_W(B, Q^H)_0$.

Lemma 2.2. *Suppose that P has generically closed orbits. Then*

$$r_H : \mathrm{VEC}_G(P, Q)_0 \to \mathrm{VEC}_W(B, Q^H)_0$$

is surjective.

Proof. By the above statement, it is sufficient to show that the restriction map $res : \mathrm{VEC}_G(P, Q)_0 \ni [E] \mapsto [E|_{P^H}] \in \mathrm{VEC}_{N(H)}(P^H, Q)_0$ is surjective. Note that the map res coincides with the map on double cosets induced by $\beta : M(P)^G \to L(B)^W$;

$$M(P_f)^G \backslash M(\tilde{P}_f)^G / M(\tilde{P})^G \to L(B_f)^W \backslash L(\tilde{B}_f)^W / L(\tilde{B})^W.$$

Let $[E] \in \mathrm{VEC}_{N(H)}(P^H, Q)_0$ and let $\phi \in L(B_{fh})^W$, where $h \in \mathcal{O}(Y)$ such that $h - 1 \in (f)$, be the transition function corresponding to E. Since β is an isomorphism over Y_f, $\phi \in L(B_{fh})^W$ extends to $\overline{\phi} \in M(P_{fh})^G$. The G-vector bundle $\overline{E}$ obtained by glueing trivial bundles over P_f and P_h by $\overline{\phi}$ is mapped to E by res. ☐

REMARK. It seems that the restriction $r_H : \mathrm{VEC}_G(P, Q) \to \mathrm{VEC}_W(B, Q^H)$ is not necessarily surjective, though the author does not know any counterexamples. Every G-vector bundle over a G-module is locally trivial [3], however, it seems difficult that a set of transition functions of a W-vector bundle over B with fiber Q^H extends to a set of transition functions of some G-vector bundle over P with fiber Q; some conditions seem to be needed so that the restriction $M(X)^G \to L(X^H)^W$ is surjective for a G-stable open set X of P such that $X \not\subset \pi_P^{-1}(U)$ (cf. [17, III,11]).

For any reductive subgroup K of G, we can construct a map r_K similarly;

$$r_K : \mathrm{VEC}_G(P, Q) \ni [E] \mapsto [E^K] \in \mathrm{VEC}_{W_K}(P^K, Q^K)$$

where $W_K := N(K)/K$. Assume that W_K contains a subgroup isomorphic to D_3 and that P^K and Q^K contain V_1 as D_3-modules, say, as D_3-modules $P^K = V_1 \oplus P'$ and $Q^K = V_1 \oplus Q'$ for D_3-modules P' and Q'. Restricting the group W_K to D_3, we have a map

$$\mathrm{VEC}_{W_K}(P^K, Q^K) \to \mathrm{VEC}_{D_3}(V_1 \oplus P', V_1 \oplus Q'). \tag{1}$$

Furthermore, the natural inclusion $V_1 \to V_1 \oplus P'$ induces a surjection

$$\mathrm{VEC}_{D_3}(V_1 \oplus P', V_1 \oplus Q') \to \mathrm{VEC}_{D_3}(V_1, V_1 \oplus Q'). \tag{2}$$

By taking a composite of the maps r_K, (1) and (2), we obtain a map $\Phi_K : \mathrm{VEC}_G(P, Q) \to \mathrm{VEC}_{D_3}(V_1, V_1 \oplus Q')$. By Mederer [14], $\mathrm{VEC}_{D_3}(V_1, V_1 \oplus Q') \cong \Omega^1_{\mathbb{C}}/S_{Q'}$ where $S_{Q'}$ is a subspace of $\Omega^1_{\mathbb{C}}$, but unfortunately, $S_{Q'}$ is not known so far except when $Q' = \{0\}$. When $Q' = \{0\}$, i.e. $Q^K \cong V_1$ as a D_3-module, we have a map

$$\Phi_K : \mathrm{VEC}_G(P, Q) \to \mathrm{VEC}_{D_3}(V_1, V_1) \cong \Omega^1_{\mathbb{C}}.$$

In the case where $K = H$ and $N(H)/H \cong D_3$, the map Φ_H constructed as above can be surjective.

Proposition 2.3. *Let H be a principal isotropy group of P and let $N(H)/H \cong D_3$. Suppose that P has generically closed orbits. If P^H contains V_1 as a D_3-module and $Q^H \cong V_1$ as a D_3-module, then the map*

$$\Phi_H : \mathrm{VEC}_G(P, Q) \to \Omega^1_{\mathbb{C}}$$

is surjective.

Proof. The assertion follows from Lemma 2.2 and the fact that $\mathrm{VEC}_{D_3}(V_1, V_1)_0 = \mathrm{VEC}_{D_3}(V_1, V_1) \cong \Omega^1_{\mathbb{C}}$ [14]. □

The condition on the fiber Q in Proposition 2.3 is rather strict. However, by Proposition 2.3, we obtain the first example of a moduli space of uncountably-infinite dimension for a connected group G.

Proof of Theorem 1.1. Let $G = SL_3$ and let $\mathfrak{sl}_3$ be the Lie algebra with adjoint action. A principal isotropy group of $\mathfrak{sl}_3$ is a maximal torus $T \cong (\mathbb{C}^*)^2$ and ${\mathfrak{sl}_3}^T$ is the Lie algebra $\mathfrak{t}$ of T. $N(T)/T$ is the Weyl group which is isomorphic to the symmetric group $S_3 \cong D_3$ and ${\mathfrak{sl}_3}^T = \mathfrak{t} \cong V_1$ as a D_3-module. The algebraic quotient space is $\mathfrak{sl}_3/\!/G \cong \mathfrak{t}/\!/S_3 \cong \mathbb{A}^2$. The complement of the principal stratum in $\mathfrak{sl}_3/\!/G \cong \mathbb{A}^2$ is defined by $y^2 - x^3 = 0$. The general fiber of the quotient map $\mathfrak{sl}_3 \to \mathfrak{sl}_3/\!/G$ is isomorphic to G/T and $\mathfrak{sl}_3$ has generically closed orbits. Applying Proposition 2.3 to the case where $P = \mathfrak{sl}_3$ and $Q = \mathfrak{sl}_3$, we obtain a surjection $\mathrm{VEC}_G(\mathfrak{sl}_3, \mathfrak{sl}_3) \to \Omega^1_{\mathbb{C}}$. Since there is a surjection $\mathrm{VEC}_G(\mathfrak{sl}_3 \oplus R, \mathfrak{sl}_3) \to \mathrm{VEC}_G(\mathfrak{sl}_3, \mathfrak{sl}_3)$ induced by the inclusion $\mathfrak{sl}_3 \to \mathfrak{sl}_3 \oplus R$ for any G-module R, Theorem 1.1 follows. □

References

[1] H. Bass and S. Haboush: *Linearizing certain reductive group actions,* Trans. Amer. Math. Soc. **292** (1985), 463–482.

[2] F. Knop: *Nichtlinearisierbare Operationen halbeinfacher Gruppen auf affinen Räumen,* Invent. Math. **105** (1991), 217–220.

[3] H. Kraft: *G-vector bundles and the linearization problem in* "Group actions and invariant theory", CMS Conference Proceedings, **10** (1989), 111–123.

[4] H. Kraft: Geometrische Methoden in der Invariantentheorie, Aspecte der Mathematik D1, Vieweg Verlag, Braunschweig, 1984.

[5] H. Kraft and G.W. Schwarz: *Reductive group actions with one-dimensional quotient,* Publ. Math. IHES, **76** (1992), 1–97.

[6] D. Luna: *Slice etales,* Bull.Soc.Math.France, Memoire, **33** (1973), 81–105.

[7] D. Luna: *Adhérences d'orbite et invariants,* Invent. Math. **29** (1975), 231–238.

[8] K. Masuda: *Moduli of equivariant algebraic vector bundles over affine cones with one-dimensional quotient*, Osaka J. Math. **32** (1995), 1065–1085.

[9] K. Masuda: *Moduli of equivariant algebraic vector bundles over a product of affine varieties*, Duke Math. J. **88** (1997), 181–199.

[10] K. Masuda: *Certain moduli of algebraic G-vector bundles over affine G-varieties*, preprint.

[11] M. Masuda, L. Moser-Jauslin and T. Petrie: *Invariants of equivariant algebraic vector bundles and inequalities for dominant weights*, Topology, **37** (1998), 161–177.

[12] M. Masuda, L. Moser-Jauslin and T. Petrie: *The equivariant Serre problem for abelian groups*, Topology, **35** (1996), 329–334.

[13] M. Masuda and T. Petrie: *Stably trivial equivariant algebraic vector bundles*, J. Amer. Math. Soc. **8** (1995), 687–714.

[14] K. Mederer: *Moduli of G-equivariant vector bundles*, Ph.D thesis, Brandeis University (1995).

[15] G.W. Schwarz: *Exotic algebraic group actions*, C. R. Acad. Sci. Paris, **309** (1989), 89–94.

[16] G.W. Schwarz: *Representaions of simple Lie groups with a free module of covariants*, Invent. Math., **50** (1978), 1–12.

[17] G.W. Schwarz: *Lifting smooth homotopies of orbit spaces*, Publ. Math. IHES, **51** (1980), 37–135.

Mathematical Science II
Himeji Institute of Technology
2167 Shosha, Himeji 671-2201
Japan
e-mail: kayo@sci.himeji-tech.ac.jp